A Look at Saturated Graphs

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Definition

Given a graph $H$, a graph $G$ of order $n$ is said to be $H$-saturated provided $G$ contains no copy of $H$, but the addition of any edge from the complement of $G$ creates a copy of $H$. That is, given $e \in \overline{G}$, then $G + e$ contains a copy of $H$. 
**Definition**

The maximum number of edges in an $H$-saturated graph of order $n$ is called the **extremal number (or the Turan number)** for $H$, and is denoted as $ex(n, H)$.

**Definition**

The minimum size of an $H$-saturated graph of order $n$ is called the **saturation number** and is denoted as $sat(n, H)$. 
1. Given a graph $G$, what is $\text{ex}(n, G)$?

2. Given a graph $G$, what is $\text{sat}(n, G)$?

3. What sizes of $G$-saturated graphs are possible between the saturation number and extremal number? This set of values is called the **saturation spectrum of $G$**.

4. When are all values possible?

5. What general theory can we build for $\text{ex}(n, G)$ or $\text{sat}(n, G)$?
Turan (1941) provided the extremal number for complete graphs:

**Theorem**

Among graphs of order $n$ which do not contain $K_t$, there exists exactly one with the maximum number of edges, the complete, balanced, $(t - 1)$-partite graph.

For triangles, this is the balanced complete bipartite graph, thus $\text{ext}(n, K_3) = \lceil n/2 \rceil \lceil n/2 \rceil$. (Determined by W. Mantel in 1906.)
The Erdős-Stone Theorem, 1946

Established the magnitude of the extremal number for all graphs with chromatic number at least 3.

**Theorem**

$$\lim_{n \to \infty} \frac{\text{ex}(n, G)}{n^2} = \frac{1}{2} \left(1 - \frac{1}{\chi(G)-1}\right).$$
In 1964 Erdős, Hajnal and Moon determined that:

**Theorem**

\[
sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}.
\]

This arises from the graph \( K_{t-2} + \overline{K}_{n-t+2} \), where + denotes join.
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This means that for a triangle, \( sat(n, K_3) = n - 1 \).
Let $F$ be a family of graphs. Then $\text{ex}(n, F)$ satisfies:

1. $\text{ex}(n, F) \leq \text{ex}(n + 1, F)$.
2. If $F_1 \subset F$ then $\text{ex}(n, F_1) \leq \text{ex}(n, F)$.
3. If $H \subseteq G$, then $\text{ex}(n, H) \leq \text{ex}(n, G)$. 
However, these rules do not hold in general for saturation numbers. Example of 3. Consider $K_4$ and a supergraph $H$ obtained by attaching an additional edge to $K_4$. We know that $sat(n, K_4) = 2n - 3$. But for $H$ we have:

$$sat(n, H) \leq \frac{3n}{2}.$$
Thus we have seen that the extremal number for triangles is $O(n^2)$ while the saturation number is $O(n)$.

This is no accident!

Kászonyi and Tuza provided a very general upper bound on saturation numbers and used it to show the following:
Theorem

For every graph $F$ there exists a constant $c$ such that

$$\text{sat}(n, F) < cn.$$
In 1995 Barefoot, Casey, Fisher, Fraughnaugh and Harary, showed the following:

**Theorem**

For \( n \geq 5 \), there exists a \( K_3 \)-saturated graph of order \( n \) with \( m \) edges if and only if it is complete bipartite or

\[
2n - 5 \leq m \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.
\]

Note: A gap at the bottom, between \( n - 1 \) and \( 2n - 5 \). This is a result of a combination of connectivity and the fact that triangle saturated graphs have diameter two. It is then easy to show the gap at the bottom exists. At the top, extremal theory and convexity suffice.
This is a result of a combination of connectivity and the fact that triangle saturated graphs have diameter two. At the top, extremal theory and convexity suffice.

**Theorem**

(Barefoot, et al) Every 2-connected graph of order $n$ and diameter two has at least $2n - 5$ edges.

**Note**

This result works well for triangles, $K_4 - e$ and other graphs where the saturation graph must have diameter 2.
With **K. Amin, J. Faudree and E. Sidorowicz** (2013) we were able to generalize this result for all $t \geq 3$.

**Theorem**

For $n \geq 3t + 4$ and $t \geq 3$, there is a $K_t$-saturated graph $G$ of order $n$ with $m$ edges if, and only if, $G$ is complete $(t - 1)$-partite or

$$(t - 1)(n - t/2) - 2 \leq m \leq \left\lfloor \frac{(t-2)n^2-2n+(t-2)}{2(t-1)} \right\rfloor + 1.$$ 

Note, same sort of gaps exist. Also note this reduces to the Barefoot et al result when $t = 3$. 
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K₃

K₄

isolated values from complete bipartite graphs

no values
Kászonyi and Tuza, 1986:

**Theorem**

1. For $n \geq 3$, $\text{sat}(n, P_3) = \lfloor n/2 \rfloor$.

2. For $n \geq 4$,

$$\text{sat}(n, P_4) = \begin{cases} n/2 & \text{if } n \text{ even} \\ (n+3)/2 & \text{if } n \text{ odd.} \end{cases}$$

3. For $n \geq 5$, $\text{sat}(n, P_5) = \lceil \frac{5n-4}{6} \rceil$.

4. Let

$$a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if } k = 2t \\ 4 \cdot 2^{t-1} - 2 & \text{if } k = 2t + 1. \end{cases}$$

then if $n \geq a_k$ and $k \geq 6$, $\text{sat}(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$. 
For all $n \geq 3$,

1. $\text{ext}(n, P_4) = \begin{cases} n & \text{if } n \equiv 0 \pmod{3} \\ n - 1 & \text{if } n \equiv 1, 2 \pmod{3} \end{cases}$

2. $\text{ext}(n, P_5) = \begin{cases} 3n/2 & \text{if } n \equiv 0 \pmod{4} \\ 3n/2 - 2 & \text{if } n \equiv 2 \pmod{4} \\ 3(n - 1)/2 & \text{if } n \equiv 1, 3 \pmod{4} \end{cases}$

3. $\text{ext}(n, P_6) = \begin{cases} 2n & \text{if } n \equiv 1 \pmod{5} \\ 2n - 2 & \text{if } n \equiv 1, 4 \pmod{5} \\ 2n - 3 & \text{if } n \equiv 2, 3 \pmod{5} \end{cases}$
Work with W. Tang, E. Wei, C.Q. Zhang (2012). If we consider $P_3$ it is simple to see that:

**Theorem**

$$sat(n, P_3) = ex(n, P_3) = \lfloor n/2 \rfloor.$$

There is a simple procedure for evolving a $P_4$-saturated graph from the saturation number to the extremal number, one edge at a time. Thus, the spectrum for $P_4$ is complete.
$P_4$ has a continuous saturation spectrum

$m = k$

$n = 2k$

$m = k + 1$

etc.

**Theorem**

If $t \geq 3$ and $n \geq t + 1$, then the spectrum of the star $K_{1,t}$ is continuous from $\text{sat}(n, K_{1,t})$ to $\text{ex}(n, K_{1,t})$. 
Kászonyi and Tuza:

**Theorem**

\[
\text{sat}(n, K_{1,t}) = \begin{cases} 
\left(\frac{t}{2}\right) + \binom{n-t}{2} & \text{if } t + 1 \leq n \leq t + t/2 \\
\left\lceil \frac{t-1}{2} n \right\rceil - t^2/8 & \text{if } t + 1/2 \leq n.
\end{cases}
\]

Folklore??? Obvious!

**Theorem**

\[
\text{ex}(n, K_{1,t}) = \left\lfloor \frac{t-1}{2} n \right\rfloor. \text{ That is, a graph that is } t - 1\text{-regular or nearly regular.}
\]
Here things get a little bit more complicated.

**Theorem**

Let $n \geq 5$ and $\text{sat}(n, P_5) \leq m \leq \text{ext}(n, P_5)$ be integers. Then there exists an $(n, m)$ $P_5$-saturated graph if and only if $n \equiv 1, 2 (mod 4)$, or

$$m \neq \begin{cases} 
\frac{3n-5}{2} & \text{if } n \equiv 3 (mod 4) \\
\frac{3n}{2} - j, j = 1, 2, \text{ or } 3 & \text{if } n \equiv 0 (mod 4).
\end{cases}$$
For $P_6$ there are 5 cases depending on the value of $n \text{ (mod 5)}$.

There are gaps in the spectrum for each case. We will see more on this later.
The extremal number for $K_4 - e$ is achieved by the complete bipartite graph.

$$ sat(n, K_4 - e) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor, $$ and is achieved by:

(a) [Diagram of a complete bipartite graph]

(b) [Diagram of another graph]
With Jessica Fuller we showed:

**Theorem**

If $G$ is a $K_4 - e$ saturated graph on $n$ vertices, then either $G$ is a complete bipartite graph, a 3-partite graph (like the saturation graph of the previous frame), or has size in the interval

$$[2n - 4, \left\lceil \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rfloor - n + 6]$$

Here the gap between the saturation number and $2n-4$ happens for reasons similar to that for triangles.
A look at the proof

Case: Suppose $4n - 18 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 5$.
Here $|A| = n - |B| - |C| - 5$, $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$.

Then $m = (n - c)(c + 2) - 5c + b - 4$. So as $b$ increases by 1, with $c$ fixed, then $m$ increases by 1.

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Case: Suppose $4n - 18 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 5$.
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Case: Suppose $4n - 18 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 5$.
Here $|A| = n - |B| - |C| - 5$, $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$.

Then $m = (n - c)(c + 2) - 5c + b - 4$. So as $b$ increases by 1, with $c$ fixed, then $m$ increases by 1.
It is straightforward to extend the $K_4 - e$ interval to larger cliques:

**Theorem**

There are $K_t - e$ saturated graphs in the interval

\[
(t - 2)n - \left(\frac{t-1}{2}\right) - 1, \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t - 3)n - \left(\frac{t-2}{2}\right) - 1.
\]

Also, there are $(K_t - e)$-saturated graphs for sporadic values of $m$ in

\[
\left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t - 3)n - \left(\frac{t-2}{2}\right) + 4, \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil \right.
\]
\[
+ (t - 2)n - \left(\frac{t-1}{2}\right) - 1.
\]
Definition

The fan $F_t$ is the graph consisting of $t$ edge-disjoint triangles that intersect at a single vertex $v$.

Note: The fan is sometimes called a friendship graph.
With Erdős, Furedi and Gunderson (1995) we determined the extremal number for fans $F_t$.

**Theorem**

For every $t \geq 1$, and for every $n \geq 50t^2$, if a graph $G$ on $n$ vertices has more than

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even} \end{cases}$$

edges, then $G$ contains a copy of the $t$-fan, $F_t$. Furthermore, the number of edges is best possible.
With J. Fuller we showed the following:

**Theorem**

For $t \geq 2$, and $n \geq 3t - 1$, $sat(n, F_t) = n + 3t - 4$.

**Theorem**

There exists an $F_2$-saturated graph $G$ on $n \geq 7$ vertices and $m$ edges where $m = n + 2$, or $2n - 4 \leq m \leq \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + 2$, or $m$ is the size of a complete bipartite graph with one additional edge.
We also showed:

**Theorem**

There exists an $F_3$-saturated graph $G$ of order $n$ with $m$ edges for $m = n + 5$, or $2n + 2 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + 5$, or $m$ equals the size of a complete bipartite graph with six added edges. Further, the only unknown possible sizes are from $2n - 5$ to $2n + 1$. 
$F_4$ gets even more interesting

**Theorem**

There exists an $F_4$-saturated graph $G$ on $n$ vertices and $m$ edges for $m = n + 8$, or $3n + 2 \leq m \leq \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 10$ except for $4n - 8$, or $G$ is a complete bipartite graph with the proper 10 additional edges.
First $tK_p$.

**Theorem**

Let $t \geq 1$, $p \geq 3$ and $n \geq p(p + 1)t - p^2 + 2p - 6$ be integers. Then

$$sat(tK_p, n) = (t - 1) \binom{p + 1}{2} + \binom{p - 2}{2} + (p - 2)(n - p + 2).$$
Theorem

Let $2 \leq p \leq q$ and $n \geq q(q + 1) + 3(p - 2)$ be integers. Then

$$sat(K_p \cup K_q, n) = (p - 2)(n - p + 2) + \binom{p - 2}{2} + \binom{q + 1}{2}.$$
Proof Idea

$$(t - 1)K_{p+1} \quad \bar{K}_{n-p+1-t+3}$$

$K_{p-2}$
Proof Idea

\[(t - 1)K_{p+1}\]

\[\overline{K}_{n-pt-t+3}\]

\[K_{p-2}\]
Definition

Let the graph comprised of $t$ copies of $K_p$ intersecting in a common $K_\ell$ be called a generalized fan and be denoted $F_{p,\ell}$. 
Theorem

Let \( p \geq 3, t \geq 2 \) and \( p - 2 \geq \ell \geq 1 \) be integers. Then, for sufficiently large \( n \),

\[
sat(F_{p,\ell}, n) = (p - 2)(n - p + 2) + \binom{p - 2}{2} + (t - 1)\binom{p - \ell + 1}{2}.
\]
Definition

A graph on \((r - 1)k + 1\) vertices consisting of \(k\) cliques each with \(r\) vertices, which intersect in exactly one common vertex, is called a \((k, r) - \text{fan}\).

Theorem

For every \(k \geq 1\), and for every \(n \geq 16k^3r^8\), if a graph \(G\) on \(n\) vertices has more than

\[
\text{ex}(n, K_r) + \begin{cases} 
  k^2 - k & \text{if } k \text{ is odd} \\
  k^2 - \frac{3}{2}k & \text{if } t \text{ is even}
\end{cases}
\]

edges, then \(G\) contains a copy of the \((k, r)\)-fan. Furthermore, the number of edges is best possible.
To see the last result is best possible consider:

For odd $k$ take the Turan graph and embed two vertex disjoint copies of $K_k$ in one partite set.

For even $k$ take the Turan graph and embed a graph with $2k - 1$ vertices and $k^2 - (3/2)k$ edges with max degree $k - 1$ in one partite set.
Definition

A tree $T$ of order $\ell$, $T \neq K_{1,\ell-1}$, having a vertex that is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor$ leaves is called a scrub-grass tree.

Theorem

Let $T$ be a path or scrub-grass tree on $\ell \geq 6$ vertices and $n = |G| \equiv 0 \ mod(\ell - 1)$ and $m$ be an integer such that $1 \leq m \leq \lfloor \frac{\ell-2}{2} \rfloor - 1$. There is no graph of size $\frac{n}{\ell-1} \left( \binom{\ell-1}{2} - m \right)$ in the spectrum of $T$. Hence, there is a gap in the spectrum.
Definition

A graph $F$ is weakly $G$-saturated if $F$ does not contain a copy of $G$, but there is an ordering of the missing edges of $G$ so that if they are added one at a time, each edge creates a new copy of $F$. The minimum size of a weakly $F$-saturated graph $G$ of order $n$ is denoted $wsat(N, F)$. 
Question:

For which graphs $G$ is $\text{sat}(n, G) = \text{wsat}(n, G)$?

Bollobás (1967) showed the following:

Theorem

$$\text{wsat}(n, K_p) = \text{sat}(n, K_p).$$
If $F$ is a graph of order $p$ and size $q$:

**Theorem**

$$\frac{\delta n}{2} - \frac{n}{\delta + 1} \leq wsat(n, F) \leq (\delta - 1)n + (p - 1)\frac{p - 2\delta}{2}.$$
wsat for Trees

We further showed that for any tree $T_p$ on $p$ vertices:

**Theorem**

$$p - 2 \leq \text{wsat}(n, T_p) \leq \binom{p - 1}{2}.$$
For a labeled tree $T_p$

**Theorem**

$$\lim_{n \to \infty} P(\text{wsat}(n, T_p)) = p - 2 \to 1.$$ 

Proof uses Cayley tree formula.
with R. Faudree (2014):

**Theorem**

\[ \text{wsat}(n, kK_t) = (t - 2)n + k - \frac{(t^2 - 3t + 4)}{2}. \]
Theorem

\[ wsat(n, kC_t) = \begin{cases} 
  n + k - 2 & \text{if } t \text{ is odd} \\
  n + k - 1 & \text{if } t \text{ is even}. 
\end{cases} \]