# Linear Arboricity of Degenerate Graphs

Guantao Chen*, Yanli Hao†, Guoning Yu†‡
Department of Mathematics and Statistics, Georgia State University
Atlanta, GA 30303, USA

**Abstract**

A *linear forest* is a union of vertex-disjoint paths, and the *linear arboricity* of a graph $G$, denoted by $\mathrm{la}(G)$, is the minimum number of linear forests needed to partition the edge set of $G$. Clearly, $\mathrm{la}(G) \geq \lceil \Delta(G)/2 \rceil$ for a graph $G$ with maximum degree $\Delta(G)$. On the other hand, the famous *Linear Arboricity Conjecture* (LAC) due to Akiyama, Exoo, and Harary from 1981 asserts that $\mathrm{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$ for every graph $G$. This conjecture has been verified for planar graphs and graphs whose maximum degree is at most 6, or is equal to 8 or 10.

A graph $G$ is *$k$-degenerate* for a positive integer $k$ if it can be reduced to a trivial graph by successive removal of vertices with degree at most $k$. We prove that for any $k$-degenerate graph $G$, $\mathrm{la}(G) = \lceil \Delta(G)/2 \rceil$ if $\Delta(G) \geq 2k^2 - k$.

*Keywords:* linear forest partition; linear arboricity; degenerate graphs

## 1 Introduction

All graphs in this paper are simple, i.e., finite, undirected, and without loops or multiple edges. A *linear forest* is a union of vertex-disjoint paths. Denote by $\Delta(G)$ the maximum degree of a graph $G$. The *linear arboricity* of a graph $G$, denoted by $\mathrm{la}(G)$, is the minimum number of linear forests needed to partition its edge set. Clearly, in order to cover all edges incident to a vertex of maximum degree, we need at least $\lceil \Delta(G)/2 \rceil$ linear forests, i.e., $\mathrm{la}(G) \geq \lceil \Delta(G)/2 \rceil$. Moreover, it is not difficult to see that

*Partially supported by NSF grant DMS-1855716, gchen@gsu.edu
†Partially sopported by the University Graduate Fellowship of Georgia State University
‡the corresponding author, gyu6@gsu.edu

$\mathrm{la}(G) \geq \lceil (\Delta(G)+1)/2 \rceil$ for regular graphs $G$ with even degree. The following conjecture, commonly referred to as the *linear arboricity conjecture* (LAC), of Akiyama, Exoo and Harary [2] in 1981 asserted this bound to be sharp.

**Conjecture 1.1** (LAC)**.** *For every graph $G$,* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

If the LAC is true, we have $\lceil \Delta(G)/2 \rceil \leq \mathrm{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$ for every graph $G$. An edge coloring of a graph is a partition of its edge set into matchings, which can be considered as a linear forest partition whose each component is a single edge. Hence, the LAC can be viewed as an analog to Vizing's theorem. The conjecture is still wide open, although there has been some progress in the past nearly 30 years. It was only verified for graphs with maximum degree at most 6, and equal to 8 or 10: $\Delta(G) = 3, 4$ by Akiyama, Exoo, and Harary [1, 2], $\Delta(G) = 5, 6, 8$ by Enomoto and Péroche [8]; and $\Delta(G) = 10$ by Guldan [11]. the LAC was confirmed for planar graphs by Wu in 1999 [16] and Wu and Wu in 2008 [17]. Furthermore, Cygan, Hou, Kowalik, Lužar and Wu in 2011 [6] conjectured that the linear arboricity is completely determined for planar graphs provided the maximum degree is large and verified their conjecture for planar graphs with $\Delta(G) \geq 9$, leaving open only the case $\Delta(G) = 6, 8$.

**Conjecture 1.2** (Cygan et al.)**.** *For any planar graph $G$ with $\Delta(G) \geq 5$,* $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Approximately and asymptotically, Alon in 1988 [3] proved that $\mathrm{la}(G) \leq \frac{\Delta(G)}{2} + O\left(\frac{\Delta(G) \log\log \Delta(G)}{\log \Delta(G)}\right)$. In the same paper, he also showed that the LAC holds for graphs with girth $\Omega(\Delta(G))$. Alon and Spencer in 1992 (see [14]) further improved this bound. In 2019, Ferber, Fox and Jain [9] further narrowed it to $\Delta(G)/2 + \beta\Delta(G)^{2/3-\alpha}$, where $\alpha, \beta$ are two positive constants. McDiarmid and Reed [13] confirmed the LAC for random regular graphs with fixed degrees. Glock, Kühn and Osthus [10] showed that, for a large range of $p$, a.a.s. the random graph $G \sim G_{n,p}$ can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests. Moreover, they also verified the LAC for large and sufficiently dense regular graphs.

For any positive integer $k$, a graph $G$ is *$k$-degenerate* if it can be reduced to a trivial graph by successive removal of vertices of degree at most $k$; equivalently, every subgraph of $G$ has a vertex of degree at most $k$. Clearly, trees are 1-degenerate; outerplanar graphs are 2-degenerate; and planar graphs are 5-degenerate. Moreover, for any surface $\Pi$, oriented or non-oriented, there is $k = k(\Pi)$ such that all graphs embeddable in $\Pi$ are

$k$-degenerate. Borodin, Kostochka and Woodall [5] studied the list edge colorings and list total colorings for degenerate graphs. Zhou, Nakano and Nishizeki [18] gave a linear time algorithm for computing the chromatic index of degenerate graphs. Isobe, Zhou and Nishizeki [12] showed that the Total Coloring Conjecture holds for $k$-degenerate graphs $G$ if $\Delta(G) \geq 4k + 3$. Inspired by their result, we show that the LAC holds for $k$-degenerate graphs with large maximum degrees. More precisely, we completely determine $\mathrm{la}(G)$ for these graphs as follows.

**Theorem 1.3.** *Let $G$ be a $k$-degenerate graph. If $\Delta(G) \geq 2k^2-k$, then* $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Basavaraju, Bishnu, Francis and Pattanayak [4] recently showed that the LAC holds for 3-degenerate graphs. We also like to make some comments about our proof techniques. Previously, except for planar graphs, all known proofs are based on converting the graphs to regular graphs and use the property that all vertices have the same degree to find a desired linear forest partition. In our proof, we take advantage of the gaps of degrees. More precisely, we reserve some vertices with small degrees as *representatives* to give some specific partitions for vertices with large degrees, which in turn gives a linear forest partition step by step for the entire graph. We hope that this new approach may shed some light on the issues that arise when trying to solve for the LAC. We will present some auxiliary lemmas in Section 2 and prove Theorem 1.3 in Section 3.

## 2 Preliminaries

In this section, we state some basic notation and terminology, present a brute-force method of adding a pair of adjacent vertices with "low degree" to a linear forest partition (Lemma 2.1), and introduce some properties of "high degree" vertices in degenerate graphs (Lemma 2.2).

We mainly use the notation from West [15]. Let $G$ be a graph, and let $V(G), E(G)$ and $\Delta(G)$ be the vertex set, the edge set and the maximum degree of $G$, respectively. The degree of a vertex $v$ in $G$, written $d_G(v)$ or $d(v)$, is the number of edges incident to $v$. Denote by $N_G(v)$ or $N(v)$ the neighborhood of $v$. Clearly, $d(v) = |N_G(v)|$. And we denote the set of edges that connects $v$ to a vertex set $W \subseteq V(G)$ by $E(v, W) = \{vw \in E(G) : w \in W\}$. Let $G \pm v, G \pm e$ and $G \pm F$ denote the addition or deletion of a vertex $v$, an edge $e$ and an edge set $F$ of $G$, respectively.

A partition $\mathcal{F} := F_1 \mid F_2 \mid \cdots \mid F_t$ of $E(G)$ is called a *linear forest partition* of $G$ if the spanning subgraph induced by $F_i$ is a linear forest for each $i \in \{1, \ldots, t\}$. For the sake of convenience, we also use $F_i$ to denote the spanning subgraph induced by $F_i$ when we work on the linear forest partition $\mathcal{F}$. Notice that graph $F_i$ may contain isolated vertices. In fact, for any vertex $v \in V(G)$ and $F_i \in \mathcal{F}$, we have $d_{F_i}(v) = 0, 1$ or $2$, that is, $v$ is an isolated vertex, an end-vertex of a path, or an internal vertex of a path in $F_i$. For each $v \in V(G)$ and $p \in \{0, 1, 2\}$, let $\mathcal{F}(v, p) = \{F_j \in \mathcal{F} : d_{F_j}(v) = p\}$. It is readily seen that $2|\mathcal{F}(v, 2)| + |\mathcal{F}(v, 1)| = d(v)$ and $|\mathcal{F}(v, 2)| + |\mathcal{F}(v, 1)| + |\mathcal{F}(v, 0)| = t$. In our constructional proof, we want to restrict the number of the vertices $v$ with $\mathcal{F}(v, 2) \neq \emptyset$ for a given linear forest partition $\mathcal{F}$.

**Lemma 2.1.** *Let $G$ be a graph and edge $xy \in E(G)$. If $2d(x) + d(y) \leq 2t + 2$, then for any linear forest partition $\mathcal{F} = F_1 \mid F_2 \mid \cdots \mid F_t$ of $G - xy$, there exists an $i \in \{1, \ldots, t\}$ such that $\mathcal{F}^* = F_1 \mid \cdots \mid F_{i-1} \mid F_i + xy \mid F_{i+1} \mid \cdots \mid F_t$ is a linear forest partition of $G$. Moreover, we have $\mathcal{F}^*(v, 2) = \mathcal{F}(v, 2)$ for any $v \in V(G) \setminus \{y\}$.*

*Proof.* Since $\mathcal{F} = F_1 \mid \cdots \mid F_t$ partitions $E(G - xy)$, we have $\sum_{i=1}^{t} d_{F_i}(x) = d(x) - 1$ and $\sum_{i=1}^{t} d_{F_i}(y) = d(y) - 1$. And by the inequality $2d(x) + d(y) \leq 2t + 2$, we have

$$\sum_{i=1}^{t} (2d_{F_i}(x) + d_{F_i}(y)) = 2d(x) + d(y) - 3 \leq (2t + 2) - 3 < 2t$$

From the Pigeonhole Principle, there exists an $i \in \{1, \ldots, t\}$ such that $2d_{F_i}(x) + d_{F_i}(y) \leq 1$. Hence, $d_{F_i}(x) = 0$ and $d_{F_i}(y) \leq 1$. Equivalently, $x$ is an isolated vertex in $F_i$ and $y$ is either an isolated vertex or an end-vertex of a path in $F_i$. As a result, $F_i^* := F_i + xy$ is still a linear forest, in which $d_{F_i^*}(x) = 1$ and $d_{F_i^*}(y) \leq 2$. Note that only the degree of $x$ and $y$ change, so that $\mathcal{F}^*(v, 2) = \mathcal{F}(v, 2)$ for any $v \neq y$. Therefore, $\mathcal{F}^* := F_1 \mid \cdots \mid F_{i-1} \mid F_i^* \mid F_{i+1} \mid \cdots \mid F_t$ is the desired linear forest partition for $G$. □

For a given vertex ordering $v_1, v_2, \ldots, v_n$ of a graph $G$, let $L(v_i) = \{v_1, v_2, \ldots, v_{i-1}\}$ and $R(v_i) = \{v_{i+1}, \ldots, v_n\}$ for any vertex $v_i$. For each $v_i \in V(G)$, let $N_L(v_i) = N(v_i) \cap L(v_i)$, $N_R(v_i) = N(v_i) \cap R(v_i)$. Every $k$-degenerate graph $G$ admits a *$k$-degenerate vertex ordering*, that is, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $G$ such that $|N_L(v_i)| \leq k$ for every vertex $v_i \in V(G)$.

The following variant of Hall's classic marriage theorem will be applied in the proof of our ensuing result. (See [7, ex. 9].) Given a family of subsets of a finite set $\mathcal{A} = \{A_1, \ldots, A_m\}$, there exist $m$ mutually disjoint subsets $A_i^* \subseteq A_i$ with size

$|A_i^*| \geq r$ for each $i \in \{1, \ldots, m\}$ if and only if $|\cup_{i\in I} A_i| \geq r|I|$ for every index set $I \subseteq \{1, \ldots, m\}$. We call this $m$-tuple a *system of distinct representatives of size* $r$ (or an $r$-*SDR*) of family $\mathcal{A}$, and say that $A_i^*$ *represents* $A_i$ for each $i \in \{1, \ldots, m\}$.

**Lemma 2.2.** *Let $G$ be a $k$-degenerate graph and $v_1, \ldots, v_n$ be a $k$-degenerate vertex ordering of $G$. For every integer $d$ with $k \leq d \leq \Delta(G)$, the family of vertex sets $\mathcal{R} := \{N_R(v_i) : d(v_i) \geq d\}$ has a $\lfloor \frac{d-k}{k} \rfloor$-SDR.*

*Proof.* Let $V_d(G) = \{v \in V(G) : d(v) \geq d\}$. For each $v_i \in V_d(G)$, since $d_G(v_i) = |N_L(v_i)| + |N_R(v_i)|$ and $|N_L(v_i)| \leq k$, we have $|N_R(v_i)| \geq d(v_i) - k \geq d - k$. For any vertex $v_j$, if $v_j \in N_R(v_i)$, then $v_i \in N_L(v_j)$. Since $|N_L(v_j)| \leq k$, each vertex $v_j \in V(G)$ can only appear in at most $k$ sets $N_R(v_i)$ for all $v_i \in V(G)$. Hence, for any $S \subseteq V_d(G)$, we have

$$\left| \bigcup_{v_i \in S} N_R(v_i) \right| \geq \frac{1}{k} \sum_{v_i \in S} |N_R(v_i)| \geq \frac{1}{k}(d-k)|S| \geq \left\lfloor \frac{d-k}{k} \right\rfloor |S|.$$

By Hall's marriage matching theorem, the family $\mathcal{R}$ has a $\lfloor \frac{d-k}{k} \rfloor$-SDR with $R^*(v_i)$ for each $v_i \in V_d(G)$ satisfying $R^*(v_i) \subseteq N_R(v_i)$ and $|R^*(v_i)| \geq \lfloor \frac{d-k}{k} \rfloor$. $\square$

# 3 Proof of Theorem 1.3

A graph $G$ is 1-degenerate if and only if it is a forest, in which case a linear forest partition of size $\lceil \Delta(G)/2 \rceil$ can be easily found. We assume $k > 1$ in the remainder of our proof. Since the addition or deletion of isolated vertices does not change the linear arboricity, we further assume that the graph being considered does not have isolated vertices.

For any positive integer $d > 1$, we call a graph $(d, 1)$-*regular* if all its vertices are of degree either $d$ or 1. Note that if graph $G$ is $k$-degenerate with $\Delta := \Delta(G)$, there is a $(\Delta, 1)$-regular $k$-degenerate graph $G^*$ containing $G$ as a subgraph. We can construct $G^*$ from $G$ by adding $\Delta - d(v)$ new vertices as neighbors for each $v \in V(G)$ satisfying $1 < d(v) < \Delta$. Clearly, $\text{la}(G) \leq \text{la}(G^*)$ since $G \subseteq G^*$. With this, it suffices that Theorem 1.3 holds for $(\Delta, 1)$-regular $k$-degenerate graphs. We state the following theorem which gives Theorem 1.3.

**Theorem 1.3*.** *Let $G$ be a $(\Delta, 1)$-regular $k$-degenerate graph. If $\Delta \geq 2k^2 - k$, then there exists a linear forest partition $\mathcal{F} = F_1 \mid F_2 \mid \cdots \mid F_t$ of $G$ with $t = \lceil \frac{\Delta}{2} \rceil$.*

*Proof.* Suppose $\Delta \geq 2k^2 - k$. Let $G$ be a $(\Delta, 1)$-regular $k$-degenerate graph, and $V_\Delta = \{v : d(v) = \Delta\}$. Then, $d(v) = 1$ if $v \notin V_\Delta$. Let $v_1, \ldots, v_n$ be a $k$-degenerate vertex ordering of $G$. Clearly, we may assume that $G$ is connected and, moreover, $i < j$ whenever $d(v_i) = \Delta$ and $d(v_j) = 1$.

Applying Lemma 2.2, we get an $r$-SDR of the family $\{N_R(v_i) : v_i \in V_\Delta\}$ where

$$r = \left\lfloor \frac{\Delta - k}{k} \right\rfloor \geq 2k - 2 \geq 2,$$

that is, there are mutually disjoint vertex sets $R^*(v_i) \subseteq N_R(v_i)$ for all $v_i \in V_\Delta$ such that $|R^*(v_i)| = r$ for all $v_i \in V_\Delta$. Additionally, we assume $R^*(v_i) = \emptyset$ when $v_i \notin V_\Delta$ for consistency.

For each $i \in \{1, 2, \ldots, n\}$, let $G_i$ be the spanning subgraph of $G$ induced by edges incident to at least one vertex in $\{v_1, v_2, \ldots, v_i\}$. Clearly, $G_1$ is the union of a star centered at $v_1$ and isolated vertices, and $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$. The following technical result implies Theorem 1.3*.

**Proposition 3.1.** *For each $i \in \{1, \ldots, n\}$, $G_i$ has a linear forest partition $\mathcal{F}^i = F_1^i \mid F_2^i \mid \cdots \mid F_t^i$, where $t = \lceil \frac{\Delta}{2} \rceil$. And even further, such linear forest partition satisfies*

$$\mathcal{F}^i(v, 2) = \emptyset \text{ for all } v \in \cup_{j>i} R^*(v_j). \qquad (*)$$

We notice that $G_{n-1} = G_n = G$, so that $\mathcal{F}^{n-1}$ gives the disred partition which completes the proof of Theorem 1.3*. We prove Proposition 3.1 by constructing each $\mathcal{F}^i$ inductively. The property $(*)$ will facilitate each step of the construction.

First, for $G_1$, by the definition, all its edges are incident to $v_1$ and $d_{G_1}(v_1) = d(v_1) = \Delta \leq 2t$. By pairing edges in the star, we get a linear forest partition $\mathcal{F}^1 = F_1^1 \mid \cdots \mid F_t^1$. Since no vertex other than $v_1$ can be an internal vertex of a path in any $F_j^1$ for $j \in \{1, \ldots, t\}$, it follows that $\mathcal{F}^1(v, 2) = \emptyset$ for all $v \neq v_1$. Hence, $(*)$ is satisfied.

Suppose that $i \geq 2$, and we have constructed a linear forest partition $\mathcal{F}^{i-1} = F_1^{i-1} \mid \cdots \mid F_t^{i-1}$ of $G_{i-1}$ such that $\mathcal{F}^{i-1}(v, 2) = \emptyset$ for each $v \in \cup_{j>i-1} R^*(v_j)$. Next, we construct a desired linear forest partition $\mathcal{F}^i$ based on $\mathcal{F}^{i-1}$.

Note that for the vertex $v_i$, we have either $v_i \notin V_\Delta$ or $v_i \in V_\Delta$. If $v_i \notin V_\Delta$, then $d(v_i) = d_{G_i}(v_i) = 1$. Because $G$ is connected and $k > 1$, no two vertices of degree 1 in $G$ can be adjacent. Based on the assumption that every vertex of degree

1 appears after every vertex of degree $\Delta$ in the $k$-degenerate ordering $v_1, v_2, \ldots, v_n$ of $G$, the only neighbor $v_j$ of $v_i$ must be in $L(v_i)$, i.e., $E(v_i, R(v_i)) = \emptyset$. Since $E(G_i) = E(G_{i-1}) \cup E(v_i, N_R(v_i))$, it follows that $G_i = G_{i-1}$. Hence, $\mathcal{F}^i = \mathcal{F}^{i-1}$ gives a desired partition of $G_{i-1}$. We now assume $v_i \in V_\Delta$, i.e., $d(v_i) = \Delta$.

Let $W = N_R(v_i) \setminus R^*(v_i)$. We first add edges from $E(v_i, W)$ one by one to $G_{i-1}$. We denote by $H_\ell$ the graph after adding $\ell$ edges for each $\ell \in \{1, 2, \ldots, |W|\}$, and by $H$ the graph $H_{|W|}$ we get eventually. The following claim implies that $H$ has a desired linear forest partition.

**Claim 3.2.** *Let $\mathcal{F}^{(i-1)*0} = \mathcal{F}^{i-1}$. For each $\ell \in \{1, 2, \ldots, |W|\}$, if $\mathcal{F}^{(i-1)*(\ell-1)}$ is a linear forest partition of $H_{\ell-1}$ with the property (∗), then there exist a linear forest partition $\mathcal{F}^{(i-1)*\ell}$ of $H_\ell$ with the property (∗).*

*Proof.* For each $\ell$, it is clear that $H_l \subseteq G_i$. Then, $d_{H_\ell}(u) \leq d_{G_i}(u) \leq k$ for any $u \in W$. Using the equality $|W| = |N_R(v_i)| - |R^*(v_i)| = \Delta - d_{G_{i-1}}(v_i) - |R^*(v_i)|$, we have $d_{H_\ell}(v_i) = d_{G_{i-1}}(v_i) + \ell \leq d_{G_{i-1}}(v_i) + |W| = \Delta - |R^*(v_i)| = \Delta - r$. Since $r \geq 2k-2$, we have

$$d_{H_\ell}(v_i) + 2d_{H_\ell}(v_j) \leq (\Delta - r) + 2k \leq \Delta + 2 \leq 2t + 2.$$

Applying Lemma 2.1, we add an edge $v_i u \in E(v_i, W)$ to $\mathcal{F}^{(i-1)*(\ell-1)}$ such $\mathcal{F}^{(i-1)*\ell}$ is a linear forest partition of $H_\ell$ satisfying $\mathcal{F}^{(i-1)*\ell}(v, 2) = \mathcal{F}^{(i-1)*(\ell-1)}(v, 2)$ for every $v \neq v_i$. Therefore, if $\mathcal{F}^{(i-1)*(\ell-1)}(v, 2) = \emptyset$ for all $v \in \cup_{j>i} R^*(v_j)$, then $\mathcal{F}^{(i-1)*\ell}(v, 2) = \emptyset$ for all $v \in \cup_{j>i} R^*(v_j)$, which implies the property (∗). □

From Claim 3.2, the graph $H = G_{i-1} + E(v_i, W)$ has a linear forest partition $\mathcal{F}^{(i-1)*|W|} = F_1^{(i-1)*|W|} \mid \cdots \mid F_t^{(i-1)*|W|}$ satisfying the property (∗) inherited from $\mathcal{F}^{i-1}$. To avoid cumbersome notation, we denote $\mathcal{F}^{(i-1)*|W|} = F_1^{(i-1)*|W|} \mid \cdots \mid F_t^{(i-1)*|W|}$ by $\mathcal{F} = F_1 \mid \cdots \mid F_t$. Note that none of the edge in $E(v_i, R^*(v_i))$ has been added, we actually have $\mathcal{F}(v, 2) = \emptyset$ for all $v \in \cup_{j>i-1} R^*(v_j)$.

In the remainder of the proof, it is convenient to view a linear forest partition $\mathcal{F} = F_1 \mid \cdots \mid F_t$ of graph $H$ as a *linear forest coloring* $\phi_0 : E(H) \to \{1, \ldots, t\}$ such that $\phi_0(e) = j$ if and only if $e \in F_j$ for $j \in \{1, \ldots, t\}$. And from the discussion above, there exists a linear forest coloring $\phi_0$ of $H$ satisfying $\mathcal{F}(v, 2) = \emptyset$ for all $v \in \cup_{j>i-1} R^*(v_j)$. We call such coloring a *desired linear forest coloring*, and denote the set of all desired linear forest colorings of $H$ by $\Phi_0(H)$.

Let $\phi_0 \in \Phi_0(H)$ which generates a desired linear forest partition $\mathcal{F} = F_1 \mid \cdots \mid F_t$. For each $p \in \{0,1,2\}$, we denote by $C_p(v) = \{j \mid d_{F_j}(v) = p\}$ which is the set of colors that appear $p$ times at the edges incident to $v$. In order to color edges in $E(G_i) \setminus E(H) = E(v_i, R^*(v_i))$, we first assign colors arbitrarily from $C_0(v_i) \cup C_1(v_i)$. We call a coloring $\phi$ of edges in $E(G_i)$ a $\phi_0$-*extension* if $\phi(e) = \phi_0(e)$ for any edge $e \in E(H)$ and $\phi(e) \in C_0(v_i) \cup C_1(v_i)$ for any edge $e \in E(G_i) \setminus E(H) = E(v_i, R^*(v_i))$. A $\phi_0$-extension $\phi$ is called $v_i$-*saturated* if for edges in $E(v_i, R^*(v_i))$, each color in $C_0(v_i)$ is used at most twice and each color in $C_1(v_i)$ is used at most once. By the coloring $\phi_0$, since $C_0(v_i)$, $C_1(v_i)$ and $C_2(v_i)$ give a partition of $\{1, \ldots, t\}$, we have $|C_2(v_i)|+|C_1(v_i)|+|C_0(v_i)| = t$. And since $\phi_0$ is a coloring of edges in $E(H)$, we have $2|C_2(v_i)| + |C_1(v_i)| = d_H(v_i) = d(v_i) - r = \Delta - r$. Hence, $2|C_0(v_i)| + |C_1(v_i)| = 2t - (\Delta - r)$. Applying the fact that $\Delta \leq 2t = 2\left\lceil \frac{\Delta}{2} \right\rceil \leq \Delta + 1$, we have

$$r \leq 2|C_0(v_i)| + |C_1(v_i)| \leq r + 1, \tag{1}$$

where $r = |E(v_i, R^*(v_i))|$. Therefore, there exists a coloring of $E(v_i, R^*(v_i))$ such that each color in $C_0(v_i)$ is used at most twice and each color in $C_1(v_i)$ is used at most once, which shows that there exists a $\phi_0$-extension which is $v_i$-saturated. Further, we call a coloring $\phi$ a *feasible* coloring of $E(G_i)$ if $d_{F'_j}(v) \leq 2$ for all $v \in V(G)$ and $j \in \{1, \ldots, t\}$.

**Claim 3.3.** *Let $\phi_0 \in \Phi_0(H)$ which generates a desired linear forest partition $\mathcal{F} = F_1 \mid \cdots \mid F_t$, and let $\phi$ be a $\phi_0$-extension with the corresponding partition $\mathcal{F}' = F'_1 \mid \cdots \mid F'_t$ of $E(G_i)$. We have that $\phi$ is $v_i$-saturated if and only if it is feasible.*

*Proof.* The necessity is clear. We prove the sufficiency. Suppose that $\phi$ is $v_i$-saturated. Let $v$ be any vertex in $V(G)$. If $v \notin \{v_i\} \cup R^*(v_i)$, we have $d_{F'_j}(v) = d_{F_j}(v) \leq 2$ for all $j \in \{1, \ldots, t\}$. Also if $v \in R^*(v_i)$, then $d_{F_j}(v) \leq 1$ since $\mathcal{F}(v, 2) = \emptyset$, which gives $d_{F'_j}(v) \leq d_{F_j}(v) + 1 \leq 2$ for all $j \in \{1, \ldots, t\}$. Now suppose that $v = v_i$. Since $\phi$ is a $\phi_0$-extension which colors the edges in $E(v_i, R^*(v_i))$ only using colors in $C_0(v_i) \cup C_1(v_i)$, we have $d_{F'_j}(v) = d_{F_j}(v) \leq 2$ for any $j \in C_2(v_i)$. Moreover, given that $\phi$ is $v_i$-saturated, each color in $C_0(v_i)$ is used at most twice and each color in $C_1(v_i)$ is used at most once. Therefore, if $j \in C_0(v_i)$, then $d_{F'_j}(v) \leq d_{F_j}(v) + 2 = 0 + 2 = 2$. And if $j \in C_1(v_i)$, then $d_{F'_j}(v) \leq d_{F_j}(v) + 1 = 1 + 1 = 2$. □

Note that a feasible coloring of a graph generates a linear forest partition of $G_i$ if and only if there is no monochromatic cycle. Therefore, for any $v_i$-saturated $\phi_0$-extension

$\phi$, if we let $B(\phi) = \{v \in R^*(v_i) \mid v_iv$ is in a cycle of $F'_j$ for some $j \in \{1, \dots, t\}\}$, then $\phi$ gives a linear forest partition $\mathcal{F}' = F'_1 \mid \cdots \mid F'_t$ of $G_i$ when $B(\phi) = \emptyset$. Moreover, since $R^*(v_j) \cap R^*(v_i) = \emptyset$ for all $j > i$ and $\phi$ is a $\phi_0$-extension which keeps the coloring of every edge $e \notin E(v_i, R^*(v_i))$. If follows that $\mathcal{F}'(v, 2) = \mathcal{F}(v, 2) = \emptyset$ for all $v \in \cup_{j>i} R^*(v_j)$. Thus additionally, the linear forest partition $\mathcal{F}'$ of $G_i$ satisfies the property $(*)$, which gives $\mathcal{F}^i$ in Proposition 3.1. The following claim implies that such a linear forest partition $\mathcal{F}'$ exists.

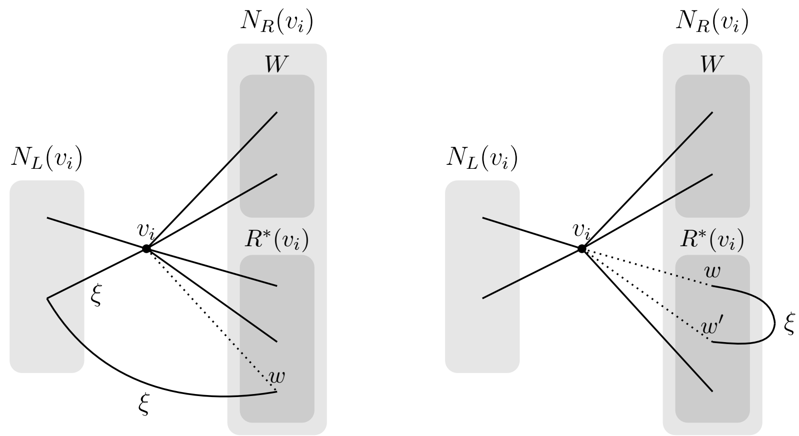
**Claim 3.4.** *There exists a $v_i$-saturated $\phi_0$-extension $\phi$ such that $B(\phi) = \emptyset$, where $\phi_0 \in \Phi_0(H)$.*

*Proof.* Suppose on the contrary that $B(\phi) \neq \emptyset$ for any $v_i$-saturated $\phi_0$-extension $\phi$, where $\phi_0$ is any desired linear forest coloring of $H$. Let $\phi$ be a coloring such that $|B(\phi)|$ is the minimum among all such colorings. Let $\mathcal{F} = F_1 \mid \cdots \mid F_t$ and $\mathcal{F}' = F'_1 \mid \cdots \mid F'_t$ be the corresponding partitions obtained from $\phi_0$ and $\phi$, respectively. Then, there exist $w \in B(\phi)$ and $\xi = \phi(v_iw)$, and $F'_\xi$ has a cycle $C$ containing edge $v_iw$. Let $P = C - v_i$. Clearly, $P$ is a path in $F_\xi$. We claim that $|V(P) \cap (R^*(v_i) \setminus \{w\})| \leq 1$. Since $\mathcal{F}(u, 2) = \emptyset$ for any vertex $u \in R^*(v_i)$, we have $d_{F_\xi}(u) \leq 1$. Hence, any vertex in $V(P) \cap R^*(v_i)$ is an end-vertex of $P$. It follows that $|V(P) \cap R^*(v_i)| \leq 2$. Since $w$ is obviously an end-vertex of $P$, there exists at most one vertex $w' \in V(P) \cap (R^*(v_i) \setminus \{w\})$, in other words, $|V(P) \cap (R^*(v_i) \setminus \{w\})| \leq 1$. There are only two possible types of cycles in $F'_\xi$ as illustrated in figure 1: if $F_\xi \in \mathcal{F}(v_i, 1)$, $P$ contains no vertices in $R^*(v_i)$ other than $w$ (see figure 1a); and if $F_\xi \in \mathcal{F}(v_i, 0)$, the equality $|V(P) \cap (R^*(v_i) \setminus \{w\})| = 1$ holds. In the latter case, we denote the unique vertex in the intersection by $w'$ (see figure 1b). Clearly, $\phi(v_iw') = \phi(v_iw) = \xi$.

Note that in either case above, we have $F_\xi \in \mathcal{F}(w, 1)$. To complete the proof, we prove that there exists a $v_i$-saturated $\phi_0$-extension $\phi^*$ obtained from $\phi$, by finding a color $\eta$ such that replacing the color $\xi$ on $v_iw$ by $\eta$ does not produce any new monochromatic cycles. We consider three cases as follows.

**Case 1.** $\mathcal{F}(w, 0) \setminus \mathcal{F}(v_i, 2) \neq \emptyset$.

Let $F_\eta \in \mathcal{F}(w, 0) \setminus \mathcal{F}(v_i, 2)$. If there is no edge $v_iu \in E(v_i, R^*(v_i))$ such that $\phi(v_iu) = \eta$, let $\phi^*$ be the coloring obtained from $\phi$ by assigning $v_iw$ the color $\eta$ to replace $\xi$. If there exists $v_iu \in E(v_i, R^*(v_i))$ such that $\phi(v_iu) = \eta$, let $\phi^*$ be obtained from $\phi$ by swapping colors on $v_iw$ and $v_iu$, i.e., $\phi^*(v_iw) = \eta$ and $\phi^*(v_iu) = \xi$. We




(a) $\xi \in \mathcal{F}(v_i, 1)$ (b) $\xi \in \mathcal{F}(v_i, 0)$

Figure 1: Two possible monochromatic cycles in $\mathcal{F}'$

denote by $\mathcal{F}^* = F_1^* \mid \cdots \mid F_t^*$ the partition given by the coloring $\phi^*$. Clearly, $\phi^*$ is $\phi_0$-saturated since no edge color in $H$ has been changed. Moreover, $\phi^*$ is $v_i$-saturated since neither color $\xi$ or $\eta$ has been used more than twice.

Further, we have $F_\eta^* \in \mathcal{F}^*(w, 1)$, which implies that there is no monochromatic cycle containing the edge $v_iw$ in $\phi^*$. Thus $w \notin B(\phi^*)$. Now if the edge $v_iu$ does not exist, then we have $|B(\phi^*)| \leq |B(\phi)| - 1$, which contradicts our choice of $\phi$. So, let us assume that the edge $v_iu$ exists. Suppose that there is a monochromatic cycle containing the edge $v_iu$ in $\phi^*$ with color $\xi$. Then, there is a path $P'$ between $u$ and $v_i$ in $F_\xi^*$ that does not contain the edge $v_iu$. Clearly, $P'$ also does not contain $v_iw$ (since $\phi^*(v_iw) = \eta$), which means that $P'$ is also a path in $F'_\xi - v_iw$. As $v_i$ belongs to $P'$, it must be the case that $P'$ is a subpath of $P$. By our earlier observation that the only vertices in $P$ that belong to $R^*(v_i)$ are $w$ and $w'$, it follows that $u \in \{w, w'\}$. But clearly, $u \neq w$ and since $\phi(v_iw') = \xi \neq \eta = \phi(v_iu)$, we have $u \neq w'$. We thus have a contradiction, and therefore we can conclude that there is no monochromatic cycle containing the edge $v_iu$ in $\phi^*$. Then we again get $|B(\phi^*)| \leq |B(\phi)| - 1$, which contradicts the choice of $\phi$.

**Case 2.** $\mathcal{F}(v_i, 0) \setminus \{F_\xi\} \neq \emptyset$.

Let $F_\eta \in \mathcal{F}(v_i, 0) \setminus \{F_\xi\}$. We choose a vertex $u \in R^*(v_i)$ such that $\phi(v_iu) = \eta$ as follows. If there exists a path from $v_i$ to $w$ in $F'_\eta$, then as $F_\eta \in \mathcal{F}(v_i, 0)$, the vertex adjacent to $v_i$ on this path belongs to $R^*(v_i)$ and we choose that vertex as $u$. If there is no path from $v_i$ to $w$ in $F'_\eta$, then we choose $u$ to be any vertex in $R^*(v_i)$ such that $\phi(v_iu) = \eta$. Note that by (1) and $|R^*(v_i)| = r$, all colors in $C_0(v_i)$ have been used under the coloring $\phi$. Such a vertex $u$ always exists as $\eta \in C_0(v_i)$. We now obtain a new coloring $\phi^*$ from $\phi$ by swapping colors on $v_iw$ and $v_iu$, i.e., $\phi^*(v_iw) = \eta$ and $\phi^*(v_iu) = \xi$. We denote by $\mathcal{F}^* = F_1^* \,|F_2^*| \cdots \mid F_t^*$ the partition given by the coloring $\phi^*$. As we have just exchanged the colors on two edges in $E(v_i, R^*(v_i))$, the coloring $\phi^*$ remains to be a $v_i$-saturated $\phi_0$-extension.

Suppose that there exists a monochromatic cycle containing the edge $v_iw$ in $\phi^*$. Then, there is a path $P'$ from $v_i$ to $w$ in $F_\eta^* - v_iw$. As $\phi^*(v_iu) \neq \eta$, we know that the edge $v_iu$ does not belong to $P'$. Thus, $P'$ is also a path in $F'_\eta$ from $v_i$ to $w$. Note there exists at most one such path $P'$ since $d_{F'_\eta}(w) = d_{F'_\eta}(v_i) = 1$ and $d_{F'_\eta}(v) \leq 2$ for all $v \in V(G)$. However, by our choice of $u$, if such $P'$ exists, the edge $v_iu$ belongs to $P'$, which is a contradiction. We can therefore conclude that there is no monochromatic cycle containing the edge $v_iw$ in $\phi^*$.

Next, suppose that there is a monochromatic cycle containing the edge $v_iu$ in $\phi^*$. Then there is a path $P''$ from $v_i$ to $u$ in $F_\xi^* - v_iu$. Clearly, this path does not contain $v_iw$, and therefore $P''$ is also a path from $v_i$ to $u$ in $F'_\xi - v_iw$. Then, as before, $P''$ is a subpath of $P$. Note that $u \neq w'$ since $\phi(v_iw') = \xi \neq \eta = \phi(v_iu)$. However, this implies that $P$ contains the vertex $u \in R^*(v_i)$ that is different from $w'$ and $w$, which is a contradiction. Therefore, there is no monochromatic cycle containing the edge $v_iu$ in $\phi^*$. We then have $|B(\phi^*)| \leq |B(\phi)| - 1$, contradicting the choice of $\phi$.

**Case 3.** $\mathcal{F}(w,0) \subseteq \mathcal{F}(v_i,2)$ and $\mathcal{F}(v_i,0) \setminus \{F_\xi\} = \emptyset$.

We claim $k = 2$ in this case. Since $\mathcal{F}(w,0) \subseteq \mathcal{F}(v_i,2)$, it follows that $\mathcal{F}(v_i,0) \cup \mathcal{F}(v_i,1) \subseteq \mathcal{F}(w,1)$, which gives $|\mathcal{F}(v_i,0)| + |\mathcal{F}(v_i,1)| \leq |\mathcal{F}(w,1)| \leq d_H(w) \leq k-1$. Then, $2|\mathcal{F}(v_i,0)| + 2|\mathcal{F}(v_i,1)| \leq 2k-2 \leq r$. From (1), since $|C_p(v_i)| = |\mathcal{F}(v_i,p)|$ for $p = 0,1,2$ (note that $C_p(v_i)$ is the index set of $\mathcal{F}(v_i,p)$), we have $2|\mathcal{F}(v_i,0)| + |\mathcal{F}(v_i,1)| \geq r$. Combining these two inequalities, we have $|\mathcal{F}(v_i,1)| = 0$, and $2|\mathcal{F}(v_i,0)| = r \geq 2k-2$, which implies $|\mathcal{F}(v_i,0)| \geq k-1$. Since we also have $|\mathcal{F}(v_i,0)| \leq |\mathcal{F}(w,1)| \leq k-1$, we now have $|\mathcal{F}(v_i,0)| = |\mathcal{F}(w,1)| = k-1$ and $r = 2k-2$. As $\mathcal{F}(v_i,0) \setminus \{F_\xi\} = \emptyset$, we have $|\mathcal{F}(v_i,0)| \leq 1$. Therefore, we get $k-1 \leq 1$, or in other words, $k \leq 2$. Since we have assumed $k \geq 2$, we now have $k = 2$.

The fact that $k = 2$ further implies $|\mathcal{F}(v_i,0)| = |\mathcal{F}(w,1)| = k-1 = 1$ and $r = 2k-2 = 2$. We then have $\mathcal{F}(v_i,0) = \{\xi\}$, which then gives $R^*(v_i) = \{w, w'\}$. Note that the degree of every vertex in $N_R(v_i)$ is at most $k$. As $\Delta \geq 2k^2 - k \geq 6$, we know that $|N_R(v_i)| \geq \Delta - k \geq 4$, which implies that $|W| \geq 2$. Let $u$ be any vertex in $W$ and let $\eta = \phi(v_i u)$. Clearly, $\eta \neq \xi$. Let $\phi^*$ be the coloring obtained from $\phi$ by swapping colors on $v_i w$ and $v_i u$, i.e., $\phi^*(v_i w) = \eta$ and $\phi^*(v_i u) = \xi$. We denote by $\mathcal{F}^* = F_1^* | F_2^* | \cdots | F_t^*$ the partition given by the coloring $\phi^*$. Since we changed the coloring of $H$, we let $\phi_0'$ be the new coloring of $H$ in which $\phi_0'(v_i u) = \xi$ and $\phi_0'(v_i v) = \phi_0(v_i v)$ for the neighbors $v \neq u$. As we have just exchanged the colors of two edges incident on $v_i$, we have $d_{F_j^*}(v_i) \leq 2$ for each $j \in \{1, 2, \ldots, t\}$. Therefore, $\phi^*$ is $v_i$-saturated.

There is no monochromatic cycle containing the edge $v_i w$ in $\phi^*$ as the only edge incident on $w$ other than $v_i w$ is colored $\xi$ in $\phi$ and therefore also in $\phi^*$. Suppose that there is a monochromatic cycle containing the edge $v_i u$ in $\phi^*$. Then as before, there is a path $P'$ from $v_i$ to $u$ in $F_\xi^* - v_i u$. Note that $v_i w$ is not in $P'$ as $\phi^*(v_i w) \neq \xi$. Then $P'$ is also a path from $v_i$ to $u$ in $F_\xi' - v_i w$. Since $v_i$ belongs to $P'$, we can then conclude that $P'$ is a subpath of $P$. Since the degree of $u$ is at most 2 , and one edge incident on it is colored $\eta$ in $\phi$, we can conclude that $u$ is an end-vertex of $P$, which contradicts the fact that the end-vertices of $P$ are $v_i$ and $w$. Therefore, there is no monochromatic cycle containing the edge $v_i u$ in $\phi^*$. Then we have $|B(\phi^*)| \leq |B(\phi)| - 1$, contradicting our choice of $\phi$. This completes our proof. □

# 4 Remark

In addition, along a similar approach as the proof of Theorem 1.3, the original upper bound stated in the LAC holds for a slightly smaller maximum degree than in Theorem 1.3. We state it as the following theorem.

**Theorem 4.1.** *Let $G$ be a $k$-degenerate graph. If $\Delta(G) \geq 2k^2 - 2k$, then* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Similarly, now we let $t = \lceil \Delta + 1/2 \rceil$, and state the following equivalent theorem in terms of $(\Delta, 1)$-regular $k$-degenerate graphs.

**Theorem 4.1*.** *Let $G$ be a $(\Delta, 1)$-regular $k$-degenerate graph. If $\Delta \geq 2k^2 - 2k$, then there exists a linear forest partition $\mathcal{F} = F_1 \mid F_2 \mid \cdots \mid F_t$ with $t = \left\lceil \frac{\Delta+1}{2} \right\rceil$.*

*Proof.* There are some slight differences from the proof of Section 3 in terms of the computation related to $t$. Suppose $\Delta \geq 2k^2 - 2k$. Then the size of each representative of the family $\{N_R(v_i) : v_i \in V_\Delta\}$ satisfies $r \geq 2k-3$. It is readily seen that all conditions related to $\Delta, r$ and $t$ still hold.

Therefore, the same construction works. We can find all desired linear forest partitions $\mathcal{F}^i = F_1^i \mid F_2^i \mid \cdots \mid F_t^i$ for any $G_i$, $i = 1, \ldots, n$ such that $\mathcal{F}^i(v, 2) = \emptyset$ for all $v \in \cup_{j>i} R^*(v_j)$. □

We strongly believe the quadratic lower bound $2k^2 - k$ for the maximum degree in Theorem 1.3 may be reduced to $2k$ since it is only required in our proof of Lemma 2.2. We wonder if there is a way to reduce it to a linear bound in terms of $k$.

## 5 Acknowlegement

Thanks to Mathew Francis for his detailed review report. His report not only corrected typos and suggested on grammars, but also helped us a lot on making our paper more readable and understandable.

## References

[1] Jin Akiyama, Geoffrey Exoo, and Frank Harary. Covering and packing in graphs. III: Cyclic and acyclic invariants. *Mathematica Slovaca*, 30(4):405–417, 1980.

[2] Jin Akiyama, Geoffrey Exoo, and Frank Harary. Covering and packing in graphs iv: Linear arboricity. *Networks*, 11(1):69–72, 1981.

[3] Noga Alon. The linear arboricity of graphs. *Israel Journal of Mathematics*, 62(3):311–325, 1988.

[4] Manu Basavaraju, Arijit Bishnu, Mathew Francis, and Drimit Pattanayak. The linear arboricity conjecture for 3-degenerate graphs, 2020.

[5] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. List edge and list total colourings of multigraphs. *Journal of Combinatorial Theory, Series B*, 71(2):184–204, 1997.

[6] Marek Cygan, Jian-Feng Hou, Łukasz Kowalik, Borut Lužar, and Jian-Liang Wu. A planar linear arboricity conjecture. *Journal of Graph Theory*, 69(4):403–425, 2012.

[7] Reinhard Diestel. *Matching Covering and Packing*, pages 35–58. Springer Berlin Heidelberg, Berlin, Heidelberg, 2017.

[8] Hikoe Enomoto and Bernard Péroche. The linear arboricity of some regular graphs. *J. Graph Theory*, 8(2):309–324, 1984.

[9] Asaf Ferber, Jacob Fox, and Vishesh Jain. Towards the linear arboricity conjecture. *Journal of Combinatorial Theory. Series B*, 142:56–79, 2020.

[10] Stefan Glock, Daniela Kühn, and Deryk Osthus. Optimal path and cycle decompositions of dense quasirandom graphs. *J. Combin. Theory Ser. B*, 118:88–108, 2016.

[11] Filip Guldan. The linear arboricity of 10-regular graphs. *Math. Slovaca*, 36(3):225–228, 1986.

[12] Shuji Isobe, Xiao Zhou, and Takao Nishizeki. Total colorings of degenerate graphs. *Combinatorica*, 27(2):167–182, 2007.

[13] Colin McDiarmid and Bruce Reed. Linear arboricity of random regular graphs. *Random Structures Algorithms*, 1(4):443–445, 1990.

[14] Joel H. Spencer Noga Alon. *The probabilistic method*. Wiley-Interscience series in discrete mathematics and optimization. Wiley, 3rd ed edition, 2008.

[15] Douglas Brent West. *Introduction to graph theory*. Prentice Hall, Upper Saddle River, N.J., 2nd edition, 2001.

[16] Jian-Liang Wu. On the linear arboricity of planar graphs. *Journal of Graph Theory*, 31(2):129–134, 1999.

[17] Jian-Liang Wu and Yu-Wen Wu. The linear arboricity of planar graphs of maximum degree seven is four. *Journal of Graph Theory*, 58(3):210–220, 2008.

[18] Xiao Zhou, Shin-ichi Nakano, and Takao Nishizeki. Edge-coloring partialk-trees. *Journal of Algorithms*, 21(3):598–617, 1996.