Disjoint odd cycles in cubic solid bricks*

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Abstract. Carvalho, Lucchesi and Murty (Journal of Combinatorial Theory, Series B 92 (2004) 319-324, Theorem 3.5) presented a proof of a theorem of Reed and Wakabayashi that *a brick G is non-solid if and only if there exists two vertex-disjoint odd cycles C_1 and C_2 such that $G - V(C_1 \cup C_2)$ has a perfect matching*. Consequently, every brick with no two vertex-disjoint odd cycles is solid. Recently, Lucchesi, Carvalho, Kothari and Murty (SIAM Journal on Discrete Mathematics, 32(2): (2018)1478-1501) constructed infinite families of solid bricks containing two vertex-disjoint odd cycles. Noticing that none of these graphs is cubic, they conjectured that *no cubic solid brick contains two vertex-disjoint odd cycles*. In this note, we present an infinite family graphs showing this conjecture fails. We further show that the minimum counterexample is unique, which has 12 vertices.

Keywords cubic graph, matching covered graphs, separating cut, tight cut, and solid brick.

1 Introduction

All graphs considered in this paper are simple graphs, i.e., graphs with finite number of vertices and no loops or parallel edges. We will generally follow the notation and terminology used by Bondy and Murty in [1]. Let $G = (V, E)$ be a graph, $X \subset V$, and $\overline{X} = V - X$ be the complement of X. The set of *boundary edges* of X, denoted by $\partial(X)$, is the set of edges with exact one end in X and one end in $\overline{X}$. Clearly, $\partial(X) = \partial(\overline{X})$ is an *edge-cut* of G, which is called *simple edge-cut* (SEC) of G. We call X and $\overline{X}$ the *shores* of $\partial(X)$. Let G/X and $G/\overline{X}$ be obtained from G by contracting X and $\overline{X}$, respectively, and call them $\partial(X)$-*contractions* of G. We call SEC $\partial(X)$ *trivial* if $|X| = 1$ or $|\overline{X}| = 1$.

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A connected graph with at least two vertices is **matching covered** if each of its edges is contained in some perfect matching. Let G be a matching covered graph. An SEC $\partial(X)$ of G is called a *separating cut* if both the $\partial(X)$-contractions of G are also matching covered, and is called a *tight cut* if $|\partial(X) \cap M| = 1$ for each perfect matching M of G. Obviously, every trivial cut is a tight cut and every tight cut is separating. We call a matching covered graph a **brick** if it is *nonbipartite* and free of *nontrivial tight cuts*, and a **brace** if it is *bipartite* and free of *nontrivial tight cuts*. Edmonds et al. [5] (also see Lovász [7], Szigeti[10] and Carvalho et al. [4]) showed that a graph G is a *brick* if and only if G is 3-connected and $G - x - y$ has a perfect matching for any two distinct vertices $x, y \in V(G)$.

A matching covered graph is **solid** if each of its separating cuts is tight. It can be shown that every bipartite matching covered graph is solid. So, a **solid brick** is a nonbipartite matching covered graph containing no non-trivial separating cut. Solid bricks have an interesting interpretation in terms of their perfect matching polytopes. For instance, Lovász [7] proved that any matching covered graph can be decomposed into a unique list of bricks and braces by a procedure called the tight cut decomposition procedure. We refer to [2, 3, 8] for literatures on solid bricks.

A graph is *odd-intercyclic* if it does not contain two vertex-disjoint odd cycles. Reed and Wakabayashi obtained the following characterization of solid bricks (unpublished), and Carvalho, Lucchesi and Murty [2] presented a proof of it.

Theorem 1. *A brick G is non-solid if and only if it has two vertex-disjoint odd cycles C_1 and C_2 such that $G - (V(C_1) \cup V(C_2))$ has a perfect matching.*

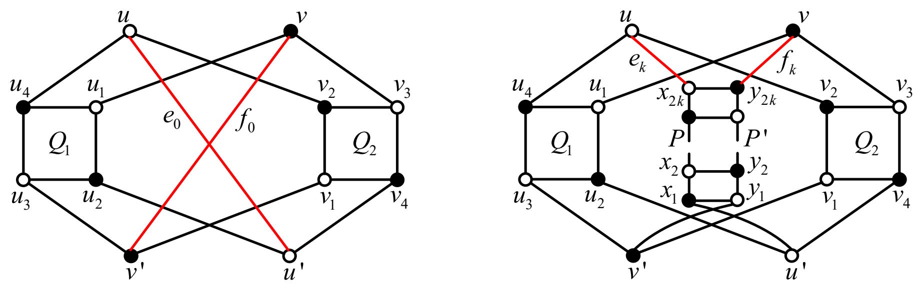
Theorem 1 has the following consequence.

Corollary 2. *Every odd-intercyclic brick is solid.*

Recently, Lucchesi, Carvalho, Kothari and Murty [8] showed the converse of Corollary 2 is not true by providing an infinite family of solid non-odd-intercyclic bricks. They noticed that none of these examples are cubic graphs (3-regular graphs). This observation led them to believe the converse of Corollary 2 holds for cubic graphs and made the following conjecture.

Conjecture 3. *Every cubic solid brick is odd-intercyclic.*

In this note, we construct an infinite family of graphs $\mathcal{G} = \{G_k \mid k \geq 0\}$ and prove that every G_k is a counterexample to Conjecture 3.



(a) (b)

Figure 1: (a) G_0; (b) G_k, $(k \geq 1)$.

Construction of $\mathcal{G}$. We start with G_0, let $Q_1 = u_1u_2u_3u_4u_1$ and $Q_2 = v_1v_2v_3v_4v_1$ be two 4-cycles, and let $P = u'u$ and $P' = v'v$ be two new edges, that are vertex-disjoint from Q_1 and Q_2. Let G_0 be obtained from $P \cup P' \cup Q_1 \cup Q_2$ by adding edges $uu_4, uv_2, u'u_2, u'v_4, vu_1, vv_3, v'u_3$ and $v'v_1$. Graph G_0 is depicted in Figure 1 (a). In G_0, $C_1 := uu'u_2u_3u_4u$ and $C_2 := vv'v_1v_2v_3v$ are two vertex-disjoint 5-cycles.

For each integer $k \geq 1$, let G_k be obtained from G_0 by replacing edges $P = u'u$ and $P' = v'v$ with two new paths of $2k+2$ vertices: $P = u'x_1x_2 \dots x_{2k}u$ and $P' = v'y_1y_2 \cdots y_{2k}v$, and adding $2k$ edges $\{x_1y_1, x_2y_2, \dots, x_{2k}y_{2k}\}$. Clearly, in G_k, $C_1 := uPu'u_2u_3u_4u$ and $C_2 := vP'v'v_1v_2v_3v$ are two vertex-disjoint odd cycles with $2k+5$ vertices.

The following result shows that each G_k in $\mathcal{G}$ is a counterexample to Conjecture 3.

Theorem 4. *For each $k \geq 0$, G_k is a solid brick.*

Moreover, the following result shows that G_0 is the unique counterexample with minimum number of vertices.

Theorem 5. *Every cubic solid brick that is not odd-intercyclic has 12 or more vertices. Moreover, if it has precisely 12 vertices then it is isomorphic to G_0.*

The proofs of Theorems 4 and 5 will be given in Section 3 after we present some properties concerning matching covered cubic graphs in Section 2.

2 Preliminaries

In this section, we present some properties of cubic matching covered graphs which will be used in the proof of the main result. Let k be a positive integer. Recall that a graph G is *k-edge-connected* if $|\partial(X)| \geq k$ for every edge cut of G. A graph G is *essentially $(k+1)$-edge-connected* if it is k-edge-connected, and $|\partial(X)| \geq k+1$ for every non-trivial edge cut $\partial(X)$. For any m, an edge cut with m edges is called an *m-cut*. The following is a corollary of the well-known Tutte's Perfect Matching Theorem.

Lemma 6 (Plesník [9])**.** *Every* 2*-edge-connected cubic graph is matching covered.*

Note that if a 2-edge-connected cubic graph contains a 2-cut, then it contains a non-trivial 3-cut. So, a 2-edge-connected cubic graph is *essentially* 4*-edge-connected* if it does not contain nontrivial 3-cuts. The brick $\overline{C}_6$ (the complement graph of a cycle with six vertices) shows that a cubic brick maybe not essentially 4-edge-connected. However, by adding the condition of being solid, we have the following result.

Lemma 7. *Every solid cubic brick is essentially* 4*-edge-connected.*

Proof. Suppose on the contrary there exists a solid cubic brick G that contains a nontrivial edge cut $D = \partial(X)$ with $|D| \leq 3$. Since all bricks are 3-connected, D contains three independent edges. Thus, each D-contraction of G is a simple cubic graph. Now, we claim that each D-contraction of G is 2-edge-connected. Otherwise, assume without loss generality that e is a cut-edge of G/X. Since G is 3-connected, one end of e must be new contracted vertex of X. Again, since G is 3-connected, $G[\overline{X}]$ is connected, which in return shows e is not a cut-edge of G/X, giving a contradiction. By Lemma 6, each D-contraction of G is matching covered. Whence D is a non-trivial separating cut of G. So G is not solid, giving a contradiction. □

Since a triangle of a cubic graph always leads to a 3-cut, the following simple observation is an immediate consequence of the Lemma 7.

Lemma 8. *If a cubic solid brick contains a triangle, then it is K_4.*

Let G be a graph, and let $e := xy$ and $e' := x'y'$ be two independent edges of G. The following two-step operation is called a *strict edge-extension* of G: (1) subdividing edges e and e' by inserting new vertices v and v', respectively, and (2) adding a new edge vv'.

Lemma 9 (Bondy and Murty [1], Exercise 9.4.7)**.** *A strict edge-extension of an essentially 4-edge-connected cubic graph G is also essentially 4-edge-connected.*

Recently, Kothari, Carvalho, Little and Lucchesi [6] proved that each tight cut of a 2-edge-connected cubic graph is a 3-cut. This implies the following theorem immediately.

Theorem 10. [6] *Every essentially 4-edge-connected cubic nonbipartite graph is a brick.*

3 Proofs of Theorems 4 and 5

3.1 Proof of Theorem 4.

We first prove the following three claims.

Claim 1. *Every graph in $\mathcal{G}$ is essentially 4-edge-connected.*

Proof. For each positive integer $n \geq 3$, let Q_n denote the n-hypercube graph. It is well-known that Q_n is essentially 4-edge-connected. Note that the graph $G_0 \in \mathcal{G}$ is a strict edge-extension of Q_3. By Lemma 9, G_0 is essentially 4-edge-connected. Furthermore, each member $G_i \in \mathcal{G}$, where $i \geq 1$, is a strict edge-extension of the preceding member G_{i-1}. Thus, by Lemma 9, each member of $\mathcal{G}$ is essentially 4-edge connected. $\square$

Claim 2. *Every graph in $\mathcal{G}$ is a brick.*

Proof. Let $G_k \in \mathcal{G}$. Clearly, $G_k \in \mathcal{G}$ is cubic. Since G_k contains two vertex-disjoint odd cycles, it is nonbipartite. By Claim 1, G_k is essentially 4-edge-connected. Then, by Theorem 10, G_k is a brick. $\square$

Claim 3. *Every graph in $\mathcal{G}$ is solid.*

Proof. Let G_k be an arbitrary graph in $\mathcal{G}$. Following the definition of G_k, e_k and f_k, we see that $G - e_k - f_k$ is a bipartite graph. Let C_1 and C_2 be two arbitrary vertex-disjoint odd cycles of G_k. Then, each of C_1 and C_2 contains precisely one of e_k or f_k. We may thus assume that C_1 contains e_k and C_2 contains f_k. Notice that in $G - V(Q_1 \cup Q_2)$, two vertex-disjoint paths from $\{u, v\}$ to $\{u', v'\}$ are uniquely determined, which are $ux_{2k}x_{2k-1} \ldots x_1u'$ and $vy_{2k}y_{2k-1} \ldots y_1v'$; and in $G^* = G - \{x_1, y_2, \ldots, x_{2k}, y_{2k}\}$, if R_1 and R_2 are two vertex-disjoint paths connecting u to u' and v to v', respectively, then either $R_1 \cap Q_1 = \emptyset$ and $R_2 \cap Q_2 = \emptyset$ or $R_1 \cap Q_2 = \emptyset$ and $R_2 \cap Q_1 = \emptyset$, and in either case $G^* - V(R_1 \cup R_2) \cong \overline{K}_2$ is the union of two isolate vertices. So, $G - V(C_1 \cup C_2)$ is a union of two isolate vertices which does not have a perfect matching. By Theorem 1, G_k is solid. $\square$

Combining Claims 2 and 3, we see that G_k is a cubic solid brick, which completes the proof of Theorem 4. $\square$

Following, the proof of Theorem 5 will be given.

3.2 Proof of Theorem 5

Let $G = (V, E)$ be a non-odd-intercyclic cubic solid brick with minimum number of vertices. Let $n = |V|$. By Lemma 8, G does not contain a triangle, so the length of any odd cycle of G is at least 5. Let C_1 and C_2 be any two vertex-disjoint odd cycles of G. So, $n \geq |C_1| + |C_2| \geq 10$. If $n = 10$, then $|C_1| = |C_2| = 5$. This implies that $G - V(C_1) - V(C_2)$ is the empty graph. Therefore, G is non-solid by Theorem 1, which giving a contradiction. Thus, $n \geq 12$.

Suppose $n = 12$. We show that G is isomorphic to G_0. If one of cycles C_1 and C_2 has more than 5 vertices, then $V(G) = V(C_1 \cup C_2)$. By Theorem 1, G is non-solid, a contradiction. Thus, $|C_1| = |C_2| = 5$ and, again by Theorem 1, $G - V(C_1) - V(C_2) \cong \overline{K}_2$ is the union two isolated vertices, say x and y. Let $C_1 = x_1x_2 \cdots x_5x_1$ and $C_2 = y_1y_2 \cdots y_5y_1$. Since G is triangle-free by Lemma 8, both C_1 and C_2 are induced cycles, and the vertex x is adjacent to exactly two vertices in one of C_1 and C_2, and one vertex in the other. So does y. If both x and y are adjacent to two vertices in the same cycle, say C_1, then C_2 has a chord (since G is cubic), giving a contradiction. So, we may assume that $N(x) \cap V(C_1) = \{x_1, x_3\}$, $N(y) \cap V(C_2) = \{y_1, y_3\}$. Let $C_1' := G[(V(C_1) \cup x) \setminus x_2]$ and $C_2' := G[(V(C_2) \cup y) \setminus y_2]$.

Claim 4. *$\{x, y, x_2, y_2\}$ is an independent vertex set of G.*

Proof. Note that in $G[\{x, y, x_2, y_2\}]$, there are only three possible edges x_2y_2, xy_2 and yx_2.

If $x_2y_2 \in E$, then $G - V(C_1' \cup C_2')$ has a perfect matching x_2y_2; if $xy_2 \in E$, then $G - V(C_1 \cup C_2')$ has a perfect matching xy_2; if $x_2y \in E$, then $G - V(C_1' \cup C_2)$ has a perfect matching x_2y. In either of the above three cases, by Theorem 1, G is non-solid, a contradiction. $\square$

Since G is cubic and triangle-free, there is a perfect matching between $X = \{x, x_2, x_4, x_5\}$ and $Y = \{y, y_2, y_4, y_5\}$. Relabeling vertices in $V(C_1)$ and vertices in $V(C_2)$ if necessary, we may assume $xy_5 \in E$ and $yx_5 \in E$. Then, $x_2y_4 \in E$ and $x_4y_2 \in E$. So, G is isomorphic to G_0, where cycle $xx_1x_2x_3x_4x_5yy_1y_2y_3y_4y_5x$ corresponds to cycle $u_1u_2u_3u_4uu'v_4v_3v_2v_1v'vu_1$. $\square$

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References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[2] M. H. de Carvalho, C. L. Lucchesi and U. S. R. Murty. The perfect matching polytope and solid bricks. *Journal of Combinatorial Theory, Series B*, 92(2):319–324, 2004.

[3] M. H. de Carvalho, C. L. Lucchesi and U. S. R. Murty. How to build a brick. *Discrete Mathematics*, 306(19):2383–2410, 2006.

[4] M. H. de Carvalho, C. L. Lucchesi and U. S. R. Murty. On tight cuts in matching covered graphs. *Journal of Combinatorics*, 9(1):163–184, 2018.

[5] J. Edmonds, L. Lovász and W. R. Pulleyblank. Brick decompositions and the matching rank of graphs. *Combinatorica*, 2(3):247–274, 1982.

[6] N. Kothari, M. H. de Carvalho, C. H. C. Little and C. L. Lucchesi. On essentially 4-edge-connected cubic bricks. *Preprint available at https://arxiv.org/pdf/1803.08713.pdf*, 2018.

[7] L. Lovász. Matching structure and the matching lattice. *Journal of Combinatorial Theory, Series B*, 43(2):187–222, 1987.

[8] C. L. Lucchesi, M. H. de Carvalho, N. Kothari and U. S. R. Murty. On two unsolved problems concerning matching covered graphs. *SIAM Journal on Discrete Mathematics*, 32(2):1478–1504, 2018.

[9] J. Plesník. Connectivity of regular graphs and the existence of 1-factors. *Mat. Časopis Sloven. Akad. Vied*, 22:310–318, 1972.

[10] Z. Szigeti. Perfect matchings versus odd cuts. *Combinatorica*, 22(4):575–589, 2002.