ON THE COEQUAL VALUES OF TOTAL CHROMATIC NUMBER AND CHROMATIC INDEX

GUANTAO CHEN AND YANLI HAO

Abstract. The chromatic index $\chi'(G)$ of a graph $G$ is the least number of colors assigned to the edges of $G$ such that no two adjacent edges receive the same color. The total chromatic number $\chi''(G)$ of a graph $G$ is the least number of colors assigned to the edges and vertices such that no two adjacent edges receive the same color, no two adjacent vertices receive the same color and no edge has the same color as its two endpoints. The chromatic index and the total coloring number are two of few fundamental graph parameters, and their correlation has always been a subject of intensive study in graph theory.

By definition, $\chi'(G) \leq \chi''(G)$ for every graph $G$. In 1984, Goldberg conjectured that for any multigraph $G$, if $\chi'(G) \geq \Delta(G) + 3$ then $\chi''(G) = \chi'(G)$. In this paper, we show that Goldberg’s conjecture is asymptotically true. More specifically, we prove that for a multigraph $G$ with maximum degree $\Delta$ sufficiently large, $\chi''(G) = \chi'(G)$ provided $\chi'(G) \geq \Delta + 10\Delta^{33/36}$. When $\chi'(G) \geq \Delta(G) + 2$, the chromatic index $\chi'(G)$ is completely determined by the fractional chromatic index. Consequently, the total chromatic number $\chi''(G)$ can be computed in polynomial-time in this case.

Keywords: total chromatic number; chromatic index, maximum degree

1. Introduction

By a (multi)graph $G$, we mean a finite undirected graph without loops, but possibly with multiple edges. A total-coloring of a graph $G$ is an assignment of colors to the edges and vertices of $G$ such that no two adjacent edges receive the same color, no two adjacent vertices receive the same color and no edge has the same color as its two endpoints. The total chromatic number, denoted by $\chi''(G)$, is the least number of colors required for a total-coloring of $G$. Similarly, vertex-coloring and chromatic number $\chi(G)$, and edge-coloring and chromatic index $\chi'(G)$ of a graph $G$ are defined, respectively. Compared with vertex coloring and edge coloring, the theory of total coloring is less
studies with fewer results. This may be due to the fact that finding a total-coloring is much more difficult than finding a vertex-coloring or an edge-coloring.

Let \( \Delta(G) \) and \( \mu(G) \) denote the maximum degree and the maximum number of edges between any two distinct vertices of \( G \), respectively. Behzad [2] in 1965 conjectured that \( \chi''(G) \leq \Delta(G) + 2 \) if \( G \) is a simple graph. The best known upper bound for total chromatic number is due to Molloy and Reed [9] who showed that there is a universal constant \( c \) such that \( \chi''(G) \leq \Delta(G) + c \) for every simple graph \( G \). They provided a proof for \( c = 10^{26} \), so long as \( \Delta \geq \Delta_0 \) for a constant \( \Delta_0 \).

It is a common belief that the total coloring is strongly related to the edge coloring, which is the subject of this paper. Vizing [14], and independently, Gupta [7] proved that \( \chi'(G) \leq \Delta(G) + \mu(G) \). Vizing [15] in 1968 proposed that \( \chi''(G) \leq \Delta(G) + \mu(G) + 1 \). A slightly stronger version is that \( \chi''(G) \leq \chi'(G) + 1 \). (See Stiebitz et al. [13], page 262.) Goldberg [6] in 1984 conjectured that \( \chi''(G) = \chi'(G) \) provided \( \chi'(G) \geq \Delta(G) + 3 \). We are not aware of any nontrivial sufficient conditions such that the total chromatic number and the chromatic index are the same besides the following one due to Cao, Chen and Jing [3]: if \( \chi'(G) \geq \max\{\Delta(G) + 2, |V(G)| + 1\} \) then \( \chi''(G) = \chi'(G) \). The condition \( |V(G)| < \chi'(G) \) may be too strong. In this paper, we confirm Goldberg’s conjecture asymptotically as follows.

**Theorem 1.1.** Let \( G \) be a graph with maximum degree \( \Delta \) sufficiently large. If \( \chi'(G) \geq \Delta + 10 \Delta^{35/36} \), then \( \chi''(G) = \chi'(G) \).

For the condition “\( \Delta \) sufficiently large”, we actually only need \( e^{\Delta^{1/18}} > 4 \Delta^3 \) and \( \Delta \geq 1 \) from our proof. In the study of vertex-coloring and edge-coloring, there is an essential and powerful re-coloring tool – Kempe change (vertex version and edge version). However, this tool has yet been developed for total-coloring. By applying the probabilistic method (Chernoff bounds (various forms) and the Local Lemma), some upper bounds of \( \chi'' \) are obtained. (See Molloy and Reed [10].) In our proof, we develop a technique to give a total-coloring of “dense graphs” and extend this coloring to the whole graph. We then combine this technique with the probabilistic method to carry out the proof. This hybrid approach may shed some light on tackling other total coloring problems.

Apart from the maximum degree \( \Delta(G) \), there is another trivial lower bound for \( \chi'(G) \), called the density of \( G \), defined by

\[
\omega(G) = \max \{ \frac{2|E(H)|}{|H| - 1} : H \subseteq G, |H| \geq 3 \text{ odd} \},
\]

where \( |H| = |V(H)| \) is the order of \( H \). The following result, conjectured independently by Goldberg [5] and Seymour [11] in the 1970s and recently
confirmed by Chen, Jing and Zang [4], shows that the density represents the chromatic index in some common cases.

**Theorem 1.2.** Let $G$ be a graph. If $\chi'(G) \geq \Delta(G) + 2$, then $\chi'(G) = \lceil \omega(G) \rceil$.

Denote by $\chi_f(G)$ the fractional chromatic index of a graph $G$. Seymour [11] showed that $\chi_f(G) = \max\{\Delta(G), \omega(G)\}$. Since the fractional chromatic index can be computed in polynomial-time, $\chi'(G)$ can be determined in polynomial-time provided $\chi'(G) \geq \Delta(G) + 2$. Hence, Theorem 1.1 implies that for graphs $G$ with maximum degree $\Delta$ sufficiently large and $\chi'(G) \geq \Delta + 10\Delta^{35/36}$, the total chromatic number can be computed in polynomial-time.

The rest of this paper is organized as follows. In Section 2, we will give some notation and terminology, and state some technical results; In Section 3, we prove Theorem 1.1 based on the results in Section 2; And Sections 4 and 5 give the proofs of these preliminary results stated in Section 2.

## 2. Preliminaries

We generally follow Stiebitz et al. [13] for notation and terminology. For a vertex $v \in V(G)$, denote $N_G(v)$ and $E_G(v)$ the set of vertices adjacent to $v$ and the set of edges incident to $v$ in $G$, respectively. Clearly, $d_G(v) = |E_G(v)|$ is the degree of $v$ in $G$. For vertex sets $U, W \subseteq V(G)$, let $E_G(U, W)$ denote the set of all edges of $G$ joining a vertex of $U$ with a vertex of $W$. When $W = V(G) \setminus U$, we call $E_G(U, W)$ the boundary of $U$ in $G$ and denote by $\partial_G(U)$, that is, $\partial_G(U) = E_G(U, V(G) \setminus U)$. Let $d_G(U) = |\partial_G(U)|$. Let $E_G(u, w)$ and $E_G(u, W)$ denote $E_G(\{u\}, \{w\})$ and $E_G(\{u\}, W)$, respectively. For any two disjoint subgraphs $F$ and $H$ of $G$, we define $E_G(F, H) = E_G(V(F), V(H))$, $\partial_G(F) = \partial_G(V(F))$ and $d_G(F) = d_G(V(F))$. Given a graph property $P$, we say a subgraph $H \subseteq G$ is a maximal subgraph with property $P$ if $H$ is maximal among all subgraphs satisfying property $P$.

For a total-coloring $\varphi$ of a graph $G$, a vertex set $W \subseteq V(G)$ is called $\varphi$-distinct if the colors assigned to the vertices in $W$ and the edges incident to $W$ are mutually distinct.

**Theorem 2.1.** Let $G$ be a graph with maximum degree $\Delta$ sufficiently large, and let $V_1, \ldots, V_m$ be $m$ disjoint independent vertex sets of $G$. Suppose that $G_0$ is obtained from $G$ by contracting each $V_i$ to a single vertex. If $\chi'(G_0) \leq \Delta + |2\Delta^{35/36}|$ and $|V_i| \leq \Delta^{1/36}$ for $i \in \{1, \ldots, m\}$, then $G$ has a total-coloring $\varphi$ using at most $\Delta + |10\Delta^{35/36}|$ colors such that $V_i$ is $\varphi$-distinct for $i \in \{1, \ldots, m\}$.
We will use the following three theorems to prove Theorem 2.1 in Section 4. The first one, due to Shannon in 1949 [12], gives an upper bound of chromatic index that only involving in the maximum degree.

**Theorem 2.2** (Shannon’s Theorem). If $G$ is a graph with maximum degree $\Delta$, then $\chi'(G) \leq \lfloor 3\Delta/2 \rfloor$.

The second one, due to Hoeffding in 1963 [8], generalizes Chernoff bounds (the upper tail part).

**Theorem 2.3** (Hoeffding’s Inequality). Let $X_1, \ldots, X_n$ be random variables and $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. If $a \leq X_i \leq b$ for all $i \in \{1, \ldots, n\}$, then $\mathbb{P}[X \geq \mu + \gamma] \leq e^{-\frac{2\gamma^2}{n(b-a)^2}}$ for all $\gamma > 0$.

The third one is the symmetric case of the Lovász Local Lemma. (See Alon and Spencer [4], Corollary 5.1.2.)

**Theorem 2.4** (the Local Lemma, Symmetric Case). Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent of all other events $A_j$ but at most $d$, and $\mathbb{P}[A_i] \leq p$ for $i \in \{1, \ldots, n\}$. If $4pd \leq 1$, then $\mathbb{P}[\text{\bigcap}_{i=1}^{n} \overline{A_i}] > 0$.

A subgraph $H$ of a graph $G$ is an induced subgraph if $H$ contains all edges of $G$ that have both endpoints in $V(H)$. An induced subgraph $H$ of graph $G$ with $|H| \geq 3$ odd is said to be $k$-dense if $2|E(H)| > (k-1)(|H| - 1)$, $k$-near-dense if $H$ is $k$-dense and $\chi'(H) = k$, and $k$-exact-dense if $2|E(H)| = k(|H| - 1)$ and $\chi'(H) = k$.

**Lemma 2.5.** If $G$ is a graph with $\chi'(G) \geq \Delta(G) + 2$, then $G$ contains a $\chi'(G)$-near-dense subgraph.

**Proof.** Since $\chi'(G) \geq \Delta(G) + 2$, by Theorem 2.1 there exists a subgraph $H \subseteq G$ with $|H| \geq 3$ odd such that $\chi'(G) = \lceil \frac{2|E(H)|}{|H|-1} \rceil$. Then $\frac{2|E(H)|}{|H|-1} > \chi'(G) - 1$, which in turn gives that $H$ is $\chi'(G)$-dense and $\chi'(H) \geq \chi'(G)$. Combining this with the fact that $\chi'(H) \leq \chi'(G)$, we get that $H$ is $\chi'(G)$-near-dense subgraph of $G$. \qed

**Remark 2.6.** The following properties follows directly from the definition.

1. A $k$-exact-dense graph is $k$-near-dense;
2. A $k$-near-dense graph is $k_1$-dense for $k_1 \leq k$;
3. If $G$ is a $k$-dense graph, then $\chi'(G) \geq k$;
4. If $\chi'(G) = k$, then $2|E(H)| \leq k(|H| - 1)$ for every subgraph $H \subseteq G$ with $|H|$ odd, moreover, if the equality holds, then $H$ must be an induced subgraph; and
5. If $H$ is a $\chi'(G)$-dense subgraph of $G$, then $H$ is $\chi'(G)$-near-dense.
Theorem 2.7. Let $\Delta \geq 2$ be a positive integer and $r \in (\frac{1}{2}, 1)$ be a real number such that $\Delta^r \geq \lceil \Delta^{1-r} \rceil$, and let $G$ be a graph such that $\Delta(G) \leq \Delta$ and $\chi'(G) \geq \Delta + 2\Delta^r$. If $G$ itself is a $\chi'(G)$-near-dense graph, then for every $k \geq \chi'(G)$, there exists a $k$-exact-dense spanning subgraph $G^*$ such that $d_{G^*}(v) \leq k - \Delta^{1-r} - (\Delta - d_G(v))$ for each $v \in V(G)$.

The proof of Theorem 2.7 will be presented in Section 5. Let $G$ be a graph and $W \subseteq V(G)$. Let $G/W$ denote the graph obtained from $G$ by contracting the set $W$, that is, we replace in $G$ the set $W$ by a new vertex $w$, each boundary edge $e \in E_G(x, y)$ with $x \notin W$ and $y \in W$ by an edge with endpoints $x$ and $w$, and delete all edges in $G[W]$. We call $G/W$ a minor of $G$. We notice that the term minor defined here is slightly broader than the commonly used term minor which requires that $G[W]$ is connected. Given an induced subgraph $H$ of $G$, we define $G/H = G/V(H)$. Moreover, we say that $G/H$ is obtained from $G$ by contracting $H$ and $G$ is obtained from $G/H$ by uncontracting $H$. The following simple observation is needed in our proof.

Lemma 2.8. Let $G$ be a graph and $H$ be a $k$-dense subgraph of $G$. Let $v_H$ be the resulting vertex from the contraction of $H$. If $F$ is a $k$-dense subgraph of $G/H$ containing $v_H$, then the graph $F^*$ obtained from $F$ by uncontracting $v_H$ is also a $k$-dense subgraph of $G$.

Proof. On the one hand, since both $F$ and $H$ are $k$-dense, $|F|$ and $|H|$ are both at least 3 and odd, and satisfy the following inequalities.

$$2|E(F)| > (k - 1)(|F| - 1) \quad \text{and} \quad 2|E(H)| > (k - 1)(|H| - 1).$$

On the other hand, considering that $F^*$ is obtained from $F$ by uncontracting $v_H$, we gain that $|F^*| = |F| + |H| - 1$ and $|E(F^*)| = |E(F)| + |E(H)|$. Hence, $|F^*|$ is odd and

$$2|E(F^*)| = 2|E(F)| + 2|E(H)| > (k - 1)(|F| + |H| - 2) = (k - 1)(|F^*| - 1),$$

which in turn gives that $F^*$ is a $k$-dense subgraph of $G$. \qed

The following result was obtained in Cao et al. [2] (Lemma 3.2 on page 4). For completeness, we include the proof here.

Lemma 2.9. If a graph $G$ satisfies $\chi'(G) \geq \Delta(G) + 2$, then two maximal $\chi'(G)$-exact-dense subgraphs (if exists) are either the same or disjoint.

Proof. Let $k = \chi'(G)$ and $\Delta = \Delta(G)$. Let $H_1$ and $H_2$ be two distinct maximal $k$-exact-dense subgraphs of $G$ such that $V(H_1) \cap V(H_2) \neq \emptyset$. By Remark 2.6-4, both $H_1$ and $H_2$ are induced subgraphs. Let $F_1 = H_1 - V(H_1 \cap H_2)$ and $F_2 = H_2 - V(H_1 \cap H_2)$. Following the maximality of $H_1$ and $H_2$, we have $F_1 \neq \emptyset \neq F_2$. 

We claim that $|H_1 \cap H_2|$ is odd. Otherwise, both $|F_1|$ and $|F_2|$ are odd. Since $F_1$ and $F_2$ are both subgraphs of $G$, $2|E(F_i)| \leq k(|F_i| - 1)$ for $i \in \{1, 2\}$. Since $H_i$ is $k$-exact-dense for $i \in \{1, 2\}$, we have $2|E(H_i)| = k(|H_i| - 1)$. Note that $2|E(H_1)| = 2|E(F_1)| + 2|E(H_1 \cap H_2)| + 2|E(H_1 \cap H_2, F_2)|$. Thus, we have the following.

(1) $2|E(H_1 \cap H_2)| + 2|E(H_1 \cap H_2, F_2)| \geq k(|H_1| - 1) - k(|F_1| - 1) = k|H_1 \cap H_2|

Since $d_G(H_1 \cap H_2) \geq |E(H_1 \cap H_2, F_1)| + |E(H_1 \cap H_2, F_2)|$, we have

$$2|E(H_1 \cap H_2)| + d_G(H_1 \cap H_2) \geq \sum_{i=1}^{2} (|E(H_1 \cap H_2)| + |E(H_1 \cap H_2, F_i)|) \geq k|H_1 \cap H_2|,$$

where we applied (1) for the last inequality.

On the other hand, $2|E(H_1 \cap H_2)| + d_G(H_1 \cap H_2) = \sum_{v \in V(H_1 \cap H_2)} d_G(v) \leq \Delta \cdot |H_1 \cap H_2|$. So, we get $\Delta \geq k$, giving a contradiction to $k \geq \Delta + 2$.

Since $|H_1 \cap H_2|$ is odd and $H_1 \cap H_2 \subseteq G$, we have $2|E(H_1 \cap H_2)| \leq k(|H_1 \cap H_2| - 1)$. Recall that $2|E(H_i)| = k(|H_i| - 1)$ for $i \in \{1, 2\}$. Thus,

$$2|E(H_1 \cap H_2)| = 2|E(H_1)| + 2|E(H_2)| - 2|E(H_1 \cap H_2)| \geq k(|H_1| - 1) + k(|H_2| - 1) - k(|H_1 \cap H_2| - 1) = k(|H_1 \cap H_2| - 1).$$

In addition, we have $2|E(H_1 \cup H_2)| \leq k(|H_1 \cup H_2| - 1)$ because of $\chi'(H_1 \cup H_2) \leq \chi'(G) = k$. Hence, $H_1 \cup H_2$ is a $k$-exact-dense subgraph, giving a contradiction to the maximality of $H_1$ and $H_2$. □

3. Proof of Theorem 1.1

Let $\varphi$ be an edge-coloring of a graph $G$ with colors from a palette $C$. For each vertex $v \in V(G)$, we define the following two color sets

$$\varphi(v) = \{ \varphi(e) : e \in E_G(v) \} \quad \text{and} \quad \overline{\varphi}(v) = C \setminus \varphi(v),$$

and call $\varphi(v)$ the set of colors present at $v$ and $\overline{\varphi}(v)$ the set of colors missing at $v$. We call $\varphi$ a $k$-edge-coloring of $G$ if $|C| \leq k$. Similarly, a total-coloring $\varphi'$ of $G$ is called a $k$-total-coloring if it uses no more than $k$ colors.

We now present the proof of Theorem 1.1. Let $G$ be a graph with maximum degree $\Delta$ sufficiently large and $\chi'(G) \geq \Delta + 10\Delta^{35/36}$. Let $k = \chi'(G)$, $k_1 = \Delta + \lceil 10\Delta^{35/36} \rceil$ and $k_0 = \Delta + \lceil 2\Delta^{35/36} \rceil$. Clearly, $k \geq k_1 \geq k_0 \geq \Delta + 2$.

Claim 3.1. If a subgraph $H \subseteq G$ is $(k_0 + 1)$-dense, then

$$|H| < \Delta^{1/36} \quad \text{and} \quad d_G(H) < \Delta - (k_0 - \Delta)(|H| - 1).$$

Proof. By the Handshaking Lemma, $\Delta \cdot |H| \geq \sum_{v \in V(H)} d_G(v) = 2|E(H)| + d_G(H)$. Since $H$ is $(k_0 + 1)$-dense, we have $2|E(H)| > k_0(|H| - 1)$, and so
\[\Delta \cdot |H| > k_0(|H| - 1) + d_G(H).\] Hence, \(d_G(H) < \Delta - (k_0 - \Delta)(|H| - 1).\) Applying \(k_0 = \Delta + [2\Delta^{35/36}] \geq \Delta + 2\Delta^{35/36}\) and \(d_G(H) \geq 0,\) we get \(|H| < \Delta^{1/36}.\) \(\Box\)

In the following algorithm, we define a sequence \(G_t = (G_0, G_1, \ldots, G_t)\) of minors of \(G\) and a companion sequence \(H_t = (H_0, \ldots, H_{t-1})\) for \(t \geq 0.\)

**Algorithm 1.** Let \(G_0 = G.\) Initially, we set \(G_0 = (G_0)\) and \(H_0 = \emptyset.\) Note that \(\chi'(G_0) = k \geq \Delta(G_0) + 2.\) Lemma 2.5 shows that \(G_0\) has a \(k\)-near-dense subgraph. Let \(H_0\) be a maximal \(k\)-near-dense subgraph of \(G_0\) and \(G_1 = G_0/H_0.\) Set \(G_1 = (G_0, G_1)\) and \(H_1 = (H_0).\)

Suppose that we have defined a sequence \(G_t = (G_0, G_1, \ldots, G_t)\) and its companion sequence \(H_t = (H_0, \ldots, H_{t-1})\) for some \(t \geq 1.\) If \(\chi'(G_t) \leq k_0,\) then we stop and let \(T = t.\) Otherwise, we have \(\chi'(G_t) > k_0 \geq \Delta + 2.\) Using the result \(\Delta \geq \Delta(G_t)\) shown in Claim 3.3 below, we get \(\chi'(G_t) \geq \Delta + 2.\) By Lemma 2.5, \(G_t\) has a \(\chi'(G_t)\)-near-dense subgraph. Let \(H_t\) be a maximal \(\chi'(G_t)\)-near-dense subgraph in \(G_t\) and \(G_{t+1} = G_t/H_t.\) Set \(G_{t+1} = (G_0, G_1, \ldots, G_t, G_{t+1})\) and \(H_{t+1} = (H_0, \ldots, H_{t-1}, H_t).\)

In Algorithm 1, for each \(t \in \{0, \ldots, T - 1\},\) since \(H_t\) is \(\chi'(G_t)\)-near-dense subgraph of \(G_t,\) we have \(|H_t| \geq 3,\) and so \(|G_{t+1}| \leq |G_t| - 2.\) Hence, Algorithm 1 contains only finite steps. We call \(G_T = (G_0, \ldots, G_T)\) a maximal dense-minor-sequence, and \(H_t\) the companion of \(G_t\) for \(t \in \{0, \ldots, T - 1\}.\) Denote by \(v_H\) the resulting vertex from the contraction of \(H_t.\)

For each \(G_t,\) we call each vertex in \(V(G_t) \setminus V(G)\) a contracted vertex. So, the vertices in \(V(G_t)\) are divided into two classes: the original vertices of \(G\) and the contracted vertices. Algorithm 1 naturally generates an onto function \(f_t: V(G) \to V(G_t)\) such that the pre-image \(f_t^{-1}(v)\) is the vertex set of \(G\) whose contraction results in \(v\) if \(v\) is a contracted vertex, and \(f_t^{-1}(v) = \{v\}\) otherwise. We call \(f_t^{-1}(v)\) the root of \(v\) for each \(v \in V(G_t).\)

**Claim 3.2.** For each contracted vertex \(v \in V(G_t),\) the subgraph \(G[f_t^{-1}(v)]\) induced by the root of \(v\) is \((k_0 + 1)\)-dense.

**Proof.** We first notice that \(H_s\) is \((k_0 + 1)\)-dense for every \(s \in \{0, \ldots, T - 1\}\) because \(H_s\) is \(\chi'(G_s)\)-near-dense and \(\chi'(G_s) \geq k_0 + 1.\)

Suppose that \(v\) is the resulting vertex from the contraction of \(H_s\) for some \(s \leq t - 1.\) If \(H_s\) does not contain any contracted vertex, then \(f_t^{-1}(v) = V(H_s)\). Since \(H_s\) is \((k_0 + 1)\)-dense, we are done. Suppose that \(H_s\) contains some contracted vertices. Let \(r\) be the largest index such that the contraction of \(H_r\) results in a contracted vertex, say \(u,\) in \(H_s.\) Let \(F_r\) be the subgraph of \(G_r\) obtained from \(H_s\) by uncontracting \(u\) back to \(H_r.\) Applying Lemma 2.5...
Proof. For any vertex \( v \in V(G_t) \), if \( v \) is a contracted vertex, by Claim 3.2 we have that \( G[f_t^{-1}(v)] \) is \((k_0 + 1)\)-dense. Combining this with Claim 3.1 we have \( d_{G_t}(v) = d_G(G[f_t^{-1}(v)]) < \Delta \). If \( v \) is not a contracted vertex, then \( d_{G_t}(v) = d_G(v) \leq \Delta \). Hence, \( \Delta(G_i) \leq \Delta \). \( \square \)

Claim 3.3. \( \Delta(G_t) \leq \Delta \) for any \( t \in \{0, \ldots, T\} \).

Proof. For each \( t \in \{0, \ldots, T - 1\} \), we have \( \chi'(G_{t+1}) \leq \chi'(G_t) \).

Proof. Suppose on the contrary that \( \chi'(G_{t+1}) > \chi'(G_t) \). Since \( t \leq T - 1 \), we have \( \chi'(G_t) > k_0 \geq \Delta + 2 \), and so \( \chi'(G_{t+1}) > \Delta + 2 \). Combining this with \( \Delta(G_{t+1}) \leq \Delta \) (Claim 3.3), we have \( \chi'(G_{t+1}) > \Delta(G_{t+1}) + 2 \). Applying Lemma 2.5 to \( G_{t+1} \), we get a \( \chi'(G_{t+1}) \)-near-dense subgraph \( F_{t+1} \) in \( G_{t+1} \).

We claim that \( v_{H_t} \in V(F_{t+1}) \). Otherwise, \( F_{t+1} \) itself is a subgraph of \( G_t \), and so \( \chi'(G_{t+1}) = \chi'(F_{t+1}) \leq \chi'(G_t) \), giving a contradiction to the assumption \( \chi'(G_{t+1}) > \chi'(G_t) \). Let \( F_t \) be the subgraph of \( G_t \) obtained from \( F_{t+1} \) by uncontracting \( v_{H_t} \), i.e., replacing \( v_{H_t} \) by \( H_t \) in \( F_{t+1} \). Since \( F_{t+1} \) is \( \chi'(G_{t+1}) \)-dense and \( \chi'(G_{t+1}) > \chi'(G_t) \), it follows that \( F_{t+1} \) is \( \chi'(G_t) \)-dense. Combining this with that \( H_t \) is a \( \chi'(G_t) \)-dense subgraph of \( G_t \) and applying Lemma 2.8 we see that \( F_t \) is also \( \chi'(G_t) \)-dense subgraph of \( G_t \). By Remark 2.23, \( F_t \) is a \( \chi'(G_t) \)-near-dense subgraph of \( G_t \). However, \( F_t \) contains \( H_t \) as proper subgraph, which gives a contradiction to the maximality of \( H_t \). \( \square \)

We first consider the case that \( t = T \). In this case, we have \( \chi'(G_t) = k_0 = \Delta + 2\Delta^5/36 \). For any \( w \in W(G_t) \), by Claim 3.2 we have \( \chi'(G_t) \leq k_0 = \Delta + 2\Delta^5/36 \). For any \( w \in W(G_t) \), by Claim 3.2 we have \( \chi'(G_t) \leq k_0 = \Delta + 2\Delta^5/36 \).
Let $W^* = f_t^{-1}(v_{H_t}) = f_t^{-1}(V(H_t))$, $W(H_t) = \{w_1, \ldots, w_m\}$, and $W_i = f_t^{-1}(w_i)$ for $i \in \{1, \ldots, m\}$. By definition, we have

$$F_t = G - \cup_{w \in W(G_t) \setminus W(H_t)} E(G[f_t^{-1}(w)]) - \cup_{w_i \in W(H_t)} E(G[W_i])$$

$$F_{t+1} = G - \cup_{w \in W(G_{t+1}) \setminus \{v_{H_t}\}} E(G[f_{t+1}^{-1}(w)]) - E(G[W^*]).$$

Note that $W(G_t) \setminus W(H_t) = W(G_{t+1}) \setminus \{v_{H_t}\}$ and $f_t^{-1}(w) = f_{t+1}^{-1}(w)$ for any $w \in W(G_t) \setminus W(H_t)$, and $\cup_{i=1}^{m} W_i \subseteq W^*$. Hence, $F_{t+1}$ is a spanning subgraph of $F_t$ and $E(F_t - F_{t+1}) = E(G[W^*]) - \cup_{w_i \in W(H_t)} E(G[W_i]) = E(F_t[W^*])$. Moreover, $F_{t+1}[W^*]$ is an independent vertex set. Hence, the $k$-total-coloring $\varphi_{t+1}$ of $F_{t+1}$ is a partial coloring of $F_t$ except that edges in $E(F_t[W^*])$ are yet colored. We will find an edge-coloring $\pi'$ of $F_t[W^*]$ using colors $1, \ldots, k$ such that its combination with $\varphi_{t+1}$ gives a desired $k$-total-coloring of $F_t$.

Note that $H_t$ is obtained from $F_t[W^*]$ by contracting $W_1, \ldots, W_m$ into $w_1, \ldots, w_m$, respectively. Hence, an edge-coloring $\pi$ of $H_t$ gives an edge-coloring $\pi'$ of $F_t[W^*]$ with the property that for every $W_i$, all edges incident to it are assigned different colors. We will find a $k$-edge-coloring $\pi$ of $H_t$ to get an edge-coloring of $F_t[W^*]$ to fulfill our goal stated above. This strategy is depicted in Figure 1.

Since $H_t$ is a $\chi'(G_t)$-near-dense subgraph of $G_t$, we have $\chi'(H_t) = \chi'(G_t)$.

Applying it with $k = \chi'(G) \geq \chi'(G_t) > k_0$, we get $k \geq \chi'(H_t) > k_0 = \Delta + \lfloor 2\Delta^{35/36} \rfloor \geq \Delta + 2\Delta^{35/36}$. By Claim 3.3, we have $\Delta(H_t) \leq \Delta(G_t) \leq \Delta$. From the assumption of $\Delta$ being sufficiently large, we have $\Delta^{35/36} \geq \lfloor \Delta^{1/36} \rfloor$.

Applying Theorem 2.1 with $r = \frac{35}{36}$, we see that $H_t$ has a $k$-exact-dense spanning supgraph $H_t^*$ such that $d_{H_t^*}(v) \leq k - \Delta^{1/36} - (\Delta - d_{H_t}(v))$ for every $v \in V(H_t)$. By the definition of $k$-exact-dense, $2|E(H_t^*)| = k(|H_t^*| - 1)$ and $\chi'(H_t^*) = k$.

Let $\pi^*$ be a $k$-edge-coloring of $H_t^*$. Since a matching of $H_t^*$ contains at most $(|H_t^*| - 1)/2$ edges, each color class of $\pi^*$ must be a near perfect matching. So, each color is missing at exactly one vertex. Hence, $\overline{\pi^*}(v)$ are
pairwise disjoint for $v \in V(H_t)$ and the following holds.

\begin{equation}
|\pi^t(v)| = k - d_{H_t}(v) \geq \Delta^{1/36} + (\Delta - d_{H_t}(v)).
\end{equation}

Since $H_t \subseteq H_t^*$, by restricting $\pi^*$ to $H_t$ we get a $k$-edge-coloring $\pi$ of $H_t$. For each vertex $v \in V(H_t)$, we have $\pi(v) \supseteq \pi^t(v)$.

On the other hand, for each $v \in V(H_t)$, we claim that

\begin{equation}
|f_t^{-1}(v)| < \Delta^{1/36} \quad \text{and} \quad d_{F_t - W_t}(f_t^{-1}(v)) \leq \Delta - d_{H_t}(v)
\end{equation}

where $d_{F_t - W_t}(f_t^{-1}(v))$ denotes the number of the edges with one endpoint in $f_t^{-1}(v)$ and the other endpoint in $V(F_t) \setminus W^*$.

For the first inequality of (3), if $v$ is a contracted vertex, i.e., $v \in W(H_t) = \{w_1, \ldots, w_m\}$, then $G[f_t^{-1}(v)]$ is $(k_0 + 1)$-dense. By Claim 3.1 we have $|f_t^{-1}(v)| < \Delta^{1/36}$. If $v$ is not a contracted vertex, then $f_t^{-1}(v) = \{v\}$. Hence, $|f_t^{-1}(v)| = |\{v\}| = 1 < \Delta^{1/36}$.

For the second inequality of (3), we notice that $d_{F_t}(f_t^{-1}(v)) = d_{G_t}(v)$ and $d_{F_t[W_t]}(f_t^{-1}(v)) = d_{H_t}(v)$. Hence,

\begin{equation}
d_{F_t - W_t}(f_t^{-1}(v)) = d_{F_t}(f_t^{-1}(v)) - d_{F_t[W_t]}(f_t^{-1}(v)) = d_{G_t}(v) - d_{H_t}(v) \leq \Delta - d_{H_t}(v).
\end{equation}

Combining (2) and (3), we get following inequality for every $v \in V(H_t)$.

\begin{equation}
|\pi^t(v)| > |f_t^{-1}(v)| + d_{F_t - W_t}(f_t^{-1}(v))
\end{equation}

Under the $k$-total-coloring $\varphi_{t+1}$ of $F_{t+1}$, for each vertex $v \in V(H_t)$, let $C_v$ denote the set of colors assigned to the vertices in $f_t^{-1}(v)$ and edges incident
to $f_t^{-1}(v)$. Note that $f_t^{-1}(v) \subseteq W^*$ for $v \in V(H_t)$. Since $W^* = f_{t+1}^{-1}(v_{H_t})$ is an independent vertex set in $F_{t+1}$ and $W^*$ is $\varphi_{t+1}$-distinct, the color sets $C_v$ for $v \in V(H_t)$ are mutually disjoint and

$$|C_v| = |f_t^{-1}(v)| + d_{F_{t+1}}(f_t^{-1}(v)) = |f_t^{-1}(v)| + d_{F_{t-W^*}}(f_t^{-1}(v)) < |\pi'(v)|.$$ 

Recall that the color sets $\pi'(v)$ for $v \in V(H_t)$ are mutually disjoint. Hence, by permuting colors, we may assume that

$$\pi(v) \supseteq \pi'(v) \supseteq C_v \quad \text{for every } v \in V(H_t).$$

Let $\pi'$ be the edge-coloring of $F_t[W^*]$ generated from the coloring $\pi$ of $H_t$, that is, $\pi'(e') = \pi(e)$ if $e' \in F_t[W^*]$ is the image of $e \in E(H_t)$ under the natural corresponding between $E(H_t)$ and $E(F_t[W^*])$. For every vertex $v \in V(H_t)$, by (5) we have $\pi'(f_t^{-1}(v)) = \pi(v) \supseteq C_v$. So, no colors in $C_v$ are assigned to edges incident to $f_t^{-1}(v)$ in $F_t[W^*]$ under coloring $\pi'$. Hence, the combination of the coloring $\varphi_{t+1}$ and $\pi'$ gives a $k$-total-coloring coloring $\varphi_t$ of $F_t$. For each $w_i$, there are three disjoint sets of colors involving $W_i$: $\varphi_{t+1}(W_i)$, $\varphi_{t+1}(E(W_i, G - W^*))$, and $\pi'(E_G(W_i, W^* - W_i)) = \pi(E_{H_t}(w_i))$. Hence, $W_i$ is $\varphi_t$-distinct. Additionally, we note that coloring $\varphi_t$ agrees with $\varphi_{t+1}$ on every edge and every vertex not in $F_t[W^*]$. Therefore, for every contracted vertex $w \in W(G_t)$, the vertex set $f_t^{-1}(w)$ is $\varphi_t$-distinct, and so $\varphi_t$ is the desired $k$-total-coloring of $F_t$. $\square$

4. Proof of Theorem 2.1

In order to simplify the presentation of the proof, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. We reserved enough room in the calculation to absorb the differences.

Let $G$ be a graph with maximum degree $\Delta$ sufficiently large. Let $V_1, \ldots, V_m$ be $m$ mutually disjoint independent vertex sets of $G$, and let $G_0$ be obtained from $G$ by contracting each $V_i$ to a single vertex. Suppose that $\chi'(G_0) \leq \Delta + 2\Delta^{35/36}$ and $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \ldots, m\}$. We will show that $G$ has a total-coloring $\eta$ using at most $\Delta + 10\Delta^{35/36}$ colors such that all $V_\ell$ are $\eta$-distinct.

Set $k = \Delta + 10\Delta^{35/36}$ and $k_0 = \Delta + 2\Delta^{35/36}$. Let $\varphi_0$ be a $k_0$-edge-coloring of $G_0$ using the colors $1, \ldots, k_0$. Since $V_1, \ldots, V_m$ are independent vertex sets, $\varphi_0$ naturally becomes a $k_0$-edge-coloring $\varphi$ of $G$ with the additional property that for each $\ell \in \{1, \ldots, m\}$, edges incident to $V_\ell$ are assigned different colors.
Let $K$ be obtained from $G$ by adding edges to $V_\ell$ such that $K[V_\ell]$, the subgraph induced by $V_\ell$, is a complete (simple) graph for $\ell \in \{1, \ldots, m\}$. Since $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \ldots, m\}$, we have $\Delta(K) < \Delta + \Delta^{1/36}$.

Let $s = \lceil \Delta^{16/36} \rceil$ and $t = \lceil k_0/s \rceil$. By definition, we have

$$\frac{\Delta + 2\Delta^{35/36}}{\Delta^{16/36} + 1} - 1 < t \leq \Delta^{20/36} + 2\Delta^{19/36} \quad \text{and} \quad s \cdot t \leq k_0.$$ 

We specify $s$ sets of colors: $C_1 = \{1, \ldots, t\}, \ldots, C_s = \{(s-1)t + 1, \ldots, st\}$. Clearly, $C_1 \cup \cdots \cup C_s = \{1, \ldots, st\}$ is a subset of the set of colors used by the edge-coloring $\varphi$.

**Definition 4.1.** With respect to a partition $U_1, \ldots, U_s$ of $V(G)$, we say that an edge $e \in E(G)$ **discords** a vertex $u \in V(G)$ if $\varphi(e) \in C_i$ and $u \in U_i$ for some $i \in \{1, \ldots, s\}$. Moreover, we say that $e$ **conflicts** with $u$ if either $e$ discords a vertex in $V_\ell$ when $u \in V_\ell$ for some $\ell \in \{1, \ldots, m\}$ or $e$ discords $u$ itself when $u \notin \cup_{\ell=1}^m V_\ell$.

**Claim 4.2.** There is a partition $U_1, \ldots, U_s$ of $V(G)$ such that

(i). for each vertex $v$ and $U_i$, $|N_K(v) \cap U_i| \leq t - 1$,

(ii). for each vertex $v$, there are at most $4\Delta^{34/36}$ edges $e$ incident with $v$ that conflicts with the other endpoint of $e$ in $G$.

**Proof.** Assign each vertex to a uniformly random part with probability $\frac{1}{s}$ (where of course, these choices are made independently). For each pair $(v, i)$ we let $A_{v,i}$ be the event that (i) fails to hold for $(v, i)$ and $B_v$ be the event that (ii) fails to hold for $v$. We will use Hoeffding’s Inequality to prove the following inequalities by assuming that $\Delta$ is sufficiently large.

A. $\mathbb{P}[A_{v,i}] < e^{-\Delta^{1/18}} < \frac{1}{4\Delta^3}$ for every pair $(v, i)$, and

B. $\mathbb{P}[B_v] < e^{-\Delta^{1/18}} < \frac{1}{4\Delta^3}$ for every $v$.

We first complete the proof of Claim 4.2 based on (A) and (B) before giving their proofs. For each vertex $v \in V(G)$, let $D(v)$ denote the union of $N_G(v)$ and these $V_\ell$ that contain a neighbor of $v$. Since $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \ldots, m\}$, it follows that $|D(v)| \leq \Delta \cdot \Delta^{1/36} = \Delta^{37/36}$. Note that events $B_v$ and $A_{v,i}$ are determined by the partition assignments of the vertices in $D(v)$. Thus, by the Mutual Independence Principle, they are mutually independent of all events concerning vertices which are not in $D(v) \cup N_G(D(v))$. Notice that each vertex $v$ involves with one event $B_v$ and $s$ events $A_{v,1}, \ldots, A_{v,s}$. So, every event is mutually independent of all but at most $(s+1)|D(v)| \leq (s+1)(|D(v)| \cdot \Delta) < \Delta^{16/36} \cdot \Delta^{37/36} \cdot \Delta < \Delta^3$ other events. Since $4 \cdot \frac{\Delta}{\Delta^3} \cdot \Delta^3 = 1$, by Theorem 2.4 (the Local Lemma) there is a partition satisfying both (i) and (ii).
Proof of A. Given a pair \((v, i)\) with \(v \in V(G)\) and \(1 \leq i \leq s\), we will show that \(\mathbb{P}[A_{v,i}] < e^{-\Delta^{1/18}}\). For each \(u \in N_K(v)\), let \(X_u\) be a 0–1 variable such that \(X_u = 1\) if \(u \in U_i\) and \(X_u = 0\) otherwise. Let \(X = \sum_{u \in N_K(v)} X_u\). Clearly, \(X\) is the number of neighbors of \(v\) assigned to \(U_i\) in \(K\). Recall that \(|N_K(v)| < \Delta + \Delta^{1/36}\). Since \(\mathbb{E}[X_u] = \mathbb{P}[X_u = 1] = 1/s\) for each \(u \in N_K(v)\), we have

\[
\mu = \mathbb{E}[X] = \sum_{u \in N_K(v)} \mathbb{E}[X_u] = \frac{|N_K(v)|}{s} < (\Delta + \Delta^{1/36})/\Delta^{16/36} = \Delta^{20/36} + \Delta^{-15/36}.
\]

Since \(t > (\Delta + 2\Delta^{35/36})/(\Delta^{16/36} + 1) - 1\) and \(\Delta\) is large, a simple calculation gives us \(t > \Delta^{20/36} + \Delta^{-15/36} + \Delta^{19/36} > \mu + \Delta^{19/36}\). Applying Hoeffding’s Inequality with \(b = 1\) and \(a = 0\), we get the following.

\[
\mathbb{P}[A_{v,i}] = \mathbb{P}[X > t] \leq \mathbb{P}[X > \mu + \Delta^{19/36}] \leq e^{-\frac{2\Delta^{19/36}}{|N_K(v)|}} < e^{-\frac{2\Delta^{38/36}}{\Delta}} = e^{-\Delta^{1/18}}
\]

In the above inequality, we used that \(|N_K(v)| < \Delta + \Delta^{1/36} \leq 2\Delta\). \(\square\)

Proof of B. Given a vertex \(v \in V(G)\), we wish to show that \(\mathbb{P}[B_v] < e^{-\Delta^{1/18}}\). For each color set \(C_i\), let \(E_{C_i}\) denote the set of edges in \(G\) assigned colors in \(C_i\). Let \(I = N_G(v) \cap \bigcup_{\ell \in \{1, \ldots, m\}} V_\ell\) and \(L = \{\ell \in \{1, \ldots, m\} : N_G(v) \cap V_\ell \neq \emptyset\}\). For \(u \in I\), let \(X_u\) be the number of edges in \(E_G(v, u)\) that conflict with \(u\), that is, the number of edges in \(E_G(v, u)\) that discard \(u\). For \(i \in \{1, \ldots, s\}\), if \(u \in U_i\) then there are \(|E_G(v, u) \cap E_{C_i}|\) edges in \(E_G(v, u)\) that conflict with \(u\). Hence, we have the following.

\[
(6) \quad \mathbb{E}[X_u] = \sum_{i=1}^{s} |E_G(v, u) \cap E_{C_i}| \cdot \mathbb{P}[u \in U_i] = \frac{|E_G(v, u)|}{s}
\]

For each \(\ell \in L\), let \(X_\ell\) be the number of pairs \((e, w)\) with \(e \in E_G(v, V_\ell)\) and \(w \in V_\ell\) such that \(e\) discords \(w\), and let \(X_\ell^*\) be the number of edges incident with \(v\) that conflict with the other endpoint in \(V_\ell\). Since \(V_\ell = \bigcup_{i=1}^{s} V_\ell \cap U_i\), we have

\[
X_\ell^* = \sum_{1 \leq i \leq s, V_\ell \cap U_i \neq \emptyset} |E_G(v, V_\ell) \cap E_{C_i}| \leq \sum_{i=1}^{s} |E_G(v, V_\ell) \cap E_{C_i}| \cdot |V_\ell \cap U_i| = X_\ell.
\]

For each \(w \in V_\ell\), let \(X_{w}\) be the number of edges \(e \in E_G(v, V_\ell)\) discarding \(w\). Clearly, \(X_\ell = \sum_{w \in V_\ell} X_{w}\). Similar to (6), we have \(\mathbb{E}[X_{w}] = |E_G(v, V_\ell)|/s\). Hence,

\[
(7) \quad \mathbb{E}[X_\ell] = \sum_{w \in V_\ell} \mathbb{E}[X_{w}] = \frac{|E_G(v, V_\ell)| \cdot |V_\ell|}{s} \leq \frac{|E_G(v, V_\ell)| \cdot \Delta^{1/36}}{s}.
\]

We divide each of \(I\) and \(L\) into two subsets as follows.

\[
I_1 = \{u \in I : |E_G(v, u)| \leq \Delta^{14/36}\} \quad \& \quad I_2 = \{u \in I : |E_G(v, u)| > \Delta^{14/36}\}
\]

\[
L_1 = \{\ell \in L : |E_G(v, V_\ell)| \leq \Delta^{14/36}\} \quad \& \quad L_2 = \{\ell \in L : |E_G(v, V_\ell)| > \Delta^{14/36}\}
\]
Let $Y = \sum_{u \in I_1} X_u + \sum_{\ell \in L_1} X_\ell$, $Z = \sum_{u \in I_2} X_u + \sum_{\ell \in L_2} X_\ell$, and $X = \sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell$. Clearly, $X = Y + Z$. By (9) and (7), we have

$$E[Y] = \sum_{u \in I} E[X_u] + \sum_{\ell \in L} E[X_\ell] \leq \frac{1}{8} \sum_{u \in I} |E_G(v, u)| + \frac{1}{8} \sum_{\ell \in L} |E_G(v, \ell)| \cdot \Delta^{1/36} \leq \frac{d_G(v) \cdot \Delta^{1/36}}{\Delta^{16/36}} \leq \Delta^{21/36}.$$  

Consequently, we have $E[Y] \leq E[X] \leq \Delta^{21/36}$ and $E[Z] \leq E[X] \leq \Delta^{21/36}$.

For each $u \in I_1$, we have $X_u \leq |E_G(v, u)| \leq \Delta^{14/36}$. For each $\ell \in L_1$, we know $X_\ell \leq |E_G(v, \ell)| \cdot |\ell| \leq \Delta^{15/36}$. Notice that $|I_1| + |L_1| \leq |N_G(v)| \leq \Delta$. Applying Hoeffding’s Inequality with $n = |I_1| + |L_1|$, $b = \Delta^{15/36}$ and $a = 0$, we get the following inequality.

$$P[Y > 2\Delta^{34/36}] \leq P[Y > E[Y] + \Delta^{34/36}] \leq e^{-\frac{2(\Delta^{34/36})^2}{\Delta^{22/36}}} = e^{-2\Delta^{18}} < \frac{e^{-\Delta^{18}}}{2}$$

For each color set $C_i$, since $|C_i| = t \leq \Delta^{20/36} + 2\Delta^{19/36}$, it follows that $|E_G(v, u) \cap E_{C_i}| \leq t$ for each $u \in N_G(v)$. So, for each $u \in I$, there are at most $t$ edges in $E_G(v, u)$ discording $u$. Hence $X_u \leq t$. For each $\ell \in L$, there are at most $t \cdot |\ell|$ pairs $(e, w)$ with $e \in E_G(v, \ell)$ and $w \in V_\ell$ such that $w \in U_\ell$ and $\varphi(e) \in C_i$ for some $i$, and so $X_\ell \leq t \cdot |\ell| \leq t \cdot \Delta^{1/36}$. Note that $t \cdot \Delta^{1/36} \leq \Delta^{21/36} + 2\Delta^{20/36} \leq 1.4 \Delta^{21/36}$ (because $\Delta$ is large) and $|I_2| + |L_2| < d_G(v)/\Delta^{14/36} \leq \Delta^{22/36}$. Applying Hoeffding’s Inequality with $n = |I_2| + |L_2|$, $b = 1.4 \Delta^{21/36}$ and $a = 0$, we get the following inequality (where we used $1.4^2 < 2$).

$$P[Z > 2\Delta^{34/36}] \leq P[Z > E[Z] + \Delta^{34/36}] \leq e^{-\frac{2(\Delta^{34/36})^2}{\Delta^{22/36}(1.4 \Delta^{21/36})^2}} < e^{-\Delta^{18}} < \frac{e^{-\Delta^{18}}}{2}$$

By definition, $\sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell$ is exactly the number of edges $e$ incident with $v$ that conflicts with the other endpoint of $e$ in $G$. Since $X_\ell \leq X_\ell$ for $\ell \in \{1, \ldots, s\}$, we have $\sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell \leq X$. Hence,

$$P[B_v] \leq P[X > 4\Delta^{34/36}] \leq P[Y > 2\Delta^{34/36}] + P[Z > 2\Delta^{34/36}] < e^{-\Delta^{1/18}},$$

which completes the proof of Claim 12.\[\Box\]

Let $U_1, \ldots, U_s$ be a partition of $V(G)$ satisfying both (i) and (ii). By (i), for each $U_i$, by using the simple greedy procedure, we get a vertex-coloring of $K[U_i]$ with colors in $C_i$, which in turn gives a vertex-coloring $\theta$ of $K$ using at most $k_0$ colors. Since in $K$ each $V_\ell$ is a clique for each $\ell \in \{1, \ldots, m\}$, it follows that $\theta$ is also a vertex-coloring of $G$ satisfying that each vertex in any $V_\ell$ is assigned a different color. It is noteworthy that the combination of $\varphi$ and $\theta$ may not produce a total-coloring of $G$ since some edges may have the same color as their endpoints.
Let \( e \in E(G) \) and \( u \) be an endpoint of \( e \). If \( u \notin \bigcup_{\ell=1}^{m} V_\ell \), we call \( e \) reject of \( u \) if \( \varphi(e) = \theta(u) \); if \( u \in V_\ell \) for some \( \ell \in \{1, \ldots, m\} \), we call \( e \) rejects \( u \) if \( \varphi(e) \in \theta(V_\ell) \). In either case, we call \( e \) a reject edge. From the construction of vertex-coloring \( \theta \), we see that if \( e \) rejects \( u \) then \( e \) must conflict with \( u \), but the converse may not be true. For each vertex \( v \in V(G) \), the reject degree of \( v \), denoted by \( \text{Rej}_v \), is the number of edges \( e \) incident with \( v \) that rejects the other endpoint of \( e \). By (ii), we have \( \text{Rej}_v \leq 4\Delta^{34/36} \) for each \( v \in V(G) \).

Let \( R \) be a spanning subgraph of \( G \) induced by all reject edges. If \( v \notin \bigcup_{\ell=1}^{m} V_\ell \), then there is at most one edge incident with \( v \) that rejects \( v \). If \( v \in V_\ell \) for some \( \ell \), there are at most \( \Delta^{1/36} \) edges incident with \( v \) sharing a common color with a vertex in \( V_\ell \), so there are at most \( \Delta^{1/36} \) edges incident with \( v \) that rejects \( v \). For each \( v \in V(G) \), the degree \( d_R(v) \) of \( v \) in \( R \) is bounded above by the sum of \( \text{Rej}_v \) and the number of edges that rejects \( v \). Therefore, \( d_R(v) \leq 4\Delta^{34/36} + \Delta^{1/36} \leq 5\Delta^{34/36} \) for every vertex \( v \in V(G) \), and so \( \Delta(R) \leq 5\Delta^{34/36} \).

Let \( R^* \) be obtained from graph \( R \) by contracting each \( V_\ell \) to a single vertex. Since \( \lvert V_\ell \rvert \leq \Delta^{1/36} \) for \( \ell \in \{1, \ldots, m\} \), it follows that \( \Delta(R^*) \leq \Delta(R) \cdot \Delta^{1/36} \leq 5\Delta^{35/36} \). By Shannon’s Theorem, \( \chi'(R^*) \leq \frac{3}{2}(5\Delta^{35/36}) \leq 8\Delta^{35/36} \). Let \( \pi^* \) be an edge-coloring of \( R^* \) using at most \( 8\Delta^{35/36} \) new colors different from \( \Delta + 2\Delta^{35/36} \) colors used in \( \varphi \) and \( \theta \). Clearly, \( \pi^* \) gives an edge-coloring \( \pi \) of \( R \) such that for each \( V_\ell \), all edges in \( R \) incident to \( V_\ell \) have different colors. By combining vertex-coloring \( \theta \), edge-coloring \( \varphi \) on \( E(G) \setminus E(R) \) and edge-coloring \( \pi \) on \( E(R) \), we get a total-coloring \( \eta \) of \( G \) such that all \( V_\ell \) are \( \eta \)-distinct. Note that we used at most \( \Delta + 2\Delta^{35/36} + 8\Delta^{35/36} = \Delta + 10\Delta^{35/36} \) colors in total, which completes the proof of Theorem 2.1.

5. Proof of Theorem 2.7

Let \( \Delta \geq 2 \) be an integer and \( r \in (1/2, 1) \) be a real number such that \( \Delta^r \geq \lceil \Delta^{1-r} \rceil \), and let \( G \) be a graph such that \( \Delta(G) \leq \Delta \) and \( \chi'(G) \geq \Delta + 2\Delta^r \). Assume that \( G \) itself is a \( \chi'(G) \)-near-dense graph. For any \( k \geq \chi'(G) \), we will show that \( G \) has a \( k \)-exact-dense spanning subgraph \( G^* \) such that \( d_{G^*}(v) \leq k - \Delta^{1-r} - (\Delta - d_G(v)) \) for every \( v \in V(G) \).

For each \( v \in V(G) \), let \( \lambda(v) = \Delta - d_G(v) \) and call it the deficiency of \( v \) in \( G \) with respect to \( \Delta \). For any vertex set \( W \subseteq V(G) \), let \( \lambda(W) = \sum_{v \in W} \lambda(w) \); and for a subgraph \( F \subseteq G \), let \( \lambda(F) = \lambda(V(F)) \). Let \( k_0 = \Delta + \lceil 2\Delta^r \rceil \) and \( k_1 = \chi'(G) \). Clearly, \( k \geq k_1 \geq k_0 \geq \Delta + 2\Delta^r \). We first prove the following two claims.

Claim 5.1. \( \Delta > (2\Delta^r - 1)(\lvert G \rvert - 1) + \lambda(G) \). In particular, \( \lvert G \rvert - 1 < \Delta^{1-r} \leq \Delta^r \).
Proof. Since $G$ is $k_1$-near-dense, we have that $2|E(G)| > (k_1 - 1)(|G| - 1)$. According to the definition of deficiency, we have that $\Delta \cdot |G| = \sum_{v \in V(G)} (d_G(v) + \lambda(v)) = \sum_{v \in V(G)} d_G(v) + \lambda(G) = 2|E(G)| + \lambda(G)$. Hence, $\Delta \cdot |G| > (k_0 - 1)(|G| - 1) + \lambda(G)$, which gives $\Delta > (k_0 - \Delta - 1)(|G| - 1) + \lambda(G)$. Since $k_0 \geq \Delta + 2\Delta^r$, we further have that $\Delta > (2\Delta^r - 1)(|G| - 1) + \lambda(G)$. Because of $\Delta^r \geq 1$ and $\lambda(G) \geq 0$, we have $\Delta > \Delta^r(|G| - 1)$, and so $|G| - 1 < \Delta^{1-r} \leq \Delta^r$.

Claim 5.2. Let $W \subseteq V(G)$ with $|W|$ odd. If $W \neq V(G)$, then $d_G(W) > \Delta^r(2|W| - 1) + \lambda(W)$, where $W = V(G) \setminus W$.

Proof. Let $D = G[W]$ and $F = G[\overline{W}]$ be the subgraphs of $G$ induced by $W$ and $\overline{W}$, respectively. We note that $2|E(D)| \leq k_1(|D| - 1)$ due to that $\chi'(D) \leq \chi'(G) = k_1$ and $|D|$ is odd. Since $G$ is $k_1$-near-dense and $|G| = |D| + |F|$, we have

$$2|E(G)| > (k_1 - 1)(|G| - 1) = k_1(|D| - 1) + k_1 \cdot |F| - (|G| - 1).$$

Consequently, we have the following.

$$(8) \quad 2|E(G)| > 2|E(D)| + k_1 \cdot |F| - (|G| - 1)$$

Applying $\sum_{v \in V(G)} d_G(v) = \sum_{v \in V(D)} d_G(v) + \sum_{v \in V(F)} d_G(v)$, we get the following.

$$(9) \quad 2|E(G)| = 2|E(D)| + d_G(V(D)) + \Delta \cdot |F| - \lambda(F).$$

Combining (8) and (9), we get

$$d_G(V(D)) > (k_1 - \Delta)|F| + \lambda(F) - (|G| - 1) \geq 2\Delta^r \cdot |F| - (|G| - 1) + \lambda(F)$$

$$> \Delta^r(2|F| - 1) + \lambda(F) \quad \text{(since } |G| - 1 < \Delta^r)$$

Since $V(D) = W$, $|F| = |\overline{W}|$ and $\lambda(F) = \lambda(\overline{W})$, we are done. \qed

Let $G^*$ be a spanning supgraph of $G$ such that

i. $\chi'(G^*) \leq k$ and $d_{G^*}(v) \leq k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for each $v \in V(G)$, and

ii. $|E(G^*)|$ is maximum subject to the two conditions above.

Since $k \geq \chi'(G) \geq \Delta + 2\Delta^r$ and $d_G(v) = \Delta - \lambda(v) \leq k - 2\Delta^r - \lambda(v) \leq k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for every $v \in V(G)$, we see that graph $G$ itself satisfies two conditions in i. So, such a graph $G^*$ exists. Since $\Delta \geq 2 > 1$, we have $\lceil \Delta^{1-r} \rceil \geq 2$. Condition i implies that $\Delta(G^*) \leq k - \lceil \Delta^{1-r} \rceil \leq k - 2$. By the assumption that $\chi'(G^*) \leq k$ and $|G^*| = |G|$ is odd, we have $2|E(G^*)| \leq k(|G| - 1)$. Thus, in order to prove that $G^*$ is $k$-exact-dense, we only need to show that $2|E(G^*)| \geq k(|G| - 1)$. Let $U = \{u \in V(G) : d_{G^*}(u) < k - \lceil \Delta^{1-r} \rceil - \lambda(u)\}$.

Claim 5.3. If $|U| \geq 2$, then $U$ is contained in a $k$-exact-dense subgraph of $G^*$. 

Proof. For any two distinct vertices $u, v \in U$, we add a new edge $e$ with endpoints $u, v$ to $G^*$. According to the definition of $U$, we have $d_{G^*}(u) = d_{G^*}(u) + 1 \leq k - [\Delta^1 - r] - \lambda(u)$ and $d_{G^*}(v) = d_{G^*}(v) + 1 \leq k - [\Delta^1 - r] - \lambda(v)$, which in turn gives $d_{G^*}(w) \leq k - [\Delta^1 - r] - \lambda(w)$ for any $w \in V(G)$. Consequently, $\Delta(G^*) \leq k - [\Delta^1 - r] \leq k - 2$. Since $|E(G^*)| > |E(G^*)|$, by the maximality of $|E(G^*)|$ we have $\chi'(G^*) > \chi(G^*) = k \geq \Delta(G^*) + 2$.

By Theorem 1.2, $G^*$ contains a subgraph $F$ with $|F| \geq 3$ odd such that $2|E(F)| > k(|F| - 1)$, which implies that $\chi'(F) > k$. Since $F - e \subseteq G^*$ and $\chi'(G^*) \leq k < \chi'(F)$, we have $e \in E(F) \setminus E(G^*)$, $\chi(F - e) = \chi'(G^*) = k$ and $2|E(F - e)| = k(|F| - 1)$. Hence, $F - e$ is a $k$-exact-dense subgraph of $G^*$ that contains both $u$ and $v$.

Fixing $u$, we have shown that for any vertex $v \in U$ with $v \neq u$, there is a maximal $k$-exact-dense subgraph in $G^*$ containing both $u$ and $v$. Since $\chi'(G^*) = k \geq \Delta(G^*) + 2$, by Lemma 2.9 these maximal $k$-exact-dense subgraphs are the same graph, and so Claim 5.3 holds.

By Claim 5.3, we consider the following three cases: (1) $U = \emptyset$; (2) $|U| = 1$; and (3) $|U| \geq 2$.

**Case 1:** $U = \emptyset$. In this case, we have $d_{G^*}(v) = k - [\Delta^1 - r] - \lambda(v)$ for all $v \in V(G)$. Hence,

$$2|E(G^*)| = \sum_{v \in V(G)} d_{G^*}(v) = (k - [\Delta^1 - r])|G| - \lambda(G)$$

$$= k(|G| - 1) + k - [\Delta^1 - r] \cdot |G| - \lambda(G).$$

Applying $\Delta > (2\Delta^r - 1)(|G| - 1) + \lambda(G) \geq k \geq \Delta + 2\Delta^r$, we get

$$k > (2\Delta^r - 1)|G| + \lambda(G) + 1 \geq [\Delta^1 - r] \cdot |G| + \lambda(G)$$

since $2\Delta^r - 1 > \Delta^r \geq [\Delta^1 - r]$. Therefore, $2|E(G^*)| > k(|G| - 1)$, giving a contradiction.

**Case 2:** $|U| = 1$. Let $U = \{u\}$ and $F^* = G^* - u$. Applying Claim 5.2 with $W = \{u\}$, we get $\Delta^* = \Delta^r(|G| - 1) + \lambda(G - u) \geq \Delta^r(|G| - 1) + \lambda(G - u)$. Hence,

$$2|E(G^*)| = \sum_{v \in V(G)} d_{G^*}(v) = \sum_{v \in V(G) \setminus \{u\}} (k - [\Delta^1 - r] - \lambda(v)) + d_{G^*}(u)$$

$$> (k - [\Delta^1 - r])(|G| - 1) - \lambda(G - u) + \Delta^r(|G| - 1) + \lambda(G - u)$$

$$\geq (k - [\Delta^1 - r] + \Delta^r)(|G| - 1) \geq k(|G| - 1),$$

which gives a contradiction.

**Case 3:** $|U| \geq 2$. By Claim 5.3, there is a maximal $k$-exact-dense subgraph $D^*$ containing $U$. If $D^* = G^*$, then we are done. We now suppose that
$D^* \neq G^*$, that is, $G^*$ is not $k$-exact-dense. Since $k \geq \chi'(G^*)$ \geq $\chi'(D^*) = k$, we have $\chi'(G^*) = k$. Not being $k$-exact-dense gives $2|E(G^*)| < k(|G| - 1)$. Let $F^* = G^* - V(D^*)$. Hence,

$$2|E(G^*)| < k(|D^*| + |F^*| - 1) = k(|D^*| - 1) + k \cdot |F^*|.$$ 

Since $D^*$ is $k$-exact-dense, $k(|D^*| - 1) = 2|E(D^*)|$. Given $U \subseteq V(D^*)$, we have $V(F^*) \cap U = \emptyset$. Consequently, $d_{G^*}(v) = k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for every vertex $v \in V(F^*)$. Hence,

$$k|F^*| = \sum_{v \in V(F^*)} d_{G^*}(v) + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$$

$$= 2|E(F^*)| + |E_{G^*}(D^*, F^*)| + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$$.

Therefore,

$$2|E(G^*)| < 2|E(D^*)| + 2|E(F^*)| + |E_{G^*}(D^*, F^*)| + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$$.

Combining this with $2|E(G^*)| = 2|E(D^*)| + 2|E(F^*)| + |E_{G^*}(D^*, F^*)|$, we get the following.

$$(10) \quad |E_{G^*}(D^*, F^*)| < \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$$.

Note that $|E_{G^*}(D^*, F^*)| = d_{G^*}(V(D^*)) \geq d_{G^*}(V(D^*))$. Applying Claim 5.2 with $W = V(D^*)$ and $W = V(F^*)$ and using $|F^*| \geq 2$, we get

$$d_{G^*}(V(D^*)) > \Delta^*(2|F^*| - 1) + \lambda(F^*) > \Delta^*|F^*| + \lambda(F^*) \geq \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$$.

Hence, $|E_{G^*}(D^*, F^*)| > \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$, giving a contradiction to (10). $\square$

6. Conclusion and Acknowledgment

A natural strength of Theorem 1.1 is whether there is a universal constant $c$ such that for a graph with maximum degree $\Delta$ sufficiently large, if $\chi'(G) \geq \Delta + c$ then $\chi''(G) = \chi'(G)$. Unfortunately, there are a couple of barriers in our proof such as Theorem 2.7 which limits the best bounds we might get is $\Delta + \Delta^{1/2}$. We thank Molloy and Reed for their book [10] that gave us several inspirational ideas.

References


*Email address: gchen@gsu.edu, yhao4@gsu.edu*

*Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303*