Precoloring extension of Vizing's Theorem for multigraphs

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Abstract

Let G be a graph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Vizing and Gupta, independently, proved in the 1960s that the chromatic index of G is at most $\Delta(G) + \mu(G)$. The distance between two edges in G is the number of edges contained in a shortest path in G between any of their endvertices. A *distance-t matching* is a set of edges having pairwise distance at least t. Edwards et al. proposed a conjecture: For any graph G, using the palette $\{1, \ldots, \Delta(G)+\mu(G)\}$, any precolored distance-2 matching can be extended to a proper edge coloring of G. Girão and Kang verified this conjecture for distance-9 matchings. In this paper, we improve the required distance from 9 to 3 for multigraphs G with $\mu(G) \geq 2$.

Keywords: Edge coloring; Precoloring extension; Vizing's Theorem; Dense subgraph; Multi-fan

1 Introduction

In this paper, we generally follow the book [15] of Stiebitz et al. for notation and terminology. Graphs in this paper are finite, undirected, and without loops, but may have multiple edges.

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Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are respectively the vertex set and the edge set of the graph G. Let $\Delta(G)$ and $\mu(G)$ be respectively maximum degree and maximum multiplicity of graph G. Let $[k] := \{1, \dots, k\}$ be a palette of k available colors. A *k-edge-coloring* of G is a map φ that assigns to every edge e of G a color from the palette $[k]$ such that no two adjacent edges receive the same color (the edge coloring is also called *proper*). Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer k such that $\mathcal{C}^k(G) \neq \emptyset$.

In the 1960s, Vizing [17] and, independently, Gupta [13] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ which is always called Vizing's Theorem. Using the palette $[\Delta(G) + \mu(G)]$, when can we extend a precolored edge set $F \subseteq E(G)$ to a proper edge coloring of G? To address this natural generalization of Vizing's Theorem, we consider edge set F such that its edges are far apart from each other. The distance between two edges in G is the number of edges contained in a shortest path in G between any of their endvertices. A distance-t matching is a set of edges having pairwise distance at least t. Following this definition, a matching is a distance-1 matching and an induced matching is a distance-2 matching.

Albertson and Moore [2] conjectured that if G is a simple graph, using the palette $[\Delta(G)+1]$, any precolored distance-3 matching can be extended to a proper edge coloring of G. Edwards et al. [8] proposed a stronger conjecture: *For any graph G, using the palette $[\Delta(G)+\mu(G)]$, any precolored distance-2 matching can be extended to a proper edge coloring of G.* Girão and Kang [9] verified this conjecture for distance-9 matchings. In this paper, we improve the required distance from 9 to 3 for multigraphs with maximum multiplicity at least 2 as below.

Theorem 1.1. *Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$, and let M be a subset of $E(G)$ such that the minimum distance between two edges of M is at least 3. If $\mu(G) \geq 2$ and M is arbitrarily precolored from the palette $\mathcal{K} = [\Delta(G)+\mu(G)]$, then there is a proper edge coloring of G using colors from $\mathcal{K}$ that agrees with the precoloring on M.*

The *density* of a graph G, denoted by $\omega(G)$, is defined as

$$\omega(G) = max \left\{ \frac{2|E(H)|}{|V(H)| - 1} : H \subseteq G, |V(H)| \geq 3 \ and \ |V(H)| \ is \ odd \right\}$$

if $|V(G)| \geq 3$ and $\omega(G) = 0$ otherwise. By counting the number of edges in color classes, we have $\chi'(G) \geq \lceil \omega(G) \rceil$. So, besides the maximum degree, the density provides another lower bound for the chromatic index of a graph. In the 1970s, Goldberg [10] and Seymour [14] independently conjectured that actuarally $\chi'(G) = \lceil \omega(G) \rceil$ provided $\chi'(G) \geq \Delta(G)+2$. The conjecture was commonly referred to as one of most challenging problems in graph chromatic theory [15], and it was confirmed recently by Chen et al. [7].

Our proof of Theorem 1.1 is based on the assumption of the above Goldberg-Seymour Conjecture. We will present the proof of Theorem 1.1 in Section 4, before which we need some new structural properties of dense subgraphs and multi-fans, and some generalizations of Vizing's Theorem introduced in Sections 2 and 3.

2 Dense subgraphs

Throughout the rest of this paper, we reserve the notation Δ and μ for maximum degree and maximum multiplicity of the graph G, respectively. For a vertex set $N \subseteq V(G)$, let $G-N$ be the graph obtained from G by deleting all the vertices in N and edges incident with them. For an edge set $F \subseteq E(G)$, let $G-F$ be the graph obtained from G by deleting all the edges in F but keeping their endvertices. If $F = \{e\}$, we simply write $G-e$. Similarly, we let $G+e$ be the graph obtained from G by adding the edge e to $E(G)$. For disjoint $X, Y \subseteq V(G)$, $E_G(X,Y)$ is the set of edges of G with one endvertex in X and the other in Y. If $X=\{x\}$ and $Y=\{y\}$, we simply write $E_G(x,y)$. For two disjoint subgraphs H_1 and H_2 of G, we simply write $E(H_1,H_2)$ for $E_G(V(H_1),V(H_2))$. For $X \subseteq V(G)$, the edge set $\partial_G(X) = E_G(X, V(G)\backslash X)$ is called the *boundary* of X in G. For a subgraph H of G, we simply write $\partial(H)$ for $\partial_G(V(H))$.

For $u \in V(G)$, let $d_G(u)$ denote the *degree* of u in G. A *k-vertex* in G is a vertex with degree exactly k in G. A *k-neighbor* of a vertex v in G is a neighbor of v that is a k-vertex in G. A *α-edge* is an edge colored with the color α. For $e \in E(G)$, $V(e)$ is the set of endvertices of e. The *diameter* of a graph G, denoted by *$diam(G)$*, is the greatest distance between any pair of vertices in $V(G)$.

An edge e of a graph G is called a *k-critical edge* if $k=\chi'(G-e)<\chi'(G)=k+1$. A graph G is called *$k$-critical* if $\chi'(H)<\chi'(G)=k+1$ for each proper subgraph H of G. It is easy to see that a connected graph G is critical if and only if every edge of G is critical.

For a graph G, a vertex $v \in V(G)$ and an edge coloring $\varphi \in \mathcal{C}^k(G)$ with some positive integer k, define the two color sets $\varphi(v) = \{\varphi(f) : f \in E(G) \text{ and } f \text{ is incident with } v\}$ and $\overline{\varphi}(v) = [k]\backslash\varphi(v)$. We call $\varphi(v)$ the set of colors *present* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v\in X}\overline{\varphi}(v)$. A vertex set $X \subseteq V(G)$ is called *φ-elementary* if $\overline{\varphi}(u)\cap\overline{\varphi}(v)=\emptyset$ for every two distinct vertices $u,v \in X$. The set X is called *φ-closed* if each color on boundary edges is present at each vertex of X. Moreover, the set X is called *strongly φ-closed* if X is φ-closed and colors on boundary edges are distinct, i.e., $\varphi(f) \neq \varphi(f')$ for every two distinct colored edges $f, f' \in \partial_G(X)$. For a subgraph H of G, let φ_H be the edge coloring of G restricted on H. We say a subgraph H of G is φ-elementary, φ-closed and strongly φ-closed, if $V(H)$ is φ-elementary, φ-closed and strongly φ-closed, respectively. Clearly, if $V(H)$ is φ_H-elementary then $V(H)$ is φ-elementary, and the converse is not true.

A subgraph H of G is *k-dense* if $|V(H)|$ is odd and $|E(H)| = (|V(H)|-1)k/2$. Moreover, H is a *maximal k-dense subgraph* if there does not exist a k-dense subgraph H' containing H as a proper subgraph. By counting edges, we see that if H is a k-dense subgraph then $\chi'(H) \geq k$. Moreover, if $\chi'(G) = k$, then $\chi'(H) = k$ and for every $\varphi \in \mathcal{C}^k(G)$, every k-dense subgraph H of G is both φ_H-elementary and strongly φ-closed.

We start with the following consequent of the Goldberg-Seymour Conjecture.

Lemma 2.1. *Let G be a multigraph and $e \in E(G)$. If e is k-critical and $k \geq \Delta(G)+1$, then $G-e$ has a k-dense subgraph H containing $V(e)$, and e is also a k-critical edge of $H+e$.*

Proof. Clearly, $\chi'(G) = k+1$ and $\chi'(G-e) = k$. By the assumption of the Goldberg-Seymour Conjecture, $\chi'(G) = \lceil \omega(G) \rceil = k+1$. So, there exists a subgraph H^* of odd order such that $|E(H^*)| > (|V(H^*)|-1)k/2$. On the other hand, we have $\frac{2|E(H^*-e)|}{|V(H^*-e)|-1} \leq \lceil \omega(H^*-e) \rceil \leq \chi'(H^*-e) \leq \chi'(G-e) = k$, which in turn gives $|E(H^*-e)| \leq (|V(H^*)|-1)k/2$. Thus $|E(H^*-e)| = (|V(H^*)|-1)k/2$. Then $k \leq \lceil \omega(H^*-e) \rceil \leq \chi'(H^*-e) \leq \chi'(G-e) = k$ and $k+1 \leq \lceil \omega(H^*) \rceil \leq \chi'(H^*) \leq \chi'(G) = k+1$, which implies that $k = \chi'(H^*-e) < \chi'(H^*) = k+1$. Thus $H = H^*-e$ is a k-dense subgraph containing $V(e)$, and e is also a k-critical edge of $H+e$. □

Lemma 2.2. *Given a graph G, if $\chi'(G) \geq \Delta(G)+1$, then maximal $\chi'(G)$-dense subgraphs are pairwise vertex-disjoint.*

Proof. Let $k = \chi'(G)$ and suppose on the contrary that there are two maximal k-dense subgraphs H_1 and H_2 with nonempty intersection. Let $H = H_1 \cap H_2$ and $H^* = H_1 \cup H_2$. For each $i = 1,2$, since $|E(H_i)| = (|V(H_i)|-1)k/2$, adding any edge to H_i will result a graph with chromatic index greater than k, and so $H_i = G[V(H_i)]$ is an induced subgraph of G. Since both H_1 and H_2 are maximal and distinct, we have $V(H_1) \setminus V(H_2) \neq \emptyset$ and $V(H_2) \setminus V(H_1) \neq \emptyset$, which in turn gives $H_1 \subsetneq H^*$ and $H_2 \subsetneq H^*$. We consider two cases according to the parity of $|V(H)|$.

Case 1: $|V(H)|$ is odd.

Since $E(H^*) = E(H_1) \cup E(H_2)$ and $E(H) = E(H_1) \cap E(H_2)$, we have

$$|E(H^*)| = |E(H_1)| + |E(H_2)| - |E(H)| = k(|V(H_1)| + |V(H_2)| - 2)/2 - |E(H)|. \qquad \mathbf{(1)}$$

On the other hand, since both H_1 and H_2 are maximal k-dense, H^* is not k-dense. Consequently, we have

$$|E(H^*)| < k(|V(H^*)|-1)/2 = k(|V(H_1)| + |V(H_2)| - |V(H)| - 1)/2. \qquad \mathbf{(2)}$$

The combination of (**1**) and (**2**) gives $|E(H)| > k(|V(H)| - 1)/2$. Consequently, we have $\chi'(G) \geq \chi'(H) > k$, giving a contradiction.

Case 2: $|V(H)|$ is even.

Let $H_1^* = H_1 - V(H)$ and $H_2^* = H_2 - V(H)$. Clearly, both H_1^* and H_2^* have odd number of vertices. Since both H_1^* and H_2^* have k-edge-colorings, the following two inequalities hold.

$$\begin{aligned} |E(H_1^*)| &\leq k(|V(H_1)| - |V(H)| - 1)/2, \\ |E(H_1^*)| &\leq k(|V(H_2)| - |V(H)| - 1)/2. \end{aligned} \qquad (\mathbf{3})$$

Since both H_1 and H_2 are k-dense, we have the following inequalities.

$$\begin{aligned} k(|V(H_1)| - 1)/2 = |E(H_1)| = |E(H)| + |E(H_1^*)| + |E(H_1^*, H)|, \\ k(|V(H_2)| - 1)/2 = |E(H_2)| = |E(H)| + |E(H_2^*)| + |E(H_2^*, H)|. \end{aligned} \qquad (\mathbf{4})$$

The combination of (**3**) and (**4**) gives

$$\begin{aligned} |E(H_1^*, H)| + |E(H)| &\geq k \cdot |V(H)|/2, \\ |E(H_2^*, H)| + |E(H)| &\geq k \cdot |V(H)|/2. \end{aligned}$$

Therefore, $\Delta(G) \cdot |V(H)| \geq \sum_{x \in V(H)} d_G(x) \geq |E(H_1^*, H)| + |E(H_2^*, H)| + 2|E(H)| \geq k|V(H)|$, contradicting the assumption $\Delta(G) < k$. □

Lemma 2.3. *Let G be a multigraph with $\chi'(G) = k + 1 \geq \Delta(G) + 2$ and e be a k-critical edge of G. We have the following statements.*

(a) $G - e$ has a unique maximal k-dense subgraph H containing $V(e)$, and e is also a k-critical edge of $H + e$;

(b) With respect to any coloring $\varphi \in \mathcal{C}^k(G-e)$, H is φ_H-elementary and strongly φ-closed;

(c) If $\chi'(G) = \Delta(G) + \mu(G)$, then $\Delta(H+e) = \Delta(G)$, $\mu(H+e) = \mu(G)$ and $diam(H+e) \leq diam(H) \leq 2$.

Proof. By Lemma 2.1, $G - e$ contains a k-dense subgraph H containing $V(e)$ and e is also a k-critical edge of $H + e$. We may assume that H is a maximal k-dense subgraph, and the uniqueness of H is a direct consequence of Lemma 2.2. This proves (a).

Since H is k-dense, by the definition, $|E(H)| = \frac{|V(H)|-1}{2}k$. Also since H has an odd order, the size of a maximum matching in H has size at most $(|V(H)| - 1)/2$. Therefore, under any k-edge-coloring φ, each color class in H is a matching of size exactly $(|V(H)|-1)/2$. Thus every color in $[k]$ is missing at exactly one vertex of H or it appears exactly once in $\partial(H)$. Consequently, $V(H)$ is φ_H-elementary and strongly φ-closed. This proves (b).

For (c), by (a) and Vizing's Theorem, $\Delta(G)+\mu(G)=\chi'(G)=\chi'(H+e)\le\Delta(H+e)+\mu(H+e)\le\Delta(G)+\mu(G)$ implying that $\Delta(H+e)=\Delta(G)=\Delta$ and $\mu(H+e)=\mu(G)=\mu$. For any coloring $\varphi\in\mathcal{C}^k(G-e)$, H is φ_H-elementary by (b). For any $x\in V(H)$, all the colors missing at other vertices present at x. Note that $k=\Delta+\mu-1$. For each vertex $v\in V(H)$, we have that $|\overline{\varphi}_H(v)|=k-d_H(v)\ge k-\Delta=\mu-1$ if $v\notin V(e)$, and $|\overline{\varphi}_H(v)|=k-d_H(v)+1\ge k-\Delta+1\ge(\mu-1)+1$ if $v\in V(e)$. Denote $|V(H)|$ by n. Thus, $d_H(x)\ge|\bigcup_{y\in V(H),y\ne x}\overline{\varphi}_H(y)|\ge(k-\Delta)(n-1)+1=(\mu-1)(n-1)+1$.

Since $\mu(H)\le\mu(G)=\mu$, we get $|N_H(x)|\ge\frac{d_H(x)}{\mu}\ge\frac{(\mu-1)(n-1)+1}{\mu}$, where $N_H(x)$ is the neighbor set of x in H. Since $\mu\ge 2$, we have $\frac{(\mu-1)(n-1)+1}{\mu}\ge\frac{n}{2}$. Hence, every vertex in H is adjacent to at least half vertices in H. Consequently, every two vertices of H share a common neighbor, which in turn gives $diam(H)\le 2$. This proves (c). □

For a subgraph H of a graph G, let G/H be the graph obtained from G by contracting $V(H)$ to a single vertex. The following technical lemma will be used several times in our proof.

Lemma 2.4. *Let G be a graph with $\chi'(G)=k\ge\Delta(G)$, H be a k-dense subgraph, and ψ and φ be k-edge-colorings of H and G/H with the same palette $[k]$, respectively. By permuting color classes of ψ on $E(H)$, we can obtain a k-edge-coloring π of G such that $\pi(f)=\varphi(f)$ for every edge in G/H. If $\chi'(G)=k\ge\Delta(G)+1$, for any fixed color $\alpha\in[k]$, then by permuting other color classes of ψ on $E(H)$ we can obtain a coloring π of G agreeing with φ such that all color classes are matchings except the edges with color α.*

Proof. We treat φ as a k-edge-coloring of $G-E(H)$. Then, edges in $\partial(H)$ have different colors. Since H is k-dense and $\chi'(G)=k$, H is ψ-elementary. For each $v\in V(H)$, we have $|\overline{\psi}(v)|=k-d_H(v)\ge\Delta(G)-d_H(v)\ge d_{G-E(H)}(v)=|\varphi(v)|$. So, by permuting color classes of ψ, we may assume that $\varphi(v)\subseteq\overline{\psi}(v)$ for each $v\in V(H)$. The combination of the modified coloring of ψ and φ gives π.

For the second part, under the condition $k\ge\Delta(G)+1$, we have $|\overline{\psi}(v)|=k-d_H(v)\ge\Delta(G)+1-d_H(v)\ge d_{G-E(H)}(v)+1=|\varphi(v)|+1$. So $|\overline{\psi}(v)\backslash\{\alpha\}|\ge|\varphi(v)\backslash\{\alpha\}|$. Notice that when $\alpha\in\overline{\psi}(v)\cap\overline{\varphi}(v)$, we need $|\overline{\psi}(v)|-1\ge|\varphi(v)|$ to ensure the inequality above, where the assumption $k\ge\Delta(G)+1$ is applied. By permuting color classes of H except α, we may assume that $\varphi(v)\backslash\{\alpha\}\subseteq\overline{\psi}(v)$ for each $v\in V(H)$. Again, the combination of the modified coloring of ψ and φ gives the desired coloring. □

3 Refinements of multi-fans and some consequences

We first recall Kempe-chains and related terminology. Let φ be a k-edge-coloring of G using the palette $[k]$. Given two distinct colors α, β, an (α, β)-*chain* is a component of the subgraph induced by edges assigned color α or β in G, which is either an even cycle or a path. We call the operation that swaps the colors α and β on an (α, β)-chain the *Kempe change*. Clearly, the resulting coloring after a Kempe change is still a proper k-edge-coloring. Furthermore, we say that a chain has *endvertices* u and v if the chain is a path joining vertices u and v. For a vertex $v \in G$, we denote by $P_v(\alpha, \beta)$ the unique (α, β)-chain containing the vertex v. For two vertices u, $v \in V(G)$, the two chains $P_u(\alpha, \beta)$ and $P_v(\alpha, \beta)$ are either identical or disjoint. More generally, let $P_{[a,b]}(\alpha, \beta)$ be a subchain of a (α, β)-chain with endvertices a and b. The operation of swapping colors α and β on the subchain P is still called a Kempe change, but the resulting coloring may no longer be a proper edge coloring.

Let G be a graph with an edge $e \in E_G(x, y)$, and φ be a proper edge coloring of G or $G - e$. A sequence $F = (x, e_0, y_0, e_1, y_1, \ldots, e_p, y_p)$ consisting of vertices and distinct edges is called a (general) *multi-fan* at x with respect to e and φ if $e_0 = e$, $y_0 = y$, and for $0 \le i \le p$, the edge $e_i \in E_G(x, y_i)$ and $\varphi(e_i) \in \overline{\varphi}(y_j)$ for some $0 \le j \le i - 1$. Notice that the definition of multi-fan in this paper is slightly general than the one in [15] since the edge e may be colored in G. We say a multi-fan F is *maximal* if there is no multi-fan containing F as a proper subsequence. Similarly, we say a multi-fan F is *maximal without any α-edge* if F does not contain any α-edge and there is no multi-fan without any α-edge containing F as a proper subsequence. Let $\mu_G(x, y) = |E_G(x, y)|$ for $x, y \in V(G)$. Note that a multi-fan may have repeated vertices, so by $\mu_F(x, y_i)$ for some $y_i \in V(F)$ we mean the number of edges joining x and y_i in F.

A *linear sequence* at x from y_0 to y_s in G, denoted by $S = (x, e_0, y_0, e_1, y_1, \ldots, e_s, y_s)$, is a sequence consisting of distinct vertices and distinct edges such that $e_i \in E_G(x, y_i)$ for $0 \le i \le s$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for $i \in [s]$. Clearly for any $y_i \in V(F)$, the multi-fan F contains a linear sequence at x from y_0 to y_i. The following local edge recoloring operation will be used in our proof. A *shifting* from y_i to y_j in the linear sequence $S = (x, e_0, y_0, e_1, y_1, \ldots, e_s, y_s)$ is an operation that replaces the current color of e_t by the color of e_{t+1} for each $i \le t \le j - 1$ with $0 \le i < j \le s$. Note that the shifting does not change the color of e_j where e_j joins x and y_j, so it will not be a proper coloring. In our proof we will uncolor or recolor the edge e_j to avoid this problem.

Lemma 3.1. [3, 11, 15] *Let G be a graph, $e \in E_G(x, y)$ be a k-critical edge and $\varphi \in \mathcal{C}^k(G - e)$ with $k \ge \Delta(G)$. And let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ, where $y_0 = y$. Then the following statements hold.*

(a) *$V(F)$ is φ-elementary, and each edge in $E(F)$ is a k-critical edge of G.*

(*b*) *If* $\alpha \in \overline{\varphi}(x)$ *and* $\beta \in \overline{\varphi}(y_i)$ *for* $0 \le i \le p$*, then* $P_x(\alpha, \beta) = P_{y_i}(\alpha, \beta)$*.*

(*c*) *If* F *is a maximal multi-fan at* x *with respect to* e *and* φ*, then* x *is adjacent to at least* $\chi'(G) - d_G(y) - \mu_G(x, y) + 1$ *vertices* z *in* $V(F)\backslash\{x, y\}$ *such that* $d_G(z) + \mu_G(x, z) = \chi'(G)$*.*

Lemma 3.2. *Let* G *be a multigraph with maximum degree* Δ *and maximum multiplicity* $\mu \ge 2$*. Let* $e \in E_G(x, y)$ *be an edge of* G *and* $k = \Delta + \mu - 1$*.*

Assume that $\chi'(G) = k + 1$*,* e *is a* k*-critical edge and* $\varphi \in \mathcal{C}^k(G - e)$*. Let* $F = (x, e, y_0, e_1, y_1, \dots, e_p, y_p)$ *be a multi-fan at* x *with respect to* e *and* φ*, where* $y_0 = y$*. We have the following statements* (*a*)*,* (*b*) *and* (*c*)*.*

(*a*) *If* F *is maximal, then* x *is adjacent to at least* $\Delta + \mu - d_G(y) - \mu_G(x, y) + 1$ *vertices* z *in* $V(F)\backslash\{x, y\}$ *such that* $d_G(z) = \Delta$ *and* $\mu_G(x, z) = \mu$*;*

(*b*) *If* F *is maximal,* $d_G(y) = \Delta$ *and* x *has only one* Δ*-neighbor* z' *in* $V(F)\backslash\{x, y\}$*, then* $\mu_F(x, z) = \mu_G(x, z) = \mu$ *for all* $z \in V(F)\backslash\{x\}$ *and* $d_G(z) = \Delta - 1$ *for all* $z \in V(F)\backslash\{x, y, z'\}$*;*

(*c*) *If* F *is maximal without any* α*-edge for* $\alpha \notin \overline{\varphi}(y)$*, then* F *not containing any* Δ*-neighbor in* $V(F)\backslash\{x, y\}$ *implies that* $d_G(y) = \Delta$*, and there exists a vertex* $z^* \in V(F)\backslash\{x, y\}$ *with* $\alpha \in \overline{\varphi}(z^*)$ *and* $d_G(z^*) = \Delta - 1$*.*

Assume that $\chi'(G) = k$*,* $\varphi \in \mathcal{C}^k(G)$ *and* $V(G)$ *is* φ*-elementary. We have the following statement* (*d*)*.*

(*d*) *If a multi-fan* F' *is maximal at* x *with respect to* e *and* φ *in* G*, then* x *has no* Δ*-neighbor in* $V(F')\backslash\{x\}$ *implies that* $d_G(z) = \Delta - 1$ *for all* $z \in V(F')\backslash\{x\}$ *and every edge in* F' *is colored by a missing color at some vertex in* $V(F')$*. Furthermore, if* F' *is maximal without any* α*-edge and* $\varphi(e) \notin \overline{\varphi}(V(F'))$*, then* F' *not containing any* Δ*-neighbor in* $V(F')\backslash\{x\}$ *implies that there exists a vertex* $z^* \in V(F')\backslash\{x\}$ *with* $\alpha \in \overline{\varphi}(z^*)$ *and* $d_G(z^*) = \Delta - 1$*.*

Proof. For statements (*a*), (*b*) and (*c*), $V(F)$ is φ-elementary by Lemma 3.1 (*a*). Statement (*a*) holds easily by Lemma 3.1 (*c*). Assume that there are q distinct vertices in $V(F)\backslash\{x\}$.

For (*b*), we have

$$\begin{aligned} q\mu &\ge \sum_{z \in V(F)\backslash\{x\}} \mu_G(x, z) \ge \sum_{z \in V(F)\backslash\{x\}} \mu_F(x, z) = 1 + \sum_{z \in V(F)\backslash\{x\}} |\overline{\varphi}(z)| \\ &\ge 1 + (k - \Delta + 1) + (k - \Delta) + (q - 2)(k - \Delta + 1) = q(k - \Delta + 1) = q\mu, \end{aligned}$$

which implies that all equalities above hold, i.e., $\mu_F(x, z) = \mu_G(x, z) = \mu$ for each $z \in V(F)\backslash\{x\}$ and $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x, y, z'\}$. This proves (*b*).

Now for (c), we must have that there exists a vertex $z^* \in V(F)\backslash\{x,y\}$ with $\alpha \in \overline{\varphi}(z^*)$, since otherwise by (a) x has at least one Δ-neighbor in $V(F)\backslash\{x,y\}$, a contradiction. Since $V(F)$ is φ-elementary, x must be incident with a α-edge. Since now there is no α-edge in F and $\alpha \in \overline{\varphi}(z^*)$, we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F)\backslash\{x\}} \mu_G(x,z) \geq \sum_{z\in V(F)\backslash\{x\}} \mu_F(x,z) = 1 + (|\overline{\varphi}(z^*)| - 1) + \sum_{z\in V(F)\backslash\{x,z^*\}} |\overline{\varphi}(z)| \\ &\geq k - \Delta + 1 + (q-1)(k-\Delta+1) = q(k-\Delta+1) = q\mu, \end{aligned}$$

which implies that all equalities above hold, i.e., $d_G(y) = \Delta$, $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x,y\}$. This proves (c).

Statement (d) follows from similar calculations as (b) and (c). □

Let G be a graph with maximum degree Δ and maximum multiplicity μ. Berge and Fournier [6] strengthened the classical Vizing's Theorem by showing that if M^* is a maximal matching of G, then $\chi'(G - M^*) \leq \Delta + \mu - 1$. An edge $e \in E_G(x,y)$ is *fully saturated* with respect to G if $d_G(x) = d_G(y) = \Delta$ and $\mu_G(x,y) = \mu$. Note that for every graph G with $\chi'(G) = \Delta + \mu$, there exists a critical subgraph H of G with $\chi'(H) = \Delta + \mu$ and $\Delta(H) = \Delta$. Moreover, every graph G with $\chi'(G) = \Delta + \mu$ contains at least two fully saturated edges in G by Lemma 3.2 (a). Stiebitz et al.[page 41 (a), [15]] obtained the following generalization of Vizing's Theorem with an elegant short proof: *Let G be a graph and let $k \geq \Delta + \mu$ be an integer. Then there is a k-edge-coloring φ of G such that every edge e with $\varphi(e) = k$ is fully saturated.* We observe that their proof actually gives a slightly stronger result which also generalizes the Berge-Fournier theorem as below.

Lemma 3.3. *Let G be a graph and M be a matching of G. If M' is a maximal matching of $G - V(M)$ such that every edge in M' is fully saturated with respect to G, then $\chi'(G - (M \cup M')) \leq \Delta(G) + \mu(G) - 1$.*

Proof. Let $G' = G - (M \cup M')$. Note that every vertex $v \in V(M \cup M')$ has $d_{G'}(v) \leq \Delta - 1$. By the maximality of M', $G - V(M \cup M')$ contains no fully saturated edges. So, G' does not have a fully saturated edge of G. By Lemma 3.2 (a), $\chi'(G') \leq \Delta + \mu - 1$, since otherwise there exist at least two fully saturated edges with respect to G in one multi-fan centered at a Δ-vertex, a contradiction. □

Lemma 3.3 has the following consequence.

Corollary 3.4. *Let G be a graph. If M is a maximal matching such that every edge in M is fully saturated with respect to G, then $\chi'(G - M) \leq \Delta(G) + \mu(G) - 1$.*

Let M be a matching of a graph G such that $\chi'(G - M) = \Delta(G) + \mu(G)$. Let $k = \Delta + \mu - 1$. By Lemma 3.3, there is a matching M' of $G - V(M)$ with fully saturated edges with respect to G such that $\chi'(G-(M\cup M'))=k$. Suppose that M' is minimal subject to the properties above. Then each edge $e\in M'$ is a k-critical edge of $G-(M\cup M'\backslash\{e\})$. Moreover, if $\mu\geq 2$, then by Lemma 2.3 (a) there is a unique maximal k-dense subgraph H_e of $G-(M\cup M')$ such that $V(e)\subseteq V(H_e)$. Clearly, every fully statured edge in H_e+e is a fully saturated edge of G, and the converse is not true. Following the above notation, we strengthen Lemma 3.3 for multigraphs with maximum multiplicity at least 2 as below.

Lemma 3.5. *For a fixed matching M of a graph G, if $\mu(G)\geq 2$ and $\chi'(G-M)=\Delta(G)+\mu(G)$, then there is a matching M^* of $G-V(M)$ such that $\chi'(G-(M\cup M^*))=\Delta(G)+\mu(G)-1$ and every edge $e\in M^*$ is fully saturated in H_e+e, where H_e is the maximal k-dense subgraph of $G-(M\cup M^*)$ containing $V(e)$.*

Proof. Let $k=\Delta+\mu-1$, and M' be defined prior to Lemma 3.5 maximizing the number m' of edges $e\in M'$ that is fully saturated in H_e+e. We claim $m'=|M'|$, which in turn gives Lemma 3.5. Suppose on the contrary there is an edge $e\in M'$ that is not fully saturated in H_e+e. By Lemma 2.3 (a), e is a k-critical edge of H_e+e. Let $\varphi\in\mathcal{C}^k(G-(M\cup M'))$.

Let $V(e)=\{x,y\}$ and F_x be a maximum multi-fan at x with respect to e and φ_{H_e}, where φ_{H_e} is the coloring induced by φ on H_e. By Lemma 3.2 (a), x contains a Δ-neighbor, say x_1, in $V(F_x)\backslash\{x,y\}$. By Lemma 3.1 (a), the edge $e_{xx_1}\in E_G(x,x_1)$ in F_x is also a critical edge of H_e+e. By Lemma 3.2 (a) again, in a maximum multi-fan at x_1 there exists a fully saturated edge e^* with respect to H_e+e. Let $M^*=(M'\backslash\{e\})\cup\{e^*\}$. Since every vertex of $V(M\cup M')$ has degree less than Δ in $G-(M\cup M')$, it follows that $M\cup M^*$ is a matching of G. Let $H_{e^*}=H_e+e-e^*$. Clearly, H_{e^*} is also k-dense. Applying Lemma 3.1 (a) again, we see that e^* is also a k-critical edge of H_e+e. Thus $\chi'(H_{e^*})=\omega(H_{e^*})=k$. By Lemma 2.4, we have $\chi'(G-(M\cup M^*))=k$.

Since maximal k-dense subgraphs of $G-(M\cup M')$ are vertex-disjoint, all other maximal k-dense subgraphs of $G-(M\cup M')$ are also maximal k-dense subgraphs of $G-(M\cup M^*)$. For any fully saturated edge $f\in M'\backslash\{e\}$, since $V(f)\cap V(e^*)=\emptyset$, f is still fully saturated with respect to the corresponding maximal k-dense subgraph. We can use M^* instead of M', which contradicts the maximality of M'. Thus $m'=|M'|$ as desired. □

4 Proof of Theorem 1.1

We rewrite Theorem 1.1 as follows.

Theorem 1.1. *Let G be a multigraph with $\mu(G)\geq 2$. Using palette $[\Delta(G)+\mu(G)]$, any precoloring of a distance-3 matching M in G can be extended to a proper edge coloring of G.*

Proof. Let $k = \Delta + \mu - 1$. We fix a precoloring of M, denoted by $\Phi : M \to [\Delta + \mu]$. Note that $\chi'(G - M) \leq k + 1$ by Vizing's Theorem. The conclusion of Theorem 1.1 holds easily if $\chi'(G - M) \leq k$ with the reason as follows. For any k-edge-coloring ψ of $G - M$, if there exists $e \in E(G - M)$ such that e is adjacent to an edge $f \in M$ and $\psi(e) = \Phi(f)$ in G, we recolor each such e with the color $\Delta + \mu$ and get a new coloring ψ' of $G - M$. Under ψ', the edges colored by $\Delta + \mu$ form a matching in G since M is a distance-3 matching. Thus the combination of Φ and ψ' is a $(k + 1)$-edge-coloring of G. Therefore, in the remainder of the proof, we assume $\chi'(G - M) = k + 1$.

Let $M_\Phi^{\Delta+\mu}$ be the set of edges colored with $\Delta+\mu$ in M. For any matching $M^* \subseteq G-V(M)$ and any $(k + 1)$-edge-coloring or k-edge-coloring φ on $G - (M \cup M^*)$, denote the $\Delta + \mu$ color class by $\bar{M}_\varphi^{\Delta+\mu}$. In particular, $\bar{M}_\varphi^{\Delta+\mu} = \emptyset$ if φ is a k-edge-coloring. We call a triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$ is **prefeasible** if it satisfies *Condition* 1: $V(M^*) \cap V(\bar{M}_\varphi^{\Delta+\mu}) = \emptyset$, i.e., all edges in M^* are not adjacent to any edge in $\bar{M}_\varphi^{\Delta+\mu}$.

With respect to a triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$, we call an edge $f \in E_G(u, v)$ in M is **first-improper** at u if there exists $f_1 \in E(G - (M \cup M^*))$ such that $\varphi(f_1) = \Phi(f)$, f is adjacent to f_1 at u, and f_1 is not adjacent to any edge in M^*; we call an edge $f \in E_G(u, v)$ in M is **second-improper** at u if there exists $f_1 \in E(G - (M \cup M^*))$ and $f_2 \in M^*$ such that $\varphi(f_1) = \Phi(f)$, f is adjacent to f_1 at u, and f_1 is adjacent to f_2. Let A_φ and B_φ respectively denote the number of first-improper edges and second-improper edges in M (counting twice if one edge is improper at both its endvertices) with respect to the triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$.

For a triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$, let $M_\varphi^A(f_1)$ ($M_\varphi^B(f_1)$, respectively) be the set of all such edges f_1 that is adjacent to some first-improper (second-improper, respectively) edge $f \in M$ with $\varphi(f_1) = \Phi(f)$. Observe that $M_\varphi^A(f_1) \cup M_\varphi^B(f_1)$ is also a matching since M is distance-3, and $|M_\varphi^A(f_1)| = A_\varphi$ and $|M_\varphi^B(f_1)| = B_\varphi$.

For any prefeasible triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$, all edges in M^* are uncolored if $|M^*| \geq 1$, $V(M^*) \cap V(M_\Phi^{\Delta+\mu}) = \emptyset$ since $M^* \subseteq G - V(M)$ and $V(M^*) \cap V(\bar{M}_\varphi^{\Delta+\mu}) = \emptyset$ by Condition 1. Recall that M is a distance-3 matching. Thus if a prefeasible triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$ also satisfies *Condition* 2: $A\varphi = B\varphi = 0$, i.e., $M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1) = \emptyset$, then $M_\Phi^{\Delta+\mu} \cup M^* \cup \bar{M}_\varphi^{\Delta+\mu}$ is a matching. Then by giving the color $\Delta + \mu$ to all the edges in M^*, we have a proper $(k + 1)$-edge-coloring Ω of G implying that Theorem 1.1 holds, where Ω is the combination of the precoloring Φ on M, the $\Delta + \mu$ coloring ϕ on M^* and the coloring φ of $G - (M \cup M^*)$. We call such desired triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$ is **feasible** if it satisfies Conditions 1 and 2.

The rest of the proof is devoted to showing the existence of a feasible triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$ of G. Our main strategy is that we first fix a particular prefeasible triple $(M_0^*, \bar{M}_{\varphi_0}^{\Delta+\mu}, \varphi_0)$, then modify it step by step to a feasible triple $(M^*, \bar{M}_\varphi^{\Delta+\mu}, \varphi)$ with $\bar{M}_\varphi^{\Delta+\mu} = M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1)$, which implies that the $\Delta + \mu$ color class in the final $(k + 1)$-edge-coloring Ω of G is $M_\Phi^{\Delta+\mu} \cup M^* \cup M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1)$.

By Lemma 3.5, there exsits a matching M_0^* of $G-V(M)$ such that $\chi'(G-(M\cup M_0^*))=k$ and each edge $e\in M_0^*$ is fully saturated and k-critical in H_e+e, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M_0^*)$ containing $V(e)$. Recall that $\chi'(G-M)=k+1$. Thus $|M_0^*|\geq 1$. Let φ_0 be a k-edge-coloring of $G-(M\cup M_0^*)$. Note that $\bar{M}_{\varphi_0}^{\Delta+\mu}=\emptyset$. Obviously, the triple $(M_0^*,\emptyset,\varphi_0)$ is prefeasible that is just our initial triple, and there is neither first-improper nor second-improper $(\Delta+\mu)$-edges in M under φ_0.

For $(M_0^*,\emptyset,\varphi_0)$, if $A_{\varphi_0}=B_{\varphi_0}=0$, i.e., $M_{\varphi_0}^A(f_1)\cup M_{\varphi_0}^B(f_1)=\emptyset$, then we are done. If $A_{\varphi_0}\geq 1$ and $B_{\varphi_0}=0$, then we give the color $\Delta+\mu$ to every edge in $M_{\varphi_0}^A(f_1)$, resulting in a new $(k+1)$-edge-coloring φ_1 of $G-(M\cup M_0^*)$ since $M_{\varphi_0}^A(f_1)$ is a matching. Thus $A_{\varphi_1}=B_{\varphi_1}=0$ and all edges in M_0^* are still not adjacent to any edge in $\bar{M}_{\varphi_1}^{\Delta+\mu}=M_{\varphi_0}^A(f_1)$, which implies that the new triple $(M_0^*,M_{\varphi_0}^A(f_1),\varphi_1)$ is feasible, so we are also done.

Now we may assume that $A_{\varphi_0}\geq 0$ and $B_{\varphi_0}\geq 1$ with respect to the initial triple $(M_0^*,\emptyset,\varphi_0)$. Let $H_1,H_2,\ldots,H_t$ be all maximal k-dense subgraphs of $G-(M\cup M_0^*)$ such that each of them contains both endvertices of some edge of M_0^*. By Lemmas 2.2-2.3, $H_1,H_2,\ldots,H_t$ are vertex-disjoint. Moreover, each H_s with $s\in[t]$ has $diam(H_s)\leq 2$ and $\chi'(H_s)=k$, and is $(\varphi_0)_{H_s}$-elementary and strongly φ_0-closed in $G-(M\cup M_0^*)$. By Lemma 3.5, each edge e in M_0^* is fully saturated in H_s+e for some $s\in[t]$, so all edges in M_0^* are only adjacent to edges inside $H_1,H_2,\ldots,H_t$. Thus for an edge $f_{uv}\in M$ with $V(f_{uv})=\{u,v\}$, if $u,v\notin V(H_s)$ for any $s\in[t]$, then f_{uv} cannot be a second-improper edge.

Since $B_{\varphi_0}\geq 1$, we consider one second-improper edge in M, say f_{uv} with $V(f_{uv})=\{u,v\}$ and $\Phi(f_{uv})=i\in[k]$, and assume that f_{uv} is second-improper at u. Hence there exists some H_s with $s\in[t]$ such that $u\in V(H_s)$, where H_s contains both endvertices x and y of one edge $e_{xy}\in M_0^*$ such that f_{uv} and e_{xy} are both adjacent to an i-edge e_{yu} in H_s. Since M is distance-3 and $diam(H_s)\leq 2$, there does not exist another edge of M whose any endvertex is also in $V(H_s)$. Notice that u and v may belong to disjoint H_s and $H_{s'}$, where $s\neq s'$ with $s,s'\in[t]$. To make f_{uv} not be second-improper, we consider the following Cases 1-3. See Figures 1 and 2.

Case 1: f_{uv} is not improper at v, or f_{uv} is first-improper at v but $v\notin V(H_s)$.

Let F_x be a maximal multi-fan at x with respect to e_{xy} and $(\varphi_0)_{H_s}$ in H_s+e_{xy}. By Lemma 3.2 (a), in F_x there exist at least one Δ-vertex in $V(F_x)\setminus\{x,y\}$, say x_1, and a linear sequence S from y to x_1 with last edge $e_{xx_1}\in E_{H_s}(x,x_1)$. Notice that x_1 is not incident with any edge in $M\cup M_0^*$ since $d_{H_s}(x_1)=\Delta$. We will do the following operations in three subcases to make sure that f_{uv} is no longer second-improper at u.

Subcase 1.1: *S does not contain both an i-edge and a boundary vertex of $V(H_s)$ that is incident with an i-edge of $\partial(H_s)$ in $G-(M\cup M_0^*)$.*

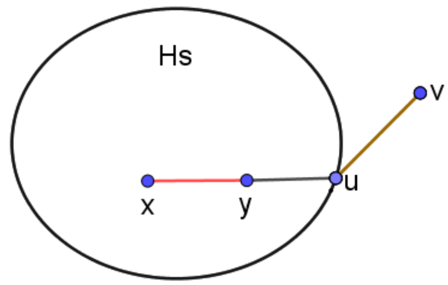

Figure 1: One possibility for the location of f_{uv} relative to H_s in Case 1.

For this subcase, we do Operation I as follows. Do a shifting in S from y to x_1 which gives a color in $[k]$ to the edge e_{xy}, uncolor the edge e_{xx_1}, and replace e_{xy} by e_{xx_1} in M_0^* since x_1 is not incident with any edge in $M \cup M_0^*$. Obviously, $H_s + e_{xy} - e_{xx_1}$ is also k-dense. By Lemma 3.1 (a), e_{xx_1} is also a k-critical edge of $H_s + e_{xy}$ and $\chi'(H_s + e_{xy} - e_{xx_1}) = k$. Thus we can permute color classes of $E(H_s + e_{xy} - e_{xx_1})$ but keep the color i unchanged to match all boundary edges by Lemma 2.4. As a result, we obtain a new matching $M_1^* = (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xx_1}\} \subseteq G - V(M)$ and a new k-edge-coloring φ_1 of $G - (M \cup M_1^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at u with respect to the new triple $(M_1^*, \emptyset, \varphi_1)$ that is also prefeasible.

Subcase 1.2: For any Δ-vertex in $V(F_x) \setminus \{x, y\}$, any linear sequence from y to this Δ-vertex contains an i-edge and a boundary vertex that is incident with one i-edge in $\partial(H_s)$.

By Lemma 3.2 (c), there exists a vertex w with $d_{H_s}(w) = \Delta - 1$ and $d_{G-(M \cup M_0^*)}(w) = \Delta$. So the i-edge, denoted by h, is the only edge in $\partial(H_s)$ at w and w is not incident with any edge in $M \cup M_0^*$. Next we fix the linear sequence S corresponding to the Δ-vertex x_1, and consider the following two subcases about the boundary i-edge h.

Subcase 1.2.1: $h \notin M_{\varphi_0}^A(f_1)$, i.e., h is not adjacent to any precolored i-edge in M.

For this subcase, we do Operation II as follows. Let $e_{xw} \in E_{H_s}(x, w)$ be the edge with $V(e_{xw}) = \{x, w\}$ in S. Do a shifting in S from y to w which gives a color in $[k]$ to the edge e_{xy}, uncolor the edge e_{xw}, and replace e_{xy} by e_{xw} in M_0^* since w is not incident with any edge in $M \cup M_0^*$. Obviously, $H_s + e_{xy} - e_{xw}$ is also k-dense. By Lemma 3.1 (a), e_{xw} is also a k-critical edge of $H_s + e_{xy}$ and $\chi'(H_s + e_{xy} - e_{xw}) = k$. Thus we can permute color classes of $E(H_s + e_{xy} - e_{xw})$ but keep the color i unchanged to match all boundary edges by Lemma 2.4. As a result, we obtain a new matching $M_1^* = (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xw}\} \subseteq G - V(M)$ and a new k-edge-coloring φ_1 of $G - (M \cup M_1^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at u with respect to the new prefeasible triple $(M_1^*, \emptyset, \varphi_1)$.

Subcase 1.2.2: $h \in M_{\varphi_0}^A(f_1)$, i.e., h is adjacent to some precolored i-edge in M.

For this subcase, we do Operation III as follows. First recolor h from the color i to the color $\Delta + \mu$. Do a shifting in S from y to x_1 which gives a color in $[k]$ to the edge e_{xy}, uncolor the edge e_{xx_1}, and permute color classes of $E(H_s + e_{xy} - e_{xx_1})$ but keep the color i unchanged to match all boundary edges by Lemma 2.4. Now we obtain a new matching $M_1^* = (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xx_1}\} \subseteq G - V(M)$ and a new $(k+1)$-edge-coloring φ_1 of $G - (M \cup M_1^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at u with respect to the new triple $(M_1^*, \bar{M}_{\varphi_1}^{\Delta+\mu}, \varphi_1)$ with $\bar{M}_{\varphi_1}^{\Delta+\mu} = \{h\}$. Notice that the triple $(M_1^*, \bar{M}_{\varphi_1}^{\Delta+\mu}, \varphi_1)$ is also prefeasible since h is not adjacent to any edge in M_1^*. Moreover, giving the color $\Delta + \mu$ to h will not be a problem since $h \in M_{\varphi_0}^A(f_1)$ and we will give the color $\Delta + \mu$ to all edges in $M_{\varphi_0}^A(f_1)$ in the final process.

For Operations I-III, we have the following observations.

(1) $M \cup M_1^* = M \cup (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xx_1}\}$ or $M \cup M_1^* = M \cup (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xw}\}$ is also a matching, where $d_{H_s}(x_1) = \Delta$, $d_{H_s}(w) = \Delta - 1$ and w is incident with one boundary i-edge h;

(2) The subgraph $H_s^1 = H_s + e_{xy} - e_{xx_1}$ or $H_s^1 = H_s + e_{xy} - e_{xw}$ is also k-dense and $(\varphi_1)_{H_s^1}$-elementary, where $V(H_s^1) = V(H_s)$, $\partial(H_s^1) = \partial(H_s)$ and $d_{H_s^1}(w) = \Delta - 2$;

(3) The new triple $(M_1^*, \bar{M}_{\varphi_1}^{\Delta+\mu}, \varphi_1)$ is also prefeasible, where $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$ or $\{h\} \subseteq (\partial(H_s) \cap M_{\varphi_0}^A(f_1))$ with some vertex $w_0 \in S$ and $i \in \overline{\varphi}_1(w_0)$.

Moreover, $B_{\varphi_1} = B_{\varphi_0} - 1$ and $A_{\varphi_1} = A_{\varphi_0} + 1$ since f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at u and the edges e_{xx_1} and e_{xw} cannot make new second-improper edges.

Case 2: f_{uv} is second-improper at v with $v \in V(H_{s'})$ for a maximal k-dense subgraph $H_{s'}$ other than H_s.

For this case, we first do the same operations for u in H_s as we did in Case 1. Recall that $V(H_s) \cap V(H_{s'}) = \emptyset$, M is distance-3 and $M_{\varphi_0}^A(f_1)$ is a matching. Then do the same operations for v in $H_{s'}$ as we did for u in H_s. Thus f_{uv} is no longer second-improper (but becomes first-improper) at both u and v with respect to one prefeasible triple $(M_2^*, \bar{M}_{\varphi_2}^{\Delta+\mu}, \varphi_2)$, where $\bar{M}_{\varphi_2}^{\Delta+\mu} \subseteq \{h_u, h_v\}$ with some edge $h_u \in \partial(H_s) \cap M_{\varphi_0}^A(f_1)$ and some edge $h_v \in \partial(H_{s'}) \cap M_{\varphi_0}^A(f_1)$ by Case 1. Moreover, $V(h_u) \cap V(h_v) = \emptyset$, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 2$.

Case 3: f_{uv} is first-improper or second-improper at v with $v \in V(H_s)$.

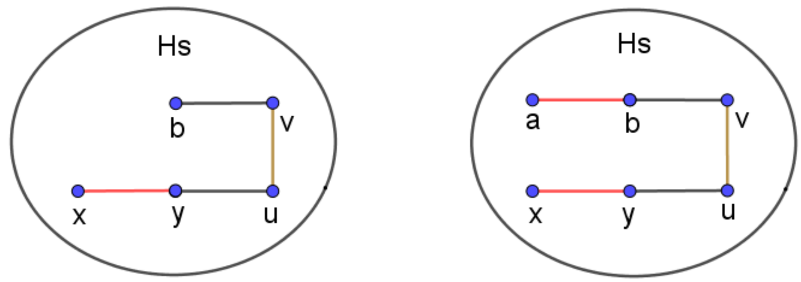

Figure 2: Two possibilities for the location of f_{uv} relative to H_s in Case 3.

If f_{uv} is a first-improper edge at v with $v \in V(H_s)$, then let $e_{bv} \in E_{H_s}(b, v)$ be the i-edge incident with v in H_s. If $d_{H_s}(b) < \Delta$, then we do the same operations for u as we did in Case 1, which does not influence the vertex b by the observation (1) in Case 1. Thus f_{uv} is no longer second-improper (but becomes first-improper) at u. We will discuss the other subcase $d_{H_s}(b) = \Delta$ in the next paragraph.

If f_{uv} is a second-improper edge at v with $v \in V(H_s)$. We use $e_{ab} \in M^*$ with $V(e_{ab}) = \{a, b\}$ to denote the edge that is adjacent to an i-edge $e_{bv} \in E_{H_s}(b, v)$. Note that $d_{H_s}(a) < \Delta$ and $d_{H_s}(b) < \Delta$. We do the same operations for u as we did in Case 1, which does not influence the vertices a and b. Thus f_{uv} is no longer second-improper (but becomes first-improper) at u with respect to one prefeasible triple $(M_1^*, \bar{M}_{\varphi_1}^{\Delta+\mu}, \varphi_1)$, where $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$ or $\{h\}$ with some boundary vertex w and its incident i-edge $h \in \partial(H_s) \cap M_{\varphi_0}^A(f_1)$ by the observation (3) in Case 1. In particular, the situation under $(M_1^*, \emptyset, \varphi_1)$ is actually the same as the subcase $d_{H_s}(b) = \Delta$ in the previous paragraph since $d_{H_s^1}(y) = \Delta$, where H_s^1 is the new k-dense subgraph after the operations for u in H_s by the observation (2) in Case 1.

Note that we also have $d_{H_s^1+e_{ab}}(a) = d_{H_s^1+e_{ab}}(b) = \Delta$ and $\varphi_1(e_{yu}) = i$. Now consider a maximal multi-fan F_a at a with respect to e_{ab} and $(\varphi_1)_{H_s^1}$ in $H_s^1 + e_{ab}$. Clearly we can do the same operations in Case 1 for v to make sure that f_{uv} is no longer a second-improper edge at v, unless these operations would have to put one edge $e_{ay} \in E_{H_s^1}(a, y)$ into M_1^*, so f_{uv} would become second-improper at u again. Therefore, by Operations I-III in Case 1 we may have the following two assumptions for the rest of our proof.

(1) y is the only Δ-vertex in $V(F_a)\backslash\{a, b\}$;

(2) If a linear sequence in F_a from b to y contains a boundary vertex w', where $d_{H_s^1}(w') = \Delta - 1$ and w' is incident with one i-edge h' in $\partial(H_s^1)$, then $h' \in M_{\varphi_0}^A(f_1)$.

Let F_b be the maximal multi-fan at b with respect to e_{ab} and $(\varphi_1)_{H_s^1}$ in $H_s^1 + e_{ab}$. We consider the following Subcases 3.1-3.3.

Subcase 3.1: F_b contains a linear sequence S from a to y with no i-edge.

Let $S = (b, e_0, a_0, e_1, a_1, \ldots, e_p, a_p)$ be a linear sequence from a to y, where $e_0 = e_{ab}$, $a_0 = a$, $e_p = e_{by} \in E_{H_s^1}(b, y)$, $a_p = y$, and S does not contain i-edges. For this subcase we do a shifting in S from a to y which gives a color in $[k]$ to e_{ab}, uncolor the edge e_{by}, and permute color classes of $E(H_s^1 + e_{ab} - e_{by})$ but keep the color i unchanged to match all the boundary edges by Lemma 2.4. Now we obtain a new matching $M_2^* = (M_1^* \backslash \{e_{ab}\}) \cup \{e_{by}\}$ and a new k-edge-coloring φ_2 of $G - (M \cup M_2^*)$, where f_{uv} is a second-improper edge at both u and v, but here $\Phi(f_{uv}) = i$, $\varphi_2(e_{bv}) = \varphi_2(e_{yu}) = i$, and the edge e_{by} is uncolored. So by giving the color i to e_{by} and recoloring e_{bv} and e_{yu} with the color $\Delta + \mu$, we obtain a new matching $M_3^* = M_2^* \backslash \{e_{by}\} = M_1^* \backslash \{e_{ab}\} \subseteq G - V(M)$ and a new $(k+1)$-edge-coloring φ_3 of $G - (M \cup M_3^*)$. Thus f_{uv} is no longer a second-improper edge or even a first-improper edge neither at u nor at v with respect to the new triple $(M_3^*, \bar{M}_{\varphi_3}^{\Delta+\mu}, \varphi_3)$, where $\bar{M}_{\varphi_3}^{\Delta+\mu} = \{e_{bv}, e_{yu}\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$ or $\bar{M}_{\varphi_3}^{\Delta+\mu} = \{h, e_{bv}, e_{yu}\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \{h\}$. Notice that $\bar{M}_{\varphi_3}^{\Delta+\mu}$ is also a matching since $\bar{M}_{\varphi_3}^{\Delta+\mu} \subseteq (M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1))$, and the triple $(M_3^*, \bar{M}_{\varphi_3}^{\Delta+\mu}, \varphi_3)$ is also prefeasible since h, e_{bv} and e_{yu} are not adjacent to any edge in M_3^*. Moreover, $B_{\varphi_2} = B_{\varphi_1} - 1 = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_1} - 1 = A_{\varphi_0}$.

Subcase 3.2: F_b contains a vertex w'' with $d_{H_s^1}(w'') = \Delta - 1$ and $i \in (\overline{\varphi}_1)_{H_s^1}(w'')$.

In this subcase, the i-edge e_{bv} is in F_b by the maximality of F_b. Note that there exists a linear sequence $S = (b, e_0, a_0, e_1, a_1, \ldots, e_{p-1}, a_{p-1}, e_p, a_p)$ from a to v in F_b, where $e_0 = e_{ab}$, $a_0 = a$, $e_{p-1} = e_{bw''} \in E_{H_s^1}(b, w'')$, $a_{p-1} = w''$, $e_p = e_{bv}$ and $a_p = v$.

If $i \in \overline{\varphi}_1(w'')$ (w'' may be the vertex a), or w'' is incident with an i-edge $h'' \in \partial(H_s) \cap M_{\varphi_0}^A(f_1)$, then we first do a shifting in S from a to v which gives a color in $[k]$ to e_{ab}, recolor the edge $e_{bw''}$ with i and uncolor the edge e_{bv}. Then recolor h'' from i to $\Delta + \mu$ if there exists h'', and permute color classes of $E(H_s^1 + e_{ab} - e_{bv})$ but keep the color i unchanged to match all the boundary edges by Lemma 2.4. Finally give the color $\Delta + \mu$ to the edge e_{bv}. Note that $h \neq h''$ since $\varphi_1(h) = \Delta + \mu \neq i = \varphi_1(h'')$, and h and h'' cannot both exist in $\partial(H_s) = \partial(H_s^1)$ since otherwise $\varphi_0(h) = \varphi_0(h'') = i$ contradicting that H_s is strongly φ_0-closed. As a result, we obtain a new matching $M_2^* = M_1^* \backslash \{e_{ab}\} \subseteq G - V(M)$ and a new $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer a second-improper edge or even a first-improper edge at v with respect to the new prefeasible triple $(M_2^*, \bar{M}_{\varphi_2}^{\Delta+\mu}, \varphi_2)$, where $\bar{M}_{\varphi_2}^{\Delta+\mu} = \{e_{bv}\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$ but h'' does not exist, $\bar{M}_{\varphi_2}^{\Delta+\mu} = \{e_{bv}, h''\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$ and h'' exists, or $\bar{M}_{\varphi_2}^{\Delta+\mu} = \{e_{bv}, h\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \{h\}$. Moreover, $\bar{M}_{\varphi_2}^{\Delta+\mu} \subseteq (M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1))$, $B_{\varphi_2} = B_{\varphi_1} - 1 = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_1} = A_{\varphi_0} + 1$.

Now we may assume that w'' is incident with an i-edge $h'' \in \partial(H_s)$ but $h'' \notin M_{\varphi_0}^A(f_1)$. Then we have $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$. Note that the vertex $w'' \notin V(F_a)$ by the assumption (2). Moreover, w'' is not incident with any edge in $M \cup M_1^*$ and w'' is only incident with the i-edge h'' in $\partial(H_s^1)$. Since $d_{G-(M \cup M_1^*)}(w'') = \Delta$ and φ_1 is a k-edge-coloring of $G - (M \cup M_1^*)$ with

$k \geq \Delta + 1$, there exists a color $\alpha \in \overline{\varphi}_1(w'')$ with $\alpha \neq i$. Since H_s^1 is $(\varphi_1)_{H_s^1}$-elementary, there exists a α-edge e_0' incident with the vertex a. Thus we can define a maximal multi-fan at a with respect to e_0' and $(\varphi_1)_{H_s^1}$ in H_s^1, denoted by $F_a' = (a, e_0', b_0, \dots, e_q', b_q)$, such that $(\varphi_1)_{H_s^1}(e_j') \in (\overline{\varphi}_1)_{H_s^1}(b_{l-1})$ for $j \in [q]$ and some $l \in [j]$. Moreover, $V(F_a')$ is $(\varphi_1)_{H_s^1}$-elementary since $V(H_s^1)$ is $(\varphi_1)_{H_s^1}$-elementary. By the assumption (1) and Lemma 3.2 (b), we have $\mu_{F_a}(a, b') = \mu_{H_s^1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a)\backslash\{a\}$. Therefore, $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint, since otherwise we have $V(F_a') \subseteq V(F_a)$ and $\alpha \in (\overline{\varphi}_1)_{H_s^1}(b')$ for some $b' \in V(F_a)$ implying $b' = w'' \in V(F_a)$, a contradiction. Note that if $w'' \notin V(F_a')$, then $V(F_a')\backslash\{a\}$ must contain a Δ-vertex in H_s^1, since otherwise Lemma 3.2 (d) and the fact $(\varphi_1)_{H_s^1}(e_0') = \alpha \in \overline{\varphi}_1(w'')$ imply that $w'' \in V(F_a')$, a contradiction. Thus F_a' contains a linear sequence $S' = (a, e_{l_1}', b_{l_1}, \dots, e_{l_t}', b_{l_t})$, where $e_{l_1}' = e_0'$, $b_{l_1} = b_0$, $b_{l_t} \in V(F_a')$ is a Δ-vertex if $w'' \notin V(F_a')$, and b_{l_t} is w'' if $w'' \in V(F_a')$. Notice that b_{l_t} is not incident with any edge in $M \cup M_1^*$ by our choice of b_{l_t}. Moreover, $b_{l_t} \neq y$ since $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint. Let β $(\beta \neq i)$ be a color in $\overline{\varphi}_1(b)$. By Lemma 3.1 (b), we have $P_b(\beta, \alpha) = P_{w''}(\beta, \alpha)$. We then consider the following two subcases according to the set $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ is either $\{b_{l_t}\}$ or $\emptyset$. If $e_0' \notin P_b(\beta, \alpha)$, then we do Kempe changes on $P_{[b,w'']}(\beta, \alpha)$, uncolor e_0' and color e_{ab} with α. If $e_0' \in P_b(\beta, \alpha)$ and $P_b(\beta, \alpha)$ meets b_0 before a, then we do Kempe changes on $P_{[b,b_0]}(\beta, \alpha)$, uncolor e_0' and color e_{ab} with α. If $e_0' \in P_b(\beta, \alpha)$ and $P_{w''}(\beta, \alpha)$ meets b_0 before a, then we uncolor e_0', do Kempe changes on $P_{[w'',b_0]}(\beta, \alpha)$, do a shifting in S from a to w'' and recolor the edge $e_{bw''}$ with β. In all three cases above, the edge e_{ab} is colored with a color in $[k]$ and e_0' is uncolored. Finally we do a shifting in S' from b_0 to b_{l_t} which gives a color in $[k]$ to e_0', and uncolor e_{l_t}'. Notice that the above shifting in S' does nothing if $b_0 = b_{l_t}$. Since $H_s^1 + e_{ab} - e_{l_t}'$ is also k-dense and $\chi'(H_s^1 + e_{ab} - e_{l_t}') = k$, we can permute color classes of $E(H_s^1 + e_{ab} - e_{l_t}')$ but keep the color i unchanged to match all the boundary edges by Lemma 2.4. Now we obtain a new matching $M_2^* = (M_1^*\backslash\{e_{ab}\}) \cup \{e_{l_t}'\}$ and a new k-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Moreover, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 2$.

Then we assume that there exists $b_{l_i} = a_j \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ for some $i \in [t-1]$. In this case we assume a_j is the closest vertex to the vertex a along S. Note that $b_{l_i} \neq b$ as $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint. Let $\alpha_i = (\varphi_1)_{H_s^1}(e_{l_{i+1}}') \in (\overline{\varphi}_1)_{H_s^1}(b_{l_i})$. By Lemma 3.1 (b), we have $P_b(\beta, \alpha_i) = P_{b_{l_i}}(\beta, \alpha_i)$. If $e_{l_{i+1}}' \notin P_b(\beta, \alpha_i)$, then we do Kempe changes on $P_{[b,b_{l_i}]}(\beta, \alpha_i)$, uncolor $e_{l_{i+1}}'$ and color e_{ab} with α_i. If $e_{l_{i+1}}' \in P_b(\beta, \alpha_i)$ and $P_b(\beta, \alpha_i)$ meets $b_{l_{i+1}}$ before a, then we do Kempe changes on $P_{[b,b_{l_{i+1}}]}(\beta, \alpha_i)$, uncolor $e_{l_{i+1}}'$ and color e_{ab} with α_i. If $e_{l_{i+1}}' \in P_b(\beta, \alpha_i)$ and $P_{b_{l_i}}(\beta, \alpha_i)$ meets $b_{l_{i+1}}$ before a, then we uncolor $e_{l_{i+1}}'$, do Kempe changes on $P_{[b_{l_i},b_{l_{i+1}}]}(\beta, \alpha_i)$, do a shifting in S from a to b_{l_i} and recolor the edge e_{l_i} with β. In all three cases above, the edge e_{ab} is colored with a color in $[k]$ and $e_{l_{i+1}}'$ is uncolored. Finally we do a shifting in S' from $b_{l_{i+1}}$ to b_{l_t}, which gives a color in $[k]$ to $e_{l_{i+1}}'$, and uncolor e_{l_t}'. Notice that the above shifting in S' does nothing if $b_{l_{i+1}} = b_{l_t}$. Since $H_s^1 + e_{ab} - e_{l_t}'$ is also k-dense and $\chi'(H_s^1 + e_{ab} - e_{l_t}') = k$, we can permute color classes of

$E(H_s^1+e_{ab}-e'_{l_t})$ but keep the color i unchanged to match all the boundary edges by Lemma 2.4. Now we obtain a new matching $M_2^* = (M_1^*\backslash\{e_{ab}\}) \cup \{e'_{l_t}\} \subseteq G - V(M)$ and a new k-edge-coloring φ_2 of $G-(M \cup M_2^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Moreover, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 2$.

Subcase 3.3: F_b does not contain a linear sequence from a to y with no i-edge, and F_b does not contain a vertex w'' with $d_{H_s^1}(w'') = \Delta - 1$ and $i \in (\overline{\varphi}_1)_{H_s^1}(w'')$.

We claim that F_b contains a linear sequence S^* from a to y^* ($y^* \neq y$), where $d_{H_s^1}(y^*) = \Delta$ and there is no i-edge in S^*. By Lemma 3.2 (a), the multi-fan F_b contains at least one Δ-vertex in H_s^1. Now if F_b does not contain any linear sequence without i-edges from a to any Δ-vertex in H_s^1, then by Lemma 3.2 (c), the multi-fan F_b contains a vertex w'' with $d_{H_s^1}(w'') = \Delta - 1$ and $i \in (\overline{\varphi}_1)_{H_s^1}(w'')$, contradicting the condition of Subcase 3.3. So F_b contains a linear sequence S^* from a to a vertex y^*, where $d_{H_s^1}(y^*) = \Delta$ and there is no i-edge in S^*. Note that $y^* \neq y$, since otherwise we also have a contradiction to the condition of Subcase 3.3. Thus the claim is proved.

Assume that $S^* = (b, e_0, a_0, e_1, a_1, \ldots, e_p, a_p)$ from a to y^*, where $e_0 = e_{ab}$, $a_0 = a$, $e_p = e_{by^*} \in E_{H_s^1}(b, y^*)$, $a_p = y^*$, and S^* contains no i-edge. Let $\theta \in \overline{\varphi}_1(y^*)$.

Subcase 3.3.1: $\theta = i$.

We do a shifting in S^* from a to y^*, uncolor the edge e_{by^*}, and permute color classes of $E(H_s^1 + e_{ab} - e_{by^*})$ but keep the color i unchanged to match all the boundary edges by Lemma 2.4. Then color the edge e_{by^*} with i and recolor the edge e_{bv} from i to $\Delta+\mu$, which results in a new matching $M_2^* = M_1^*\backslash\{e_{ab}\} \subseteq G - V(M)$ and a new $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$. Then f_{uv} is no longer a second-improper edge or even a first-improper edge at v with respect to the new prefeasible triple $(M_2^*, \bar{M}_{\varphi_2}^{\Delta+\mu}, \varphi_2)$ with $\bar{M}_{\varphi_2}^{\Delta+\mu} = \{e_{bv}\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \emptyset$, or $\bar{M}_{\varphi_2}^{\Delta+\mu} = \{e_{bv}, h\}$ if $\bar{M}_{\varphi_1}^{\Delta+\mu} = \{h\}$ (when $y^* \in V(F_x) \cap V(F_b)$) by the observation (3) in Case 1. Moreover, $\bar{M}_{\varphi_2}^{\Delta+\mu} \subseteq (M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1))$, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 1$.

Subcase 3.3.2: $\theta \neq i$.

Since $V(H_s^1)$ is $(\varphi_1)_{H_s^1}$-elementary, there exists a θ-edge e'_0 incident with the vertex a. Thus similarly as in Subcase 3.2, we can define a maximal multi-fan at a with respect to e'_0 and $(\varphi_1)_{H_s^1}$ in H_s^1, denoted by $F'_a = (a, e'_0, b_0, \ldots, e'_q, b_q)$, such that $(\varphi_1)_{H_s^1}(e'_j) \in (\overline{\varphi}_1)_{H_s^1}(b_{l-1})$ for $j \in [q]$ and some $l \in [j]$. By the assumption (1) and Lemma 3.2 (b), we have $\mu_{F_a}(a, b') = \mu_{H_s^1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a)\backslash\{a\}$. Therefore, $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint, since otherwise we have $V(F'_a) \subseteq V(F_a)$ and $(\varphi_1)_{H_s^1}(e'_0) = \theta \in (\overline{\varphi}_1)_{H_s^1}(b')$ for some $b' \in V(F_a)$ implying $y^* = b' \in V(F_a)$, which contradicts the assumption (1).

Note that $V(F'_a)\backslash\{a\}$ must contain a Δ-vertex in H_s^1, since otherwise Lemma 3.2 (d) and the fact $(\varphi_1)_{H_s^1}(e'_0) = \theta \in \overline{\varphi}_1(y^*)$ imply that $y^* \in V(F'_a)$, which contradicts $d_{H_s^1}(y^*) = \Delta$. Moreover, if F'_a does not contain any linear sequence to a Δ-vertex in H_s^1 without i-edges, then by Lemma 3.2 (d) the multi-fan F'_a contains a vertex w^* with $i \in (\overline{\varphi}_1)_{H_s^1}(w^*)$ and $d_{H_s^1}(w^*) = \Delta - 1$, so w^* is not incident with any edge in $M \cup M_1^*$. Thus F'_a contains a linear sequence $S' = (a, e'_{l_1}, b_{l_1}, \ldots, e'_{l_t}, b_{l_t})$, where $e'_{l_1} = e'_0$, $b_{l_1} = b_0$, b_{l_t} is w^* if there exists a vertex $w^* \in V(F'_a)$ with $d_{H_s^1}(w) = \Delta - 1$ such that w^* is incident with a boundary i-edge $h^* \in \partial(H_s^1)$ but $h^* \notin M_{\varphi_0}^A(f_1)$, and b_{l_t} is a Δ-vertex in H_s^1 otherwise. Notice that b_{l_t} is not incident with any edge in $M \cup M_1^*$ by our choice of b_{l_t}. Moreover, if $b_{l_t} = w^*$ as defined above, then $b_{l_t} = w^*$ is not a vertex in $V(F_b)$ by the condition of Subcase 3.3. And $b_{l_t} \neq y$ since $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint. Let β ($\beta \neq i$) be a color in $\overline{\varphi}_1(b)$. By Lemma 3.1 (b), we have $P_b(\beta, \theta) = P_{y^*}(\beta, \theta)$. We then consider the following two subcases according to the set $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$ is either $\{b_{l_t}\}$ or $\emptyset$. If $e'_0 \notin P_b(\beta, \theta)$, then we do Kempe changes on $P_{[b,y^*]}(\beta, \theta)$, uncolor e'_0 and color e_{ab} with θ. If $e'_0 \in P_b(\beta, \theta)$ and $P_b(\beta, \theta)$ meets b_0 before a, then we do Kempe changes on $P_{[b,b_0]}(\beta, \theta)$, uncolor e'_0 and color e_{ab} with θ. If $e'_0 \in P_b(\beta, \theta)$ and $P_{y^*}(\beta, \theta)$ meets b_0 before a, then we uncolor e'_0, do Kempe changes on $P_{[y^*,b_0]}(\beta, \theta)$, do a shifting in S^* from a to y^* and recolor e_{by^*} with β. In all three cases above, the edge e_{ab} is colored with a color in $[k]$ and e'_0 is uncolored. Then we do a shifting in S' from b_0 to b_{l_t} which gives a color in $[k]$ to e'_0, and uncolor e'_{l_t}, and permute color classes of $E(H_s^1 + e_{ab} - e'_{l_t})$ but keep the color i unchanged to match all the boundary edges except i-edges by Lemma 2.4. Finally recolor h^* with the color $\Delta + \mu$ if w^* is incident with a boundary i-edge $h^* \in \partial(H_s) \cap M_{\varphi_0}^A(f_1)$. Now we obtain a new matching $M_2^* = (M_1^*\backslash\{e_{ab}\}) \cup \{e'_{l_t}\} \subseteq G - V(M)$ and a new proper $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) with respect to the new prefeasible triple $(M_2^*, \bar{M}_{\varphi_2}^{\Delta+\mu}, \varphi_2)$, where $\bar{M}_{\varphi_2}^{\Delta+\mu} = \emptyset$ or $\{h\}$ or $\{h^*\}$. Moreover, $\bar{M}_{\varphi_2}^{\Delta+\mu} \subseteq M_{\varphi_0}^A(f_1)$, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 2$.

Then we assume that there exists $b_{l_i} = a_j \in (V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$ for some $i \in [t-1]$. In this case we assume a_j is the closest vertex to a along S^*. Note that $b_{l_i} \neq b$ as $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are vertex-disjoint. Let $\theta_i = (\varphi_1)_{H_s^1}(e'_{l_{i+1}}) \in (\overline{\varphi}_1)_{H_s^1}(b_{l_i})$. By Lemma 3.1 ($b$), $P_b(\beta, \theta_i) = P_{b_{l_i}}(\beta, \theta_i)$. If $e'_{l_{i+1}} \notin P_b(\beta, \theta_i)$, then we do Kempe changes on $P_{[b,b_{l_i}]}(\beta, \theta_i)$, uncolor $e'_{l_{i+1}}$ and color e_{ab} with θ_i. If $e'_{l_{i+1}} \in P_b(\beta, \theta_i)$ and $P_b(\beta, \theta_i)$ meets $b_{l_{i+1}}$ before a, then we do Kempe changes on $P_{[b,b_{l_{i+1}}]}(\beta, \theta_i)$, uncolor $e'_{l_{i+1}}$ and color e_{ab} with θ_i. If $e'_{l_{i+1}} \in P_b(\beta, \theta_i)$ and $P_{b_{l_i}}(\beta, \theta_i)$ meets $b_{l_{i+1}}$ before a, then we uncolor $e'_{l_{i+1}}$, do Kempe changes on $P_{[b_{l_i}, b_{l_{i+1}}]}(\beta, \theta_i)$, do a shifting in S^* from a to b_{l_i} and recolor the edge $e_{l_i} = e_{bb_{l_i}} \in E_{H_s^1}(b, b_{l_i})$ with β. In all three cases above, the edge e_{ab} is colored with a color in $[k]$ and $e'_{l_{i+1}}$ is uncolored. Then we do a shifting in S' from $b_{l_{i+1}}$ to b_{l_t} which gives a color in $[k]$ to $e'_{l_{i+1}}$, and uncolor the edge e'_{l_t}, and permute color classes of $E(H_s^1 + e_{ab} - e'_{l_t})$ but keep the color i unchanged to match all the boundary edges except i-edges by Lemma 2.4. Finally recolor h^* with $\Delta + \mu$ if w^* is incident with a boundary i-edge $h^* \in \partial(H_s) \cap M_{\varphi_0}^A(f_1)$. Now we obtain a new matching $M_2^* = (M_1^* \backslash \{e_{ab}\}) \cup \{e'_{l_t}\} \subseteq G - V(M)$ and a new proper $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer a second-improper edge (but becomes a first-improper edge) with respect to the new prefeasible triple $(M_2^*, \bar{M}_{\varphi_2}^{\Delta+\mu}, \varphi_2)$, where $\bar{M}_{\varphi_2}^{\Delta+\mu} = \emptyset$ or $\{h\}$ or $\{h^*\}$. Moreover, $\bar{M}_{\varphi_2}^{\Delta+\mu} \subseteq M_{\varphi_0}^A(f_1)$, $B_{\varphi_2} = B_{\varphi_0} - 2$ and $A_{\varphi_2} = A_{\varphi_0} + 2$.

In all above Cases 1-3, the second-improper edge f_{uv} in M is no longer a second-improper edge with respect to one new prefeasible triple, say $(M^{*'}, \bar{M}_{\varphi'}^{\Delta+\mu}, \varphi')$ uniformly. Observe that all our operations in Cases 1-3 are inside $G[V(H_s)]$ and $G[V(H'_s)]$, and on at most two possible edges respectively in $\partial(H_s) \cap M_{\varphi_0}^A(f_1)$ and $\partial(H_{s'}) \cap M_{\varphi_0}^A(f_1)$. Recall that $M_{\varphi_0}^A(f_1)$ is a matching and all maximal k-dense subgraphs $H_1, H_2, \ldots, H_t$ are vertex-disjoint. Thus all other maximal k-dense subgraphs of $G - (M \cup M_0^*)$ distinct with H_s and $H_{s'}$ are also maximal k-dense subgraphs of $G - (M \cup M^{*'})$. For any other edges in M_0^* is still fully saturated with respect to the corresponding maximal k-dense subgraphs distinct with H_s and $H_{s'}$. Recall that M is a distance-3 matching, and each maximal k-dense subgraph of $H_1, H_2, \ldots, H_t$ has diameter at most 2. Thus for all other second-improper edges distinct with f_{uv} in M, we can do the same operations as we did for f_{uv} in Cases 1-3 such that the number of second-improper edges becomes zero with respect to one new prefeasible triple, say $(M^{*''}, \bar{M}_{\varphi''}^{\Delta+\mu}, \varphi'')$. By operations in Cases 1-3 we have $\bar{M}_{\varphi''}^{\Delta+\mu} \subseteq (M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1))$, Then by giving the color $\Delta + \mu$ to all edges in $M_{\varphi''}^A(f_1) = (M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1)) \setminus \bar{M}_{\varphi''}^{\Delta+\mu}$, the number of first-improper edges also becomes zero, and we get the final feasible triple $(M^*, \bar{M}_{\varphi}^{\Delta+\mu}, \varphi)$, where $\bar{M}_{\varphi}^{\Delta+\mu} = M_{\varphi_0}^A(f_1) \cup M_{\varphi_0}^B(f_1)$. The proof is now finished. □

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