On the average degree of edge chromatic critical graphs

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Abstract

Let $G$ be a simple graph, and let $\chi'(G)$ and $\Delta(G)$ denote the chromatic index and the maximum degree of $G$, respectively. A graph $G$ is a critical class two graph if $\chi'(G) = \Delta(G)+1$ and $\chi'(H) \leq \Delta(G)$ for every proper subgraph $H$ of $G$. Let $\overline{d}(G)$ denote the average degree of $G$, i.e., $\overline{d}(G) = 2|E(G)|/|V(G)|$. Vizing in 1968 conjectured that $\overline{d}(G) \geq \Delta(G) - 1 + \frac{3}{n}$ if $G$ is a critical class two graph of order $n$. In this paper, we prove that $\overline{d}(G) \geq \frac{3}{4}\Delta(G) - 8$ if $G$ is a critical class two graph. Let $\delta(G)$ denote the minimum degree of $G$. We show that there exist two functions $D$ and $D_0$ such that for any $\epsilon \in (0, 1)$, if $G$ is a critical class two graph with $\Delta(G) \geq D(\epsilon)$ and $\delta(G) \geq D_0(\epsilon)$ then $\overline{d}(G) \geq (1 - \epsilon)\Delta(G)$. We will give two specific functions satisfying the statement above in the paper. Moreover, we show that if $G$ is a critical class two graph and $\delta(G) \geq (\log \Delta(G))^2$, then $\overline{d}(G) \geq \Delta(G) - o(\Delta(G))$.

Keywords: edge coloring; critical class two graphs; Vizing’s average degree conjecture

1 Introduction

All graphs in this paper are finite, undirected and simple. We will generally follow Stiebitz et al. \cite{12} for notation and terminology. An edge coloring of a graph $G$ is to color the edges of $G$ such that each edge receives a color and adjacent edges, that is, distinct edges having a common end, receive different colors. The minimum number of colors in such an edge coloring of $G$ is called the chromatic index of $G$, written $\chi'(G)$. As proved by Holyer \cite{6} the determination of the chromatic index is an NP-hard problem. On the other hand, the maximum degree $\Delta(G)$, that is, the maximum number of edges of $G$ having a common end, is an obvious lower bound of $\chi'(G)$. Vizing \cite{13}, and independently Gupta \cite{11}, proved that $\chi'(G) \leq \Delta(G) + 1$ for any graph $G$. This leads to a natural classification of graphs. Following Fiorini and Wilson \cite{4}, a graph $G$ is said to be of class one if $\chi'(G) = \Delta(G)$ and of class two if $\chi'(G) = \Delta(G) + 1$.

In investigating the classification problem, critical graphs are of particular interest. A graph $G$ is called critical if $\chi'(H) < \chi'(G)$ for every proper subgraph $H$ of $G$. An edge $e$ of $G$ is called a critical edge of $G$ if $\chi'(G - e) < \chi'(G)$. It is easy to see that a connected graph $G$ is critical if and only if every edge of $G$ is critical. The only critical class one graph with maximum degree $\Delta$ is the star $K_{1,\Delta}$. There is no critical class two graph with maximum degree $\Delta = 1$, and the only critical class two graphs with maximum degree $\Delta = 2$ are the odd cycles. Critical graphs of class two have rather more structure than arbitrary graphs of class two, and it follows from Vizing’s theorem that every graph of class two contains a critical graph of class two with the same maximum degree as a subgraph. These facts can be used when proving results in relation to the classification problem. In the remainder of this paper, a critical class two graph will be called $\Delta$-critical graph for convenience.
Let $\bar{d}(G)$ denote the average degree of a graph $G$. So $2|E(G)| = \bar{d}(G) \cdot n$ for any graph $G$ with order $n$. The following is a well-known conjecture of Vizing proposed in 1968.

Conjecture 1. [Vizing’s Average Degree Conjecture] If $G$ is a $\Delta$-critical graph, then $\bar{d}(G) \geq \Delta(G) - 1 + \frac{3}{n}$.

The conjecture has been verified for all graphs $G$ with $\Delta(G) \leq 6$, see [7, 4, 8, 9]. The lower bounds of $\bar{d}(G)$ for $\Delta$-critical graphs have been heavily studied starting with the work of Fiorini [3] and Haile [5]. For long time, the best known result was due to Woodall [15], who showed that $\bar{d}(G) \geq \frac{2}{3}(\Delta(G) + 1)$ for every $\Delta$-critical graph. His proof depends on the improvement of Vizing’s Adjacency Lemma. Woodall also provided infinitely many examples demonstrating that his result cannot be improved by the use of his new adjacency lemmas and Vizing’s Adjacency Lemma alone. Using the simple broom method developed in [2], Cao et al. [1] showed that $\bar{d}(G) \geq 0.69242\Delta(G) - 0.1701$ for every $\Delta$-critical graph with $\Delta(G) \geq 66$. In this paper we develop a few new adjacency lemmas and obtain the following three results on Vizing’s Average Degree Conjecture.

Theorem 1. If $G$ is a $\Delta$-critical graph, then $\bar{d}(G) \geq \frac{3}{4}\Delta(G) - 8$.

Let $\delta(G)$ denote the minimum degree of graph $G$. For any constant $\epsilon \in (0, 1)$, let $c_0(\epsilon) = \left\lceil \frac{1-\epsilon}{\epsilon} \right\rceil$. Let $c_1(\epsilon) = c_0(\epsilon) - 1$. We will use $c_0, c_1$ instead of $c_0(\epsilon), c_1(\epsilon)$ if $\epsilon$ is clear.

Theorem 2. There exist two functions $D$ and $D_0$ from $(0, 1)$ to $\mathbb{R}$ such that for any positive real number $\epsilon \in (0, 1)$, if $G$ is a $\Delta$-critical graph with $\Delta(G) \geq D(\epsilon)$ and $\delta(G) \geq D_0(\epsilon)$, then $\bar{d}(G) \geq (1 - \epsilon)\Delta(G)$.

In the proof, we show that Theorem 2 is true with the following two functions:

$$D_0(\epsilon) = \begin{cases} 2\left(\frac{1}{\epsilon}\right)^3 + 2\sqrt{\left(\frac{1}{\epsilon}\right)^6 - \left(\frac{1}{\epsilon}\right)^3} & \text{if } \epsilon < \frac{1}{3}, \\ 2 & \text{if } \epsilon \geq \frac{1}{3}, \end{cases}$$

$$D(\epsilon) = \max \left\{ f(\epsilon), \frac{3c_0 + 1}{\rho^2}, \frac{N + c_0}{\epsilon^3} \right\},$$

where $\rho = \frac{c^3}{2c^3 + 1}$, $N = (c_0 + 1)(\frac{1}{\rho} + 1)^{3c_0 + 1}$ and $f(\epsilon) = 32/\epsilon$ if $\epsilon > \frac{15}{16}$ and $f(\epsilon) = \frac{1}{\epsilon^2}(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1)$ otherwise.

Consider the case when $\epsilon < \frac{1}{3}$. Since $c_0 + 1 \geq \frac{1}{\rho} > c_0 = \left\lceil \frac{1-\epsilon}{\epsilon} \right\rceil \geq 3$ and $\frac{1}{\rho} = \frac{1}{\epsilon} + \frac{1}{c_0}$, we have $D(\epsilon) = \frac{N + c_0}{\epsilon^3}$ by comparing the three functions above. To avoid unnecessary and complicated calculation, we point out a rough upper bound of $\frac{N + c_0}{\epsilon^3}$ when $\epsilon < \frac{1}{3}$ which is $(\frac{3}{\epsilon^3})^\frac{2}{3}$.

Corollary 3. Let $G$ be a $\Delta$-critical graph with maximum degree $\Delta$. If $\Delta > 10^{972}$ and $\delta(G) \geq (\log \Delta)^\frac{3}{2}$, then $\bar{d}(G) \geq \Delta - \frac{2\Delta}{(\log \Delta)^\frac{3}{2}}$.

Proof. Let $\epsilon = (\log \Delta)^\frac{1}{3}$, since $\Delta > 10^{972}$, $0 < \epsilon < \frac{1}{3}$. Then $\log \Delta = \frac{12}{\epsilon^4} \geq \frac{4}{\epsilon} \log \frac{3}{\epsilon^3} \geq \log D(\epsilon)$, so that $\Delta \geq D(\epsilon)$. Also $(\log \Delta)^\frac{3}{2} > \frac{6}{\epsilon^3} > D_0(\epsilon)$. Thus the result follows from Theorem 2.

Theorem 2 will be proved in Section 3, assuming the truth of Lemmas 7 and 8, which are proved in Section 5. In a similar way, Theorem 1 will be proved in Section 4, assuming the truth of Lemmas 16 and 17, which are proved in Section 6.
2 Coloring preliminaries and adjacency lemmas

Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For a vertex $v \in V(G)$, let $N(v) = \{w \in V(G) : vw \in E(G)\}$ be the neighborhood of $v$ in $G$, and let $d(v) = |N(v)|$ be the degree of $v$ in $G$. Let $N[v] = N(v) \cup \{v\}$. A vertex $v \in V(G)$ is called a $d$-vertex if its degree is $d$. For a vertex set $T$, let $N(T) = \bigcup_{v \in T} N(v)$. For a non-negative integer $k$, let $[1,k] = \{1,2,\ldots,k\}$ and let $\mathcal{C}^k(G)$ denote the set of all edge colorings of $G$ with color set $[1,k]$. So $\varphi \in \mathcal{C}^k(G)$ if and only if $\varphi$ assigns to each edge $e \in E(G)$ a color $\varphi(e) \in [1,k]$ such that $\varphi(e) \neq \varphi(f)$ for any two adjacent edges $e$ and $f$ of $G$. Note that $\chi'(G)$ is the smallest $k$ such that $\mathcal{C}^k(G) \neq \emptyset$.

In the reminder of this section, we always assume that $G$ is a $\Delta$-critical graph with maximum degree $\Delta$, $xy \in E(G)$, and $\varphi \in \mathcal{C}^\Delta(G-xy)$. We will re-state these assumptions in each lemma for their completeness.

For a vertex $v \in V(G)$, let $\varphi(v) = \{\varphi(vw) : w \in N(v)\}$ and $\bar{\varphi}(v) = [1,\Delta] \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors present at $v$ and $\bar{\varphi}(v)$ the set of colors missing at $v$ with respect to $\varphi$. For a vertex set $T$, let $\bar{\varphi}(T) = \bigcup_{v \in T} \bar{\varphi}(v)$. A set $X \subseteq V(G)$ is called elementary with respect to $\varphi$ if the sets $\bar{\varphi}(v)$ with $v \in X$ are pairwise disjoint. For a color $\alpha \in [1,\Delta]$, let $E_\alpha$ denote the set of edges $e$ of $G$ with $\varphi(e) = \alpha$ and call it a color class with respect to $\varphi$. Clearly, $E_\alpha$ is a matching of $G$. So if $\alpha$ and $\beta$ are two colors, then the spanning subgraph $H$ of $G$ with edge set $E_\alpha \cup E_\beta$ has maximum degree at most 2, so every component of $H$ is either a path or an even cycle (whose edges are colored alternately with $\alpha$ and $\beta$) and we refer to such a component as an $(\alpha,\beta)$-chain of $G$ with respect to $\varphi$. For a vertex $v$ of $G$, let $P_v(\alpha,\beta,\varphi)$ denote the unique component of $H$ that contains the vertex $v$. Let $\varphi' = \varphi/P_v(\alpha,\beta,\varphi)$ denote the mapping obtained from $\varphi$ by switching the colors $\alpha$ and $\beta$ on the edges of $P_v(\alpha,\beta,\varphi)$. Then, clearly, $\varphi' \in \mathcal{C}^\Delta(G-xy)$ is an edge coloring of $G-xy$ with color set $[1,\Delta]$, too. This switching operation is called a Kempe change.

A multi-fan at $x$ with respect to $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$ is a sequence $F = (e_1,y_1,\ldots,e_p,y_p)$ with $p \geq 1$ consisting of edges $e_1,e_2,\ldots,e_p$ and vertices $x,y_1,y_2,\ldots,y_p$ satisfying the following two conditions:

- The edges $e_1,e_2,\ldots,e_p$ are distinct, $e_1 = e$ and $e_i \in xy_i$ for $i = 1,\ldots,p$.
- For every edge $e_i$ with $2 \leq i \leq p$, there is a vertex $y_j$ with $1 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.

Notice that multi-fan is slightly more general than Vizing-fan which requires $j = i-1$ in the second condition. The following lemma shows that a multi-fan is elementary. The proof can be found in the book [12].

**Lemma 1.** [Stiebitz, Scheide, Toft and Favhholdt [12]] Let $G$ be a $\Delta$-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. Let $F = (e_1,y_1,\ldots,e_p,y_p)$ be a multi-fan at $x$ with respect to $e$ and $\varphi$. Then the following statements hold:

(a) $\{x,y_1,y_2,\ldots,y_p\}$ is elementary.

(b) If $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y_i)$ for some $i$, then $P_x(\alpha,\beta,\varphi) = P_{y_i}(\alpha,\beta,\varphi)$.

For a vertex $v \in V(G)$ and a given positive number $q$, let

$$\varphi^{bad}(v) = \bar{\varphi}(v) \cup \{\varphi(v') : v' \in N(v) \text{ with } d(v') < q\},$$

$$\sigma_q(x,y) = |\{z \in N(y) \setminus \{x\} : d(z) \geq q\}|.$$
Lemma 2. [Vizing’s Adjacency Lemma [14]] If $G$ is a $\Delta$-critical graph, then $\sigma_{\Delta}(x, y) \geq \Delta(G) - d(x) + 1$ for every $xy \in E(G)$.

A Kierstead path with respect to $e = xy$ and $\varphi \in C^\Delta(G-e)$ is a sequence $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, e_2, \ldots, e_p$ and vertices $y_0, y_1, \ldots, y_p$ satisfying the following two conditions:

- The vertices $y_0, y_1, \ldots, y_p$ are distinct, $e_1 = e$ and $e_i = y_{i-1}y_i$ for $1 \leq i \leq p$.
- For every edge $e_i$ with $2 \leq i \leq p$, there is a vertex $y_j$ with $0 \leq j < i$ such that $\varphi(e_i) \in \varphi(y_j)$.

Notice that a Kierstead path with 3 vertices is a special multi-fan with center $y = y_1$. For a Kierstead path with 4 vertices, the following result was obtained.

Lemma 3. [Kostochka and Stiebitz [12], Luo and Zhao [10]] Let $G$ be a $\Delta$-critical graph, $e_1 = y_0y_1$ be an edge critical graph, and $\varphi \in C^\Delta(G-e_1)$. If $K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to $e_1$ and $\varphi$, then $V(K)$ is elementary unless $d(y_1) = d(y_2) = \Delta(G)$, in which case, all colors in $\varphi(y_1), \varphi(y_2)$ and $\varphi(y_3)$ are distinct except one possible common missing color in $\varphi(y_3) \cup (\varphi(y_0) \cup \varphi(y_1))$.

A simple broom with respect to $e = xy$ and $\varphi \in C^\Delta(G-e)$ is a sequence $B = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 3$ such that $(y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to $e$ and $\varphi$ whenever $3 \leq i \leq p$.

Lemma 4. [Chen, Chen and Zhao [2]] Let $G$ be a $\Delta$-critical graph, $e_1 = y_0y_1 \in E(G)$ and $\varphi \in C^\Delta(G-e_1)$ and let $B = \{y_0, e_1, y_1, e_2, y_2, \ldots, e_p, y_p\}$ be a simple broom. If $|\varphi(y_0) \cup \varphi(y_1)| \geq 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary under $\varphi$.

By applying the lemma above, Cao et al. proved the following result.

Lemma 5. [Cao, Chen, Jiang, Liu, Lu [1]] Let $G$ be a $\Delta$-critical graph with maximum degree $\Delta$, $xy \in E(G)$ and $\varphi \in C^\Delta(G-xy)$. For a positive number $q$ with $d(x) < q \leq \Delta - 1$, let $Z^*_q = \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \varphi(y) \text{ and } d(z) > q\}$. Then the following three inequalities hold.

\[
|Z^*_q| \geq \Delta - d(y) + 1 - \left\lceil \frac{d(x) + d(y) - \Delta - 2}{\Delta - q} \right\rceil \quad (1)
\]
\[
\sum_{z \in Z^*_q} (d(z) - q) \geq (\Delta - d(y) + 1)(\Delta - q) - d(x) - d(y) + \Delta + 2 \quad (2)
\]
\[
\text{For all } z \in Z^*_q, \quad \sigma_q(x, z) \geq 2\Delta - d(x) - d(y) + 1 - \left\lceil \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right\rceil \quad (3)
\]

Let $\alpha, \beta,$ and $\gamma$ denote three arbitrary colors. For any edge incident to $x$ with color $\alpha$, we use $x_\alpha$ to denote the other end of it. Moreover, for any edge incident to $x_\alpha$ with color $\beta$, we use $x_{\alpha\beta}$ to denote the other end of it. We use $y_\gamma$ to denote the other end of the edge incident with $y$ with color $\gamma$. Since $\{x, y\}$ is elementary, $\varphi(x), \varphi(y)$ and $\varphi(x) \cap \varphi(y)$ form a partition of the color set $[1, \Delta]$. We divide $\varphi(x) \cap \varphi(y)$ into two sets of colors as the following:

- $A_{\varphi}(x, y, q) = \{\alpha \in \varphi(x) \cap \varphi(y) : d(y_\alpha) < q\}$,
- $B_{\varphi}(x, y, q) = \{\beta \in \varphi(x) \cap \varphi(y) : d(y_\beta) \geq q\}$. 

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Let \( S_\varphi(x, y, q) = A_\varphi(x, y, q) \cup \bar{\varphi}(x) \cup \bar{\varphi}(y) \). We will use \( A_\varphi(q), B_\varphi(q) \) and \( S_\varphi(q) \) in short if \( x, y \) are clear. Note that \( S_\varphi(q) \cap \varphi(x) = A_\varphi(q) \cup \varphi(y) \) since the set \( \{ x, y \} \) is elementary and so \( \bar{\varphi}(y) \subseteq \varphi(x) \). When \( q \) is clear, let \( N_S(x) = \{ \alpha : \alpha \in \varphi(q) \cap \varphi(x) \} \) and \( N_B(x) = \{ \beta : \beta \in B_\varphi(q) \} \). Clearly \( N(x) = N_S(x) \cup N_B(x) \cup \{ y \} \).

In this paper, we will use the following operation very often.

\( (\eta, \theta, \delta) \)-switching: Let \( xy \) be an edge in a \( \Delta \)-critical graph \( G \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \). Assume \( \eta, \theta, \delta \) are three distinct colors such that \( \eta \in A_\varphi(y), \theta \in \bar{\varphi}(y) \) and \( \delta \in \bar{\varphi}(y) \cap \varphi(x) \). Form \( \varphi_1 = \varphi/P_{y_\eta}(\delta, \theta, \varphi) \), so that \( \theta \in \bar{\varphi}_1(y_\eta) \), and obtain \( \varphi_2 \) by recoloring the edge \( yy_\eta \) with \( \theta \). (If \( \theta \in \bar{\varphi}_1(y_\eta) \) then \( \varphi_1 = \varphi \) and the only edge recolored is \( yy_\eta \).

**Lemma 6.** Suppose that an \((\eta, \theta, \delta)\)-switching turns \( \varphi \) into \( \varphi_1 \). Then \( \varphi_1(e) = \varphi(e) \) for every edge \( e \) incident with \( x \) or \( x_\theta \), or incident with \( y \) except \( e = yy_\eta \). Moreover, \( S_{\varphi_1}(q) = S_\varphi(q) \), and for every edge \( e \), \( \varphi_1(e) \in S_{\varphi_1}(q) \) if and only if \( \varphi(e) \in S_\varphi(q) \).

**Proof.** Since \( P_x(\delta, \theta, \varphi) = P_y(\delta, \theta, \varphi) \) (by Lemma 1), \( P_{y_\eta}(\delta, \theta, \varphi) \) is disjoint from it. For every edge \( e \) incident with \( x \) or \( y \) or \( x_\theta \), \( \varphi_1(e) = \varphi(e) \) unless \( e = yy_\eta \). It follows that \( \delta \in \bar{\varphi}(x) = \bar{\varphi}_1(x), \eta \in A_\varphi(q) \cap \bar{\varphi}_1(y), \) \( \theta \in \bar{\varphi}(y) \cap A_\varphi(q) \), so that \( \{ \delta, \eta, \theta \} \subseteq S_\varphi(q) \cap S_{\varphi_1}(q) \). Since edges of other colors are unaffected by the switch, \( S_{\varphi_1}(q) = S_\varphi(q) \) and the result follows.

We briefly explain how will we use this operation. For a color \( \eta \in A_\varphi(q) \), we can show that under certain conditions there exist \( \theta \in \bar{\varphi}(y) \) and \( \delta \in \bar{\varphi}(y) \cap \bar{\varphi}(x) \) such that the switching above is valid. Then after the switching, \( \eta \) becomes to a missing color of \( y \). This means in some sense, we may treat the colors in \( A_\varphi(q) \) as same as the colors in \( \bar{\varphi}(y) \).

## 3 Proof of Theorem 2

**Theorem 2.** There exist two functions \( D \) and \( D_0 \) from \((0, 1)\) to \( \mathbb{R} \) such that for any positive real number \( \epsilon \in (0, 1) \), if \( G \) is a \( \Delta \)-critical graph with \( \Delta(G) \geq D(\epsilon) \) and \( \delta(G) \geq D_0(\epsilon) \), then \( d(G) \geq (1 - \epsilon)\Delta(G) \).

In this section we will prove Theorem 2 based on Lemma 7 and Lemma 8. Let \( \epsilon \in (0, 1) \), and let \( D(\epsilon) \) and \( D_0(\epsilon) \) be defined as in Section 1. Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \geq D(\epsilon) \) and minimum degree \( \delta(G) \geq D_0(\epsilon) \). Let \( q = (1 - \epsilon)\Delta \). For a vertex \( v \in V(G) \), let \( d_{<q}(v) = |\{ w \in N(v) : d(w) < q \}| \). We will use the discharging method to prove the theorem and initially assign to each vertex \( x \) of \( G \) a charge \( M(x) = d(x) \) and redistribute the charge according to the following rule:

- **Rule of Discharge:** Each vertex \( y \) with degree larger than \( q \) distributes \( d(y) - q \) equally among all neighbors of \( y \) with degree less than \( q \).

Denote by \( M'(x) \) the resulting charge on each vertex \( x \). It is easy to see that Theorem 2 holds if \( M'(x) \geq q \) for every \( x \in V(G) \) with \( d(x) \geq D_0(\epsilon) \). We divide this statement into two claims to prove it.

**Claim 3.1.** For every \( x \in V(G) \) with \( d(x) \geq \epsilon \Delta \), \( M'(x) \geq q \).

**Claim 3.2.** For every \( x \in V(G) \) with \( D_0(\epsilon) \leq d(x) \leq \epsilon \Delta \), \( M'(x) \geq q \).

### 3.1 Part I: Claim 3.1.

#### 3.1.1 Lemmas (Part I)

Recall that \( \varphi_{\text{bad}}(v) = \bar{\varphi}(v) \cup \{ \varphi(v') : v' \in N(v) \text{ with } d(v') < q \} \) and \( c_0 = c_0(\epsilon) = \left\lceil \frac{1}{\epsilon^2} \right\rceil \).

The following two lemmas will be used in the proof of Claim 3.1. Due to the length of the proofs,
they will be given in Section 5, which also contain the original definition of the function $f$.

**Lemma 7.** Let $\epsilon \in (0,1)$, $G$ be a $\Delta$-critical graph with maximum degree $\Delta \geq f(\epsilon)$, $xy \in E(G)$ with $d(x) < q = (1-\epsilon)\Delta$ and $\varphi \in \mathcal{C}^\Delta(G-xy)$. Then for any color $\alpha \in \varphi(y)$, there exists a set $X \subseteq \varphi^{-1}(x_{\alpha}) \cap \varphi(x) \setminus \{\alpha\}$ with $|X| \leq c_0$ such that $|\varphi^{-1}(x_{\beta}) \cap \varphi(x) \setminus \{\beta\}| < 3c_0$ for all $\beta \in \varphi^{-1}(x_{\alpha}) \cap \varphi(x) \setminus \{\alpha\}$.

**Lemma 8.** Let $\epsilon \in (0,1)$, $G$ be a $\Delta$-critical graph with maximum degree $\Delta \geq f(\epsilon)$, $xy \in E(G)$ with $d(x) < q = (1-\epsilon)\Delta$ and $\varphi \in \mathcal{C}^\Delta(G-xy)$. Then for any $c_0 + 2$ colors $\xi_1, \xi_2, \ldots, \xi_{c_0+2} \in \mathcal{C}(q) \cap \varphi(x)$, $|\bigcap_{i=1}^{c_0+2} \varphi^{-1}(x_{\xi_i}) \setminus \{\xi_i\}| < 3c_0 + 1$.

**Lemma 9.** Let $\rho \in (0,1)$ and $N_1, N_2, \Delta, l$ be positive integers such that $\Delta \geq N_2/\rho^2$ and $l > (N_1 - 1)(\frac{1}{\rho} + 1)^{N_2}$. Then for any $l$ subsets $A_1, A_2, \ldots, A_l$ of $\{1, 2, \ldots, \Delta\}$ with $|A_i| \geq \rho \Delta$ for each $i \in [1, l]$, there exist $N_1$ sets $A_{i_1}, A_{i_2}, \ldots, A_{i_{N_1}}$ such that $|\bigcap_{i=1}^{N_1} A_{i_i}| \geq N_2$.

**Proof.** We say a subset of $\{1, 2, \ldots, \Delta\}$ is a $N_2$-subset if it has size $N_2$. There are $(\Delta^N)_{N_2}$ $N_2$-subsets in total, and each $A_i$ contains at least $(\rho^N)_{N_2}$ of them. If one of the $N_2$-subset is covered by at least $N_1$ $A_i$'s, we are done. Otherwise we have $l(\rho^N)_{N_2} \leq (N_1 - 1)(\Delta^N)_{N_2}$. Since $\rho < 1$, if $0 \leq i \leq N_2 - 1 < \rho^2 \Delta$, then $\frac{\rho^2 - \rho^2 \Delta}{\Delta - i} > \frac{\rho^2 - \rho^2 \Delta}{\Delta - \rho^2 \Delta} = \frac{\rho^2}{1+\rho} = (\frac{\rho}{1+\rho})^{-1}$. Hence $(\rho^N)_{N_2} > (\frac{\rho}{1+\rho})^{-N_2}(\Delta^N)_{N_2}$.

Corollary 10. Let $\epsilon, \rho \in (0,1)$, $G$ be a $\Delta$-critical graph with maximum degree $\Delta \geq \max\{ f(\epsilon), (3c_0 + 1)/\rho^2 \}$ and $xy \in E(G)$ with $d(x) < q = (1-\epsilon)\Delta$. Then for any $\varphi \in \mathcal{C}^\Delta(G-xy)$ there exists a set $X \subseteq \mathcal{C}(q) \cap \varphi(x)$ with $|X| \leq (c_0 + 1)(\frac{1}{\rho} + 1)^{3c_0+1}$ such that $|\varphi^{-1}(x_{\xi}) \setminus \{\xi\}| < \rho \Delta$ for all $\xi \in \mathcal{C}(q) \cap \varphi(x)$.\hfill \Box

**Proof.** Suppose on the contrary that there are $l > (c_0 + 1)(\frac{1}{\rho} + 1)^{3c_0+1}$ colors $1, 2, \ldots, l$ in $\mathcal{C}(q) \cap \varphi(x)$ such that $|\varphi^{-1}(x_{\xi}) \setminus \{\xi\}| \geq \rho \Delta$ for all $i \in [1, l]$. Let $N_1 = c_0 + 2$ and $N_2 = 3c_0 + 1$. Then, $N_1, N_2, \rho, \Delta$ and $l$ satisfy the conditions of Lemma 9. Let $A_i = \varphi^{-1}(x_{\xi_i}) \setminus \{i\}$ for $i = 1, 2, \ldots, l$, by Lemma 9 there are $c_0 + 2$ colors $\xi_1, \xi_2, \ldots, \xi_{c_0+2}$ in $\mathcal{C}(q) \cap \varphi(x)$ such that $|\bigcap_{i=1}^{c_0+2} \varphi^{-1}(x_{\xi_i}) \setminus \{\xi_i\}| \geq 3c_0 + 1$. Since $\Delta \geq f(\epsilon)$, we also have the conclusion of Lemma 8, which is a contradiction.\hfill \Box

### 3.1.2 Proof of Claim 3.1

**Claim 3.1.** For every $x \in V(G)$ with $d(x) > \epsilon \Delta$, $M'(x) \geq q$.

**Proof.** Recall $M(x) = d(x)$ and $q = (1-\epsilon)\Delta$. If $d(x) \geq q = (1-\epsilon)\Delta$, then $M'(x) = M(x) - \frac{d(x) - q}{d(x)}d(x) = q$.

We assume $\epsilon \Delta \leq d(x) < q$. Choose $y \in N(x)$ such that $\sigma_q(x, y)$ is minimum over all neighbors of $x$. Let $s = \Delta - \sigma_q(x, y)$. Then for each $z \in N(x)$, $d(z) > \sigma_q(x, z) \geq \sigma_q(x, y) = \Delta - s$. Now we consider two cases.

**Case 1.** $s \leq \epsilon^2 \Delta$.

In this case, for each $z \in N(x)$, $d(z) \geq \Delta - s \geq \Delta - \epsilon^2 \Delta > q$ and $d(z) - \sigma_q(x, z) \leq d(z) + s - \Delta \leq s$. So we have:
\[ M'(x) > \sum_{z \in N(x)} \frac{d(z) - q}{d(z) - \sigma_q(x, z)} \geq \sum_{z \in N(x)} \frac{\Delta - s - q}{s} \]
\[ = d(x) \left( \frac{\Delta - s - q}{s} \right) \geq \epsilon \Delta \left( \frac{\epsilon \Delta - \epsilon^2 \Delta}{\epsilon^2 \Delta} \right) = (1 - \epsilon) \Delta = q \]

**Case 2.** \( s > \epsilon^2 \Delta. \)

For any \( y \in N(x) \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \), let \( Y(\varphi) = \{ v \in N(y) \setminus \{ x \} : \varphi(vy) \in \bar{\varphi}(x) \} \) and \( C_y(\varphi) = \{ \varphi(vy) : v \in Y(\varphi) \text{ and } d(v) < q \} \). Since \( \bar{\varphi}(x) \cap \varphi(y) = \emptyset \) by Lemma 1, \( \bar{\varphi}(x) \subseteq \varphi(y) \), \( Y(\varphi) \) and \( C_y(\varphi) \) are well-defined.

The following properties will be used soon.

1. \( |C_y(\varphi)| \leq c_0, \)
2. \( |N_S(x)| \geq s - c_0 \geq \epsilon^2 \Delta - c_0. \)

Let \( \rho = \frac{\epsilon^3}{\epsilon^2 + 1} \) and \( N = (c_0 + 1)(\frac{1}{\rho} + 1)^{3c_0 + 1} \). Since \( \Delta \geq D(\epsilon) \geq \max\{ f(\epsilon), (3c_0 + 1)/\rho^2 \} \), by Corollary 10, there is a subset \( X \) of \( N_S(x) \) with \( |X| \leq N \) such that \( |\varphi_{\text{bad}}(z)| \leq \rho \Delta \) for all \( z \in N_S(x) \setminus X \).

3. For all \( z \in N_S(x) \setminus X \), \( d(z) \geq \Delta - |\varphi_{\text{bad}}(z)| > q \) and \( d(z) - \sigma_q(x, z) \leq |\varphi_{\text{bad}}(z)|. \)

Now we prove the three statements above.

For (1), since \( \{ x, y \} \cup Y(\varphi) \) is the vertex set of a multi-fan at \( y \), it is elementary by Lemma 1. Assume the contrary that \( |C_y(\varphi)| > c_0 \), then \( \Delta \geq |\bigcup_{y_\alpha \in Y(\varphi)} \bar{\varphi}(y_\alpha)| = \sum_{y_\alpha \in Y(\varphi)} |\bar{\varphi}(y_\alpha)| \geq (c_0 + 1)\epsilon \Delta > \Delta \), giving a contradiction.

For (2), recall \( A_\varphi(x, y, q) = \{ \alpha \in \varphi(x) \cap \varphi(y) : d(y_\alpha) < q \} \) and \( N_S(x) = \{ x_\alpha : \alpha \in A_\varphi(q) \cup \bar{\varphi}(y) \} \). Then \( |A_\varphi(q) \cup \bar{\varphi}(y) \cup C_y(\varphi)| = |N_S(x)| + |C_y(\varphi)| = \Delta - \sigma_q(x, y) = s. \) Since \( |C_y(\varphi)| \leq c_0 \) and \( s = |N_S(x)| + |C_y(\varphi)| \), we have \( |N_S(x)| \geq s - c_0 \geq \epsilon^2 \Delta - c_0. \)

For (3), we have \( d(z) \geq \Delta - |\bar{\varphi}(z)| \geq \Delta - |\varphi_{\text{bad}}(z)| \geq \Delta - \rho \Delta. \) Since \( \rho < \frac{1}{\epsilon^2}, \Delta - |\varphi_{\text{bad}}(z)| > q. \) Notice that \( |\varphi_{\text{bad}}(z)| = \Delta - \sigma_q(x, z). \) So \( d(z) - \sigma_q(x, z) \leq |\varphi_{\text{bad}}(z)|. \)

By using these properties, we have

\[ M'(x) > \sum_{z \in N_S(x) \setminus X} \frac{d(z) - q}{d(z) - \sigma_q(x, z)} \geq \sum_{z \in N_S(x) \setminus X} \frac{\Delta - |\varphi_{\text{bad}}(z)| - q}{|\varphi_{\text{bad}}(z)|} \]
\[ \geq (\epsilon^2 \Delta - c_0 - N) \left( \frac{\epsilon \Delta - \rho \Delta}{\rho \Delta} \right) \]
\[ = \Delta - \frac{c_0 + N}{\epsilon^2} \quad \text{(Since } \frac{\epsilon - \rho}{\rho} = \frac{1}{\epsilon^2}. \text{)} \]
\[ \geq (1 - \epsilon) \Delta = q. \quad \text{(Since } \Delta \geq D(\epsilon) \geq \frac{N + c_0}{\epsilon^2}. \text{)} \]

\( \square \)
3.2 Part II: Claim 3.2.

3.2.1 Lemmas (Part II)

In this subsection, we still assume that \( q = (1 - \epsilon)\Delta \) and consider the vertices with degree at most \( \epsilon \Delta \). We point out that all the conclusions in Subsection 3.2 are true without assuming \( \Delta \geq D(\epsilon) \).

If \( d(x) \leq \epsilon \Delta \), then for any \( v \in V(G) \) with \( d(v) < q \) and any coloring \( \varphi \in C^\Delta(G - xy) \), \( |\varphi(x) \cap \varphi(v)| \geq 2 \). In this case we will use an operation called \((\eta, \theta, \delta)\)-switching instead of \((\eta, \theta, \delta)\)-switching in some cases if pointing out \( \delta \) is not necessary. We give the details here.

\[(\eta, \theta)\text{-switching:} \] Let \( xy \) be an edge in a \( \Delta \)-critical graph \( G \) and \( \varphi \in C^\Delta(G - xy) \). For any \( \eta \in A_\varphi(q) \) and \( \theta \in \bar{\varphi}(y) \), since \( \eta \in A_\varphi(q) \), \( y_\theta \) exists and \( d(y_\theta) < q \), there exists a common missing color \( \delta \) of \( x \) and \( y_\theta \). We do a \((\eta, \theta, \delta)\)-switching in \( \varphi \).

Notice that this operation is exactly the \((\eta, \theta, \delta)\)-switching without pointing out \( \delta \), thus it satisfies all the properties stated in Lemma 6. As we pointed out at the end of Section 2, this operation allows us to treat \( A_\varphi(q) \) and \( \bar{\varphi}(y) \) as the same set (in some sense). Recall that \( S_\varphi(q) = A_\varphi(q) \cup \bar{\varphi}(x) \cup \bar{\varphi}(y) \). In the reminder of this paper, we will write \( A_\varphi(q) \cup \bar{\varphi}(y) \) as \( S_\varphi(q) \cap \bar{\varphi}(x) \) very often.

The next lemma is a direct corollary of a result proved by Woodall [15], it is actually Claim 3.2, Claim 3.3 and Claim 3.6 in the proof of Lemma 2.4 in [15].

**Lemma 11.** Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) \leq \epsilon \Delta \) and \( \varphi \in C^\Delta(G - xy) \). Then for any two distinct colors \( \alpha \) and \( \beta \) with \( \alpha \in \bar{\varphi}(y) \) and \( \beta \in S_\varphi(q) \), \( x_{\alpha \beta} \) exists and \( d(x_{\alpha \beta}) \geq \Delta - d(x) + 2 > q \).

**Lemma 12.** Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) \leq \epsilon \Delta \) and \( \varphi \in C^\Delta(G - xy) \). Then for any \( \alpha \in S_\varphi(q) \cap \bar{\varphi}(x) \), \( \varphi^{\text{bad}}(x_{\alpha}) \setminus \{\alpha\} \subseteq B_\varphi(q) \).

**Proof.** Let \( \beta \neq \alpha \) be a color in \( S_\varphi(q) \). Recall that \( S_\varphi(q) \cup B_\varphi(q) = [1, \Delta] \), we will prove that \( x_{\alpha \beta} \) exists and \( d(x_{\alpha \beta}) \geq q \), which implies the lemma.

If \( \alpha \in \bar{\varphi}(y) \), we are done by Lemma 11. Assume that \( \alpha \in A_\varphi(q) \). Since \( xy \) is uncolored, there exists a color \( \theta \) in \( \bar{\varphi}(y) \). Now we do a \((\alpha, \theta)\)-switching in \( \varphi \) to obtain a new coloring \( \varphi_1 \) in which \( \varphi_1(x_{\alpha \theta}) = \alpha \in \bar{\varphi}(y) \).

It follows from Lemma 11 that \( S_{\varphi_1}(q) \subseteq S_{\varphi_1}(x_{\alpha \theta}) \). Now Lemma 6 implies that \( S_{\varphi}(q) = S_{\varphi_1}(q) \subseteq \varphi(x_{\alpha \theta}) \). In particular, \( x_{\alpha \beta} \) exists. By Lemma 6, the color of \( x_{\alpha \alpha \beta} \) under \( \varphi_1 \) is still in \( S_{\varphi}(q) = S_{\varphi_1}(q) \), and by Lemma 11 again we have \( d(x_{\alpha \beta}) \geq \Delta - d(x) + 2 > q \).

**Lemma 13.** Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) \leq \epsilon \Delta \) and \( \varphi \in C^\Delta(G - xy) \). Let \( \xi \in S_\varphi(q) \cap \bar{\varphi}(x) \). Then for any \( \beta \in \varphi^{\text{bad}}(x_{\xi}) \setminus \{\xi\} \), we have \( \beta \in \varphi(x) \) and \( \varphi^{\text{bad}}(x_{\beta}) \setminus \{\beta\} \subseteq B_\varphi(q) \).

**Proof.** Since \( \beta \in \varphi^{\text{bad}}(x_{\xi}) \setminus \{\xi\} \), by Lemma 12 we have \( \beta \in B_\varphi(q) \subseteq \varphi(x) \). Let \( \alpha \) be a color in \( S_{\varphi}(q) = A_{\varphi}(q) \cup \varphi(x) \cup \bar{\varphi}(y) \). We will prove that \( x_{\beta \alpha} \) exists and \( d(x_{\beta \alpha}) \geq q \), which implies the lemma. We consider the three cases separately.

**Case 1.** \( \alpha \in \bar{\varphi}(x) \).
We divide this case into two subcases.

**Subcase 1.1.** \( \xi \in \bar{\varphi}(y) \).
In this case, we recolor the edge \( xy \) by \( \xi \) and leave \( xx_\xi \) uncolored to obtain a new coloring \( \varphi_1 \). In the new coloring, we have \( \alpha \in \varphi_1(x) \subseteq S_{\varphi_1}(x, x_{\xi}, q), \beta \in \varphi^{\text{bad}}(x_{\xi}) \setminus \{\xi\} = S_{\varphi_1}(x, x_{\xi}, q) \).

Thus by Lemma 12, \( x_{\beta \alpha} \) exists and \( d(x_{\beta \alpha}) \geq \Delta - d(x) + 2 > q \).
Subcase 1.2. \( \xi \in A_\phi(q) \).

In this case, there exists \( y_\xi \in N(y) \) such that \( \phi(y_\xi) = \xi \) and \( d(y_\xi) < q \). Thus, there are two colors in \( \bar{\varphi}(x) \cap \varphi(y) \). Choose \( t \in \bar{\varphi}(x) \cap \varphi(y) \) such that \( t \neq \alpha \), choose \( \xi_0 \in \bar{\varphi}(y) \). Clearly \( \alpha, \beta \notin \{ \xi, \xi_0, t \} \), since \( \xi \in A_\varphi(q), \alpha \in \bar{\varphi}(x) \subseteq \varphi(y) \), and \( \beta \in \bar{\varphi}(q) \subseteq \varphi(x) \cap \varphi(y) \). Then we can do \((\xi, \xi_0, t)\)-switching in \( \varphi \). By Lemma 6, this operation will not affect the edge \( xx_\xi \) and other edges with color \( \alpha, \xi \) or \( \beta \) except \( y_\xi y_\xi \). After this operation, we are back to Subcase 1.1.

Case 2. \( \alpha \in \varphi(y) \).

Choose \( \delta \in \varphi(x) \); clearly \( \beta \notin \{ \alpha, \delta \} \). Assume first that at least one of \( \alpha, \delta \) is in \( \varphi(x_\beta) \). Let \( \text{Case 1} \) with \( \varphi_1, \delta \) in place of \( \varphi, \alpha \), we have \( x_{\beta \delta} \) exists and \( d(x_{\beta \delta}) \geq q \) under coloring \( \varphi_1 \). Hence under coloring \( \varphi, x_{\beta \alpha} \) exists and \( d(x_{\beta \alpha}) \geq q \).

This argument fails if \( \{ \alpha, \delta \} \subseteq \varphi(x_\beta) \). But then \( x_{\beta \alpha} \) exists and we may assume \( d(x_{\beta \alpha}) < q \).

Since \( d(x) < \epsilon \Delta = \Delta - q \), we could have chosen \( \delta \in \varphi(x) \cap \varphi(x_{\beta \alpha}) \), and the previous argument would then have worked.

Case 3. \( \alpha \in A_\varphi(q) \).

Choose \( \xi_0 \in \bar{\varphi}(y) \) and do \((\alpha, \xi_0)\)-switching of \( \varphi \) to give \( \varphi_1 \). By Lemma 6, excepting \( yy_\xi \), this operation does not affect edges incident with \( x \) or \( y \) or with color \( \alpha \) or \( \beta \), and so \( \beta \in \bar{\varphi}(q) \) and \( \xi \in \bar{\varphi}(q) \cap \varphi(x) \) even when \( \xi_0 = \xi \). Now \( \alpha \in \bar{\varphi}(y) \), and the result follows from Case 2 with \( \varphi_1 \) in place of \( \varphi \). \( \square \)

Lemma 14. Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) \leq \epsilon \Delta \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \). Then \( \varphi^{\text{bad}}(x_{\xi_1}) \cap \varphi^{\text{bad}}(x_{\xi_2}) = \emptyset \) for any two colors \( \xi_1, \xi_2 \in S_\varphi(q) \cap \varphi(x) \).

Proof. Let \( \beta \in \varphi^{\text{bad}}(x_{\xi_1}) \). Then \( \beta \in S_\varphi(q) \) by Lemma 12, so \( \beta \neq \xi_1, \xi_2 \). We will prove that \( x_{\xi_2 \beta} \) exists and \( d(x_{\xi_2 \beta}) \geq q \), which implies the lemma.

Choose \( \xi_0 \in \bar{\varphi}(y) \), we do \((\xi_1, \xi_0)\)-switching in \( \varphi \) at \( y \) if \( \xi_1 \in A_\varphi(q) \). By Lemma 6, this operation will not affect the edges \( xx_{\xi_1}, xx_{\xi_2} \) and also other edges with color \( \xi_1 \) or \( \beta \) except \( yy_\xi \). Then we can do \((\xi_2, \xi_0)\)-switching in \( \varphi \) to give \( \varphi_1 \). Notice that by Lemma 12, there exists \( x_{\xi_1 \xi_2} \in N(x_{\xi_1}) \) such that \( d(x_{\xi_1 \xi_2}) \geq q \). Thus in the new coloring, \( \xi_2 \notin \varphi(x_{\xi_1}), \beta \in \varphi^{\text{bad}}(x_{\xi_1}) \setminus \{ \xi_1 \} = S_\varphi(x_{\xi_1}, q), \varphi_1(xy) = \xi_1 \in \varphi(x_{\xi_1}) \subseteq S_\varphi(x_{\xi_1}, q) \). Moreover, we have that \( \xi_2 \in \varphi_1^{\text{bad}}(y) \) because \( \xi_2 \in S_\varphi(x,y,q) \cap \varphi(x) \). Thus by Lemma 13, \( x_{\xi_2 \beta} \) exists and \( d(x_{\xi_2 \beta}) \geq q \). This completes the proof of Lemma 14. \( \square \)

Now combining Lemma 12 and Lemma 14, we have the following corollary.

Corollary 15. Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) \leq \epsilon \Delta \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \). Then for any \( \xi \in S_\varphi(q) \cap \varphi(x) \), we have \( \varphi^{\text{bad}}(x_{\xi_1}) \setminus \{ \xi \} \subseteq B_\varphi(q) \). Moreover, \( \varphi^{\text{bad}}(x_{\xi_1}) \cap \varphi^{\text{bad}}(x_{\xi_2}) = \emptyset \) for any two colors \( \xi_1, \xi_2 \in S_\varphi(q) \cap \varphi(x) \).

3.2.2 Claim 3.2

Claim 3.2. For every \( x \in V(G) \) with \( D_0(e) \leq d(x) \leq \epsilon \Delta \), \( M'(x) \geq q \).

Proof. For any \( v \in N(x) \), we have \( \sigma_{\Delta}(x,v) \geq \Delta - d(x) + 1 > (1 - \epsilon) \Delta = q \) by Lemma 2. So \( d(v) \geq \sigma_{\Delta}(x,v) \geq \sigma_{\Delta}(x,v) \geq q \). Following the discharging rule, we have

\[
M'(x) \geq \sum_{v \in N(x)} \frac{d(v) - q}{d(v) - \sigma_{\Delta}(x,v)}.
\]
Choose \( y \in N(x) \) and \( \varphi \in \mathcal{C}^\Delta(G - xy) \) such that \( p := \|B_\varphi(q)\| \) is minimum over all neighbours of \( x \) and all \( \Delta \)-colorings of \( G - xy \). Note that \( |S_\varphi(q)| = \Delta - p \) and \( p = \|B_\varphi(q)\| \leq |A_\varphi(q)| + |B_\varphi(q)| = |\varphi(x)\backslash \bar{\varphi}(y)| = d(x) - 1 - |\bar{\varphi}(y)| \leq d(x) - 2 \). According to the discharging rule, a vertex \( x_\xi \in S_\varphi(q) \cap \varphi(x) \) will distribute charge to \( x \) and \( x_\beta \)'s where \( \beta \in \varphi^{bad}(x_\xi) \backslash \bar{\varphi}(x_\xi) \). Let \( X_\varphi(q) = \bigcup_{\xi \in S_\varphi(q) \cap \varphi(x)} (\varphi^{bad}(x_\xi) \backslash (\bar{\varphi}(x_\xi) \cup \{\xi\}) \big) \) and \( t = |X_\varphi(q)| \). Note that \( |\varphi^{bad}(x_\xi) \backslash (\bar{\varphi}(x_\xi) \cup \{\xi\})| = \{\beta \in \varphi(\xi) \cup \{\xi\} : d(x_{\xi}\beta) < q\} = d(x_{\xi}) - \sigma_q(x, x_\xi) - 1 \) denotes the number of vertices (except \( x \)) that receive charge from \( x_\xi \). Now we claim the following statements and prove them one by one:

1. \( X_\varphi(q) \subseteq B_\varphi(q) \),
2. For any \( v \in N(x) \), \( \sigma_q(x,v) \geq \Delta - d(x) + 1 + p > q \),
3. For any \( v \in N(x) \), \( \frac{d(v) - q}{d(v) - \sigma_q(x,v)} \geq \frac{\Delta - q}{\Delta - \sigma_q(x,v)} \),
4. For any \( v \in N_S(x) \), \( d(v) \geq \Delta - p + t \) and
5. \( \sum_{v \in N_S(x)} \frac{1}{d(v) - \sigma_q(x,v)} \geq \frac{|N_S(x)|^2}{|N_S(x)| + t} = \frac{(d(x) - p - 1)^2}{d(x) - p - 1 + t} \).

To prove (1), recall \( \varphi^{bad}(x) \backslash (\bar{\varphi}(x) \cup \{\xi\}) \subseteq \varphi^{bad}(x_\xi) \backslash \{\xi\} \subseteq B_\varphi(q) \) by Corollary 15.

To prove (2), let \( v \in N(x) \) and \( \varphi' \in \mathcal{C}^\Delta(G - xv) \). Then for any \( w \in N(v) \) with \( \varphi'\langle vw \rangle \in \varphi'(x) \), \( \{x,v,w\} \) is elementary by Lemma 1, so \( d(w) \geq |\varphi'(x)| = \Delta - d(x) + 1 > q \). Thus \( \sigma_q(x,v) \geq |\{w \in N(v) \setminus \{x\} : \varphi'(vw) \in \varphi'(x)\}| + |B_\varphi'(x,v,q)| \geq |\varphi'(x)| + |B_\varphi(q)| \geq \Delta - d(x) + 1 + p > q \).

To prove (3), let \( v \in N(x) \). Using the property above, we have \( d(v) \geq \sigma_q(x,v) \geq \Delta - d(x) + 1 + p > q \). Thus \( \frac{d(v) - q}{d(v) - \sigma_q(x,v)} \) is a decreasing function of \( d(v) \). Hence \( \frac{d(v) - q}{d(v) - \sigma_q(x,v)} \geq \frac{\Delta - q}{\Delta - \sigma_q(x,v)} \) holds.

To prove (4), for any \( x_\alpha \in N_S(x) \), we have \( S_\varphi(q) \subseteq \varphi(x_\alpha) \) by Lemma 12. For any \( \beta \in X_\varphi(q) \), there exists \( \xi \in S_\varphi(q) \cap \varphi(x) \) such that \( x_\beta \) exists and \( d(x_{\xi}\beta) < q \). Obviously \( \beta \in \varphi(x_\alpha) \) if \( \xi = \alpha \) and by Lemma 14, \( \beta \in \varphi(x_\alpha) \) if \( \xi \neq \alpha \). Thus, \( S_\varphi(q) \cup X_\varphi(q) \subseteq \varphi(x_\alpha) \) and so \( d(x_\alpha) \geq |S_\varphi(q)| + t = \Delta - p + t \). The fourth statement holds.

To prove (5), let \( x_\alpha \in N_S(x) \). Recall \( d(x_\alpha) - \sigma_q(x,x_\alpha) - 1 = |\varphi^{bad}(x_\alpha) \backslash (\bar{\varphi}(x_\alpha) \cup \{\alpha\})| \). Since \( \alpha \) is in \( S_\varphi(q) \), not in \( B_\varphi(q) \) and it follows from Corollary 15 that \( \varphi^{bad}(x_\xi) \)'s are disjoint for \( \xi \in S_\varphi(q) \cap \varphi(x) \). Thus \( \sum_{x_\alpha \in N_S(x)} (d(x_\alpha) - \sigma_q(x,x_\alpha) - 1) = |\bigcup_{x_\alpha \in N_S(x)} \varphi^{bad}(x_\alpha) \backslash (\bar{\varphi}(x_\alpha) \cup \{\alpha\})| = t \).

Then by the Cauchy-Schwarz inequality, we have \( \sum_{x_\alpha \in N_S(x)} \frac{1}{d(v) - \sigma_q(x,v)} \geq \frac{|N_S(x)|^2}{|N_S(x)| + t} = \frac{(d(x) - p - 1)^2}{d(x) - p - 1 + t} \). Notice that \( |N_S(x)| = d(x) - |\{y\}| - |B_\varphi(q)| = d(x) - p - 1 \), so the last equality holds.

Now by the properties above, we have
After simplifying, we have the following rule:

\[ M'(x) \geq d(x) + \sum_{v \in N_G(x)} \frac{d(v) - q}{d(v) - \sigma_q(x, v)} + \left( \sum_{v \in N_G(x)} \frac{d(v) - q}{d(v) - \sigma_q(x, v)} \right) + \frac{d(y) - q}{d(y) - \sigma_q(x, y)} \]

\[ \geq \sum_{v \in N_G(x)} \frac{\Delta - p + t - q}{d(v) - \sigma_q(x, v)} + \left( \sum_{v \in N_G(x)} \frac{\Delta - q}{\Delta - \sigma_q(x, v)} \right) + \frac{\Delta - q}{\Delta - \sigma_q(x, y)} \]

\[ \geq (\epsilon \Delta - p + t) \left( \frac{(d(x) - p - 1)^2}{d(x) - p - 1 + t} + (p + 1) \frac{\epsilon \Delta}{d(x) - p - 1} \right). \]

Since \( d(x) \leq \epsilon \Delta \), the first fraction above is a decreasing function of \( t \). Since \( t \leq p \) by (1), we have:

\[ M'(x) \geq \epsilon \Delta \left( \frac{(d(x) - p - 1)^2}{d(x) - 1} + \frac{p + 1}{d(x) - 1 - p} \right) \]

\[ = \epsilon \Delta \left( \frac{(d(x) - p - 1)^2}{d(x) - 1} + \frac{d(x)}{2(d(x) - 1 - p)} + \frac{d(x)}{2(d(x) - 1 - p)} - 1 \right) \]

\[ \geq \epsilon \Delta \left( 3 \left( \frac{d(x)^2}{4(d(x) - 1)} \right)^{\frac{1}{3}} - 1 \right). \quad (a + b + c \geq 3(abc)^{\frac{1}{3}} \text{ if } a, b, c > 0.) \]

The number above is greater than \( q \) if \( \epsilon \geq \frac{1}{3} \) or \( \epsilon < \frac{1}{3} \) and \( d(x) \geq 2(\frac{1}{3})^3 + 2(\frac{1}{3})^6 - (\frac{1}{3})^3 \), we briefly explain it. If \( \epsilon \geq \frac{1}{3} \), then since \( d(x)^2 \geq 4(d(x) - 1) \), we have \( \epsilon \Delta \left( 3 \left( \frac{d(x)^2}{4(d(x) - 1)} \right)^{\frac{1}{3}} - 1 \right) \geq \epsilon \Delta(3 - 1) \geq 2\epsilon \Delta \geq (1 - \epsilon) \Delta = q \). If \( \epsilon < \frac{1}{3} \), we solve the following inequality:

\[ \epsilon \Delta \left( 3 \left( \frac{d(x)^2}{4(d(x) - 1)} \right)^{\frac{1}{3}} - 1 \right) \geq (1 - \epsilon) \Delta. \]

After simplifying, we have \( d(x)^2 - 4(\frac{1}{3})^3 d(x) + 4(\frac{1}{3})^3 \geq 0 \). Then by the quadratic formula, we get what we want. \( \square \)

Claim 3.1 and Claim 3.2 together prove Theorem 2.

4 Proof of Theorem 1

**Theorem 1.** If \( G \) is a \( \Delta \)-critical graph, then \( \overline{d}(G) \geq \frac{3}{4} \Delta(G) - 8 \).

Let \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \). In this section we assume \( \Delta \geq 16 \) since when \( \Delta < 16 \), \( \frac{2}{3} \Delta \geq \frac{3}{4} \Delta - 8 \). Theorem 1 is implied by Woodall’s result [15]. We initially assign to each vertex \( x \in V(G) \) a charge \( M(x) = d(x) \) and redistribute the charge according to the following rule:

- **Rule of Discharge (first step):** Each vertex \( y \) with degree larger than \( \frac{3}{4} \Delta - 8 \) distributes \( d(y) - (\frac{3}{4} \Delta - 8) \) equally among all its neighbors with degree less than \( \frac{3}{4} \Delta - 8 \).
Denote by $M'(x)$ the resulting charge on each vertex $x$. We will prove the following two claims.

**Claim 4.1** For every $x \in V(G)$ with $d(x) > \frac{1}{3}\Delta$, $M'(x) \geq \frac{3}{4}\Delta - 8$.

**Claim 4.2** For every $x \in V(G)$ with $4 \leq d(x) \leq \frac{1}{3}\Delta$, $M'(x) \geq \frac{3}{4}\Delta - 8$.

The two claims above imply that $M'(x) \geq \frac{3}{4}\Delta - 8$ when $d(x) \geq 4$. But for vertices with degree 2 or 3, we only have the following result:

(*) For every $x \in V(G)$ with $d(x) = 2$, $M'(x) > \frac{1}{2}\Delta$; For every $x \in V(G)$ with $d(x) = 3$, $M'(x) > \frac{5}{8}\Delta$.

The proof of (*) can be found in the proof of Claim 4.4 in Subsection 4.3. Obviously this result is not enough for proving Theorem 1. In order to deal with the vertices with degree 2 or 3, we will adjust the discharging rule to complete the whole proof. Details of the last step will be given in Subsection 4.3.

### 4.1 Part I: Claim 4.1.

#### 4.1.1 Lemmas

The following lemma is a generalization of Lemma 12. We will use a separated section to prove it deal to its length. Recall that $c_1 = c_1(\epsilon) = \left\lceil \frac{1}{\epsilon} - 1 \right\rceil$ for any $\epsilon \in (0, 1)$.

**Lemma 16.** Let $\epsilon \in (0, 1)$, $q$ be a positive number, $G$ be a $\Delta$-critical graph with maximum degree $\Delta$, $xy \in E(G)$ with $d(x) < q$ and $\varphi \in C^\Delta(G - xy)$. If $q \leq \min\{(1 - \epsilon)(\Delta - 2c_1, \Delta - 6c_1)\}$, then for each $z \in N_S(x)$ except at most $c_1$ vertices, there exists a coloring $\varphi' \in C^\Delta(G - xy)$ such that $\varphi'(xz) \in \varphi'(y)$ and for each $\xi \in (\mathcal{S}_{\varphi'}(q) \cap \varphi'(z)) \backslash \{\varphi'(xz)\}$ except at most $4c_1$ colors, $d(x_{\xi}) \geq q$ where $1 = \varphi'(xz)$.

The next lemma is a corollary of Lemma 16, we will explain why it can be implied by Lemma 16 in Section 6 which also contain the proof of Lemma 16.

**Lemma 17.** Let $\epsilon \in (0, 1)$, $q$ be a positive number, $G$ be a $\Delta$-critical graph with maximum degree $\Delta$, $xy \in E(G)$ with $d(x) < q$ and $\varphi \in C^\Delta(G - xy)$. If $q \leq \min\{(1 - \epsilon)(\Delta - 2c_1, \Delta - 6c_1)\}$, then there are at least $\Delta - \sigma_q(x,y) - 2c_1$ vertices $z \in N(x) \backslash \{y\}$ such that $\sigma_q(x,z) \geq 2\Delta - d(x) - \sigma_q(x,y) - 5c_1$.

Now let $\epsilon = \frac{1}{4}$ in Lemma 17. In this case $c_1 = 2$. We have the following:

**Corollary 18.** Let $G$ be a $\Delta$-critical graph with maximum degree $\Delta$, $q \leq \min\{\frac{3}{2}\Delta - 4, \Delta - 12\}$ be a positive number and $xy \in E(G)$ with $d(x) < q$. Then there are at least $\Delta - \sigma_q(x,y) - 4$ vertices $z \in N(x) \backslash \{y\}$ such that $\sigma_q(x,z) \geq 2\Delta - d(x) - \sigma_q(x,y) - 5$.

In general, for any $x \in V(G)$ and positive number $q$ with $q \leq \Delta$, we define the following two parameters:

$$p(x, q) := \min_{y \in N(x)} \sigma_q(x, y) - (\Delta - d(x) + 1) \quad \text{and}$$

$$p'(x, q) := \min\{ p(x, q), \left\lceil \frac{d(x)}{2} \right\rceil - 6 \}.$$ 

**Lemma 19.** Let $G$ be a $\Delta$-critical graph with maximum degree $\Delta$, $q \leq \min\{\frac{3}{2}\Delta - 4, \Delta - 12\}$ be a positive number. Then for any $x$ with $d(x) < q$, $x$ has at least $d(x) - p'(x, q) - 5$ neighbors $z$ for which $\sigma_q(x, z) \geq \Delta - p'(x, q) - 11$.
Proof. Let $y \in N(x)$ such that $p(x, q) = \sigma_q(x, y) - \Delta + d(x) - 1$. If $p'(x, q) = p(x, q)$, by Corollary 18, there are at least $\Delta - \sigma_q(x, y) - 4 = \Delta - (\Delta - d(x) + p(x, q) + 1) - 4 = d(x) - p(x, q) - 5$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma_q(x, z) \geq 2\Delta - d(x) - 10 - \sigma_q(x, y) = \Delta - p(x, q) - 11$. If $p'(x, q) = \left\lfloor \frac{d(x)}{2} \right\rfloor - 6 < p(x, q)$, then for every $z \in N(x)$, $\sigma_q(x, z) > \Delta - d(x) + 1 + \left\lfloor \frac{d(x)}{2} \right\rfloor - 6 \geq \Delta - \left\lfloor \frac{d(x)}{2} \right\rfloor - 6 = \Delta - p'(x, q) - 12$. So $\sigma_q(x, z) \geq \Delta - p'(x, q) - 11$. □

4.1.2 Proof of Claim 4.1

In the reminder of Section 4, we fix the number $q$ to be $\frac{3}{4} \Delta - 8$.

Claim 4.1. For every $x \in V(G)$ with $d(x) > \frac{1}{4} \Delta$, $M'(x) \geq q$.

Proof. Since $M'(x) \geq q$ when $d(x) \geq q$, we will consider the case when $\frac{1}{4} \Delta < d(x) < q$. Let $y \in N(x)$ such that $d(y)$ is minimum over all neighbor of $x$, $\varphi \in \mathcal{C}(G - xy)$. Let $Z_q = \{z \in N(x) : d(z) > q\}$, $Z_y = \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \bar{\varphi}(y)\}$ and $Z_q^* = Z_q \cap Z_y$. Clearly, for each $z \in Z_q^*$, $x$ receives $\frac{d(z) - \sigma(z, x)}{d(z) - \sigma(z, x)}$ charge from it. Hence $M'(x) \geq d(x) + \sum_{z \in Z_q^*} \frac{d(z) - \sigma(z, x)}{d(z) - \sigma(z, x)}$. Since the notation $p'(x, q)$ will be used heavily in this proof, we let $p' = p'(x, q)$ for convenience.

For any $z \in Z_q^*$, the three inequalities below will be used soon.

1. $\sum_{z \in Z_q}(d(z) - q) \geq (\Delta - q)(\Delta - d(y) + 1) - (d(x) + d(y) - \Delta - 2)$,
2. $\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) - 1$ and
3. $\sigma_q(x, z) \geq 2\Delta - d(x) - d(y)$ if $d(y) \leq q$.

We give the proof of these statements. (1) is exactly Lemma 5 (2). Recall that Lemma 5 also shows that $\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) + 1 - \lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \rfloor$. Since $d(x) < q$ and $q = \frac{3}{4} \Delta - 8$, we have $\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \rfloor \leq 2$, so $\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) - 1$. Moreover, if $d(y) \leq q$, we further have $\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \rfloor \leq 1$ and $\sigma_q(x, z) \geq 2\Delta - d(x) - d(y)$.

Now we consider the following three cases to complete the proof.

Case 1. $d(y) \leq q$.

In this case, by (1) and (3) we have

$$\sum_{z \in Z_q}(d(z) - q) \geq (\Delta - q)(\Delta - d(y) + 1) - (d(x) + d(y) - \Delta - 2)$$

$$\Delta - (2\Delta - d(x) - d(y))$$

$$= (\Delta - q)(\Delta - d(y) + 1) + \frac{2}{d(x) + d(y) - \Delta} - 1.$$

Since $\bar{\varphi}(x) \cap \bar{\varphi}(y) = \emptyset$ by Lemma 1, $d(x) + d(y) \geq \Delta + 2$. Thus, $d(x) \geq \Delta - q + 2$ and
\[ M'(x) \geq d(x) + \frac{(\Delta - q)(\Delta - d(y) + 1) + 2}{d(x) + d(y) - \Delta} - 1 \]

\[ \geq d(x) + q - \Delta + \frac{(\Delta - q)(\Delta - q + 1) + 2}{d(x) + q - \Delta} - 1 - q + \Delta \quad \text{(Since } d(y) \leq q. \text{)} \]

\[ \geq 2\sqrt{(\Delta - q)(\Delta - q + 1) + 2} + \Delta - q - 1 \]

\[ \geq 3(\Delta - q) - 1 > q. \]

**Case 2.** \( d(y) > q, 2(\Delta - q) - 10 < d(x) < q \) and \( p' > \Delta - q - 11. \)

In this case, each neighbor of \( x \) has degree larger than \( q \) and \( Z_q^* = Z_q. \) So \( M'(x) \geq d(x) + \sum_{z \in Z_y} \frac{d(z) - q}{d(z) - \sigma_q(x, y)} + \sum_{u \in N(x) \setminus Z_y} \frac{d(u) - q}{d(u) - \sigma_q(x, u)}. \)

By (1) and (2) we have

\[ \sum_{z \in Z_y} \frac{d(z) - q}{d(z) - \sigma_q(x, y)} \geq \frac{(\Delta - d(y) + 1)(\Delta - q) - d(x) - d(y) + \Delta + 2}{\Delta - (2\Delta - d(x) - d(y) - 1)} > 1. \]

Since \( d(x) > 2(\Delta - q) - 10 \) and \( p' \leq \frac{d(x)}{2} - 6, \) we have \( q \geq \Delta - \frac{d(x)}{2} - 5 \geq \Delta - d(x) + 1 + p'. \)

By the definition of \( p' \), for each neighbor \( u \) of \( x \) we have \( \sigma_q(x, u) \geq \Delta - d(x) + 1 + p'. \) Since \( d(u) \geq d(y) > q \) for each \( u \in N(x) \) and \( q \geq \Delta - d(x) + 1 + p', \) we have

\[ \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq \frac{d(u) - q}{d(u) - (\Delta - d(x) + 1 + p')} \geq \frac{d(y) - q}{d(y) - (\Delta - d(x) + 1 + p')} \]

Notice that \( |N(x) \setminus Z_y| \geq d(x) - |\varphi(y)| = d(x) - (\Delta - d(y) + 1), \) we have

\[ \sum_{u \in N(x) \setminus Z_y} \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq |N(x) \setminus Z_y| \cdot \frac{d(y) - q}{d(y) - (\Delta - d(x) + 1 + p')} \geq \frac{(d(x) - (\Delta - d(y) + 1))(d(y) - q)}{d(y) - (\Delta - d(x) + 1 + p')} \]

\[ \geq \frac{(d(x) - (\Delta - d(y) + 1))(d(y) - q)}{d(x) + d(y) - \Delta + 1 + p'} \]

Where the last inequality follows from the condition that \( p' > \Delta - q - 11. \) Above all, we have:

\[ M'(x) \geq d(x) + \sum_{z \in Z_y} \frac{d(z) - q}{d(z) - \sigma_q(x, y)} + \sum_{u \in N(x) \setminus Z_y} \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \]

\[ \geq d(x) - 1 + \frac{(\Delta - d(y) + 1)(\Delta - q)}{d(x) + d(y) - \Delta + 1} + \frac{(d(x) + d(y) - \Delta - 1)(d(y) - q)}{d(x) + d(y) + 2 - \frac{5}{4}\Delta} \]

\[ = d(x) - 1 + \frac{(\Delta - d(y) + 1)(\Delta - q)}{d(x) + d(y) - \Delta + 1} + \frac{(\frac{3}{4}\Delta - 3)(d(y) - q)}{d(x) + d(y) + 2 - \frac{5}{4}\Delta} + d(y) - q \]

To prove this long expression is larger than \( q, \) we let \( A = d(y) - \frac{3}{4}\Delta + 8 \) and \( B = d(x) - \frac{1}{2}\Delta - 6 \) be stepping-stones for the calculation. Easy to check that both \( A \) and \( B \) are positive and \( A < \frac{1}{4}\Delta + 8. \) Thus
\[ M'(x) \geq B + \frac{1}{2} \Delta + 6 - 1 + \left(\frac{\frac{1}{4} \Delta - A + 9}{A + B + \frac{1}{4} \Delta - 1} + \frac{\frac{1}{4} \Delta - 3}{A + B} + A \right) \]
\[
\geq \frac{1}{4} \Delta + A + B + \frac{1}{4} \Delta + \left(\frac{\frac{1}{4} \Delta - A + 9}{A + B + \frac{1}{4} \Delta} + \frac{\frac{1}{4} \Delta - 3}{A + B} \right) + 5 \\
= \frac{1}{4} \Delta + A + B + \frac{1}{4} \Delta + \left(\frac{\frac{1}{4} \Delta - A + 9}{A + B + \frac{1}{4} \Delta} + \frac{\frac{1}{4} \Delta - 3}{A + B} \right) + 5 \\
\geq \frac{1}{4} \Delta + A + B + \frac{1}{4} \Delta + \left(\frac{\frac{1}{4} \Delta - 3}{A + B + \frac{1}{4} \Delta} \right) + 5 \\
\geq \frac{1}{4} \Delta + 2\sqrt{\frac{1}{4} \Delta - 3} \frac{\frac{1}{4} \Delta + 5}{\frac{3}{4} \Delta - 1} > q \\
\]

**Case 3.** \( d(y) > q \) and \( \frac{1}{4} \Delta < d(x) \leq 2(\Delta - q) - 10 \) or \( \Delta - q < 11 \).

Notice that if \( d(x) \leq 2(\Delta - q) - 10 \), then \( p' = \min\{ p(x, q), \left\lfloor \frac{d(x)}{2} \right\rfloor - 6 \} \leq \frac{\Delta - q - 11}{2} \). Thus in both cases, we have \( p' \leq \Delta - q - 11 \).

Since \( q = \frac{1}{2} \Delta - 8 < \frac{1}{2} \Delta - 4 \) and \( \Delta \geq 16 \), we have \( \Delta - q \geq 12 \). Thus we may apply Lemma 19, \( x \) has at least \( d(x) - p' - 5 \) neighbors \( z \) for which \( \sigma_q(x, z) \geq \Delta - p' - 11 \). For such neighbors \( z \), since \( q \leq \Delta - p' - 11 \), we have

\[ \frac{d(z) - q}{d(z) - \sigma_q(x, z)} \geq \frac{d(z) - q}{d(z) - (\Delta - p' - 11)} \geq \frac{\Delta - q}{p' + 11}. \]

**Case 3.a** \( q > \Delta - d(x) + p' + 1 \), i.e., \( p' < d(x) + q - \Delta - 1 \).

In this case, \( \frac{d(x) - p' - 5}{d(x) + q - \Delta + 10} > \frac{\Delta - q - 4}{d(x) + q - \Delta + 10} \), thus

\[ M'(x) \geq d(x) + (d(x) - p' - 5) \frac{\Delta - q}{p' + 11} \]
\[
\geq d(x) + \frac{\Delta - q - 4}{d(x) + q - \Delta + 10} (\Delta - q) \\
= (d(x) + q - \Delta + 10) + \frac{(\Delta - q)(\Delta - q - 4)}{d(x) + q - \Delta + 10} - (q - \Delta + 10) \\
\geq 2\sqrt{(\Delta - q)(\Delta - q - 4)} + \Delta - q - 10 \geq 3(\Delta - q) - 18 > q. \]

**Case 3.b** \( q \leq \Delta - d(x) + p' + 1 \).

Following the definition of \( p' \), for every \( u \in Z_q \), we have \( \sigma_q(x, u) \geq \Delta - d(x) + p' + 1 \), which in turn gives

\[ \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq \frac{d(u) - q}{d(u) - (\Delta - d(x) + p' + 1)}. \]

Since \( q \leq \Delta - d(x) + p' + 1 \), we have \( \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq \frac{\Delta - q}{d(x) - p' - 1} = \frac{\Delta - q}{d(x) - p' - 1} \). Thus,

\[ M'(x) \geq d(x) + (d(x) - p' - 5) \frac{\Delta - q}{p' + 11} + (p' + 5) \frac{\Delta - q}{d(x) - p' - 1} \]

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Let $A = d(x) - p' - 5 \geq \frac{d(x)}{2} + 1$ and $B = p' + 11$. Since $\frac{A}{B} + \frac{B}{A} \geq 2$ and $A + B = d(x) + 6$, we have:

$$M'(x) \geq d(x) + (\Delta - q)(2 - \frac{6d(x) + 14}{(\frac{d(x)}{2} + 1)(\frac{d(x)}{2} + 5)}) \geq d(x) + (\Delta - q)(2 - \frac{14}{(\frac{d(x)}{2} + 1)(\frac{d(x)}{2} + 5)}) > \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 8)(\Delta + 40)}) \geq \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 10)(\Delta + 32)}) \geq \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 32)}) = q$$

Then since $A \geq \frac{d(x)}{2} + 1$, we have

$$M'(x) \geq d(x) + (\Delta - q)(2 - \frac{6d(x) + 14}{(\frac{d(x)}{2} + 1)(\frac{d(x)}{2} + 5)}) > \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 8)(\Delta + 40)}) \geq \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 10)(\Delta + 32)}) \geq \frac{\Delta}{4} + (\Delta - q)(2 - \frac{96}{(\Delta + 32)}) = q$$

This completes the proof of Case 3 and Claim 4.1. □

### 4.2 Part II: Claim 4.2

**Claim 4.2.** For every $x \in V(G)$ with $4 \leq d(x) \leq \frac{1}{4}\Delta$, $M'(x) \geq q$.

**Proof.** Let $x \in V(G)$ with $4 \leq d(x) \leq \frac{1}{4}\Delta$. Since the proof of claim 3.2 does not contain the assumption $\Delta \geq D(\epsilon)$, we apply that claim here with $\epsilon = \frac{1}{4}$. When $\epsilon = \frac{1}{4}$, $D_0(\epsilon) = 2\left(\frac{1}{8}\right)^3 + 2\sqrt{\left(\frac{1}{8}\right)^6 - \left(\frac{1}{8}\right)^3}$ is about 8.345. Thus we only need to prove this claim when $d(x)$ is in $\{4, 5, 6, 7, 8\}$. Choose $y \in N(x)$ and $\varphi \in \mathcal{C}(G - xy)$ such that $p = |B_\varphi(x, y, q)| = |B_\varphi(q)|$ is minimum over all neighbours of $x$ and all $\Delta$-colorings of $G - xy$. We will use the following properties to prove the claim:

1. $|S_\varphi(q)| = \Delta - p$, $p \leq d(x) - 2$ and $|N_S(x)| = d(x) - 1 - p$;
2. For each $v \in N(x)$, $d(v) \geq \sigma_q(x, v) \geq \Delta - d(x) + 1 + p$ and $|\varphi^\text{bad}(v)| \leq d(x) - 1 - p$;
3. Let $X' = \bigcup_{v \in N_S(x)}(\varphi^\text{bad}(v) \setminus \varphi(xv))$, then $X' \subseteq B_\varphi(q)$ and $|X'| = \sum_{v \in N_S(x)}(|\varphi^\text{bad}(v)| - 1)$. Consequently, $|X'| \leq p$.
Thus, if one of the neighbors of the last inequality holds. Since (4), since $d(v) ≥ |\varphi'(x)| = \Delta - d(x) + 1 + p > q$ for each $v ∈ N(x)$ by (2), $M'(x) = d(x) + \sum_{v ∈ N(x)} \frac{d(v) - q}{|\varphi'(x)|}$ by the discharging rule. Also (2) implies $|\varphi'(v)| ≤ d(x) - 1 - p$, thus the last inequality holds. Since $d(x) - 1 - p < d(x) < \Delta - q = \frac{1}{2} \Delta + 8$, we have $|\varphi'(v)| < \Delta - q$. Hence the first inequality holds.

Now we start to prove $M'(x) ≥ q$ for each $x$ with $d(x) ∈ \{4, 5, 6, 7, 8\}$.

**Case 0.** $p = 0$ or $p = d(x) - 2$.

Suppose $p = 0$. Then $|S_\varphi(q)| = \Delta$ by the first property, so $S_\varphi(q)$ contains all $\Delta$ colors. Thus by Lemma 12, for any $v ∈ N(x) \setminus \{y\}$, $d(y) ≥ |S_\varphi(q)| = \Delta$ and $\varphi'(v) = \{\varphi(x)\}$. Hence by (4), $M'(x) ≥ (d(x) - 1) \frac{\Delta - q}{2} ≥ 2(\Delta - q) > q$. Suppose $p = d(x) - 2$. Then for all $v ∈ N(x)$, we have $\sigma_q(x, v) ≥ \Delta - d(x) + 1 + p = \Delta - 1$ by (2). So $x$ is the only vertex in $N(v)$ with degree less than $q$, $|\varphi'(v)| = 1$ and $d(v) = \Delta$. Hence $M'(x) ≥ d(x)(\Delta - q) ≥ 4(\Delta - q) > q$.

**Case 1.** $d(x) = 4$.

In this case, $p ≤ d(x) - 2 = 2$. By using **Case 0**, we assume that $p = 1$. Then $|N_\varphi(x)| = 2$.

Without loss of generality, assume that $\varphi'(x) = \{\xi_1, \xi_2\}$ and $B_\varphi(q) = \{\beta\}$. So $X' ⊆ \{\beta\}$. If $X' = \emptyset$, then $\varphi'(x) = \{\xi_1, \xi_2\} = \varphi'(x) = \{\beta\}$. So $M'(x) ≥ \sum_{v ∈ N_\varphi(x)} \frac{\Delta - q}{|\varphi'(v)|} + \sum_{v ∈ N_\varphi(x) \cup \{y\}} \frac{\Delta - q}{d(x) - 1 - p} ≥ 2(\Delta - q) + 2(\Delta - q) > q$. Suppose $X' = \{\beta\}$, let’s say $\beta ∈ \varphi'(x)$. Then $\varphi'(x) = \{\xi_1, \xi_2\} = \emptyset$. By Lemma 13 and Lemma 14, we have $\varphi'(x) = \emptyset$. Thus, $M'(x) ≥ \sum_{v ∈ N_\varphi(x) \cup \{y\}} \frac{\Delta - q}{|\varphi'(v)|} + \sum_{v ∈ N_\varphi(x) \cup \{y\}} \frac{\Delta - q}{d(x) - 1 - p} ≥ 2(\Delta - q) + 2(\Delta - q) > q$. This proves **Case 1**.

**Case 2.** $d(x) = 5$.

In this case, $p ≤ d(x) - 2 = 3$. By **Case 0**, we are done if $p = 0$ or 3.

Suppose $p = 1$, then $|N_\varphi(x)| = 3$ by (1) and $\sum_{v ∈ N_\varphi(x)} (|\varphi'(v)| - 1) = |X'| ≤ 1$. Hence $M'(x) ≥ \sum_{v ∈ N_\varphi(x)} \frac{\Delta - q}{|\varphi'(v)|} + \sum_{v ∈ N_\varphi(x) \cup \{y\}} \frac{\Delta - q}{d(x) - 1 - p} ≥ 2(\Delta - q) + 2(\Delta - q) > q$.

Suppose $p = 2$, then for all $v ∈ N(x)$, $\sigma_q(x, v) ≥ \Delta - d(x) + 1 + p = \Delta - 2$, so $|\varphi'(v)| ≤ 2$. If one of the neighbors of $x$ has no neighbors other than $x$ with degree less than $q$, we are done because $M'(x) ≥ 5 + (\Delta - 1 - q) + 4(\Delta - q) > q$. Thus the only remaining case is: Each neighbors of $x$ has exactly 1 neighbor other than $x$ with degree less than $q$. Under these assumptions, $N_\varphi(x) = 2$ and $|X'| = 2$. Let $S_\varphi(q) ∩ x = \{\xi_1, \xi_2\}$ and $X' = B_\varphi(q) = \{\beta_1, \beta_2\}$. Notice that
$\varphi(y) \subseteq S_\varphi(q) \cap \varphi(x)$ and $|\varphi(y)| \geq 1$, we may assume $\xi_1 \in \varphi(y)$. Without loss of generality, by Lemma 12, let’s say $x_{\xi_1}y, x_{\xi_2}y$ have degrees less than $q$. By Lemma 13, the unique neighbor of $x_{\xi_1}$ with degree less than $q$ is $x_{\beta_1}$; The unique neighbor of $x_{\beta_2}$ with degree less than $q$ is $x_{\beta_2}$. Now, recolor $xy$ by $\xi_1$ and leave $xx_{\xi_1}$ uncolored to obtain a new coloring $\varphi_1$. Then in the new coloring, $\beta_1 \in S_{\varphi_1}(x, x_{\xi_1}, q), x_{\beta_1}\beta_2$ has degree less than $q$, i.e. $\beta_2 \in \varphi_1(\beta_1)$. By Lemma 13, $x_{\beta_2}\beta_1$ should have degree at least $q$, giving a contradiction. This proves Case 2.

**Case 3.** $d(x) = 6$.

In this case, $p \leq d(x) - 2 = 4$. If $p = 0$ or $p = 4$, we are done by Case 0.

If $p = 1$, then $|N_S(x)| = 4$, $\sum v_{N_S(x)}(|\varphi_{bad}(v)| - 1) = |X'| \leq 1$. Thus

$$M'(x) \geq \sum v_{N_S(x)}\left(\frac{\Delta - q}{|\varphi_{bad}(v)|}\right) + \sum v_{N_B(x) \cup \{y\}}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq 3(\Delta - q + \frac{\Delta - q}{2}) + 2\left(\frac{\Delta - q}{4}\right) > q.$$  

If $p = 2$, then $|N_S(x)| = 3$, $\sum v_{N_S(x)}(|\varphi_{bad}(v)| - 1) = |X'| \leq 2$. Thus

$$M'(x) \geq \sum v_{N_S(x)}\left(\frac{\Delta - q}{|\varphi_{bad}(v)|}\right) + \sum v_{N_B(x) \cup \{y\}}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq (\Delta - q) + 2\left(\frac{\Delta - q}{2}\right) + 3\left(\frac{\Delta - q}{3}\right) > q.$$  

If $p = 3$, then $M'(x) \geq \sum v_{N(x)}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq 6\left(\frac{\Delta - q}{2}\right) > q$. This proves Case 3.

**Case 4.** $d(x) = 7$.

In this case, $p \leq d(x) - 2 = 5$. Similarly, if $p = 0$ or $p = 5$ we are done. Also by similar methods we used in Case 3, it is routine to check the cases when $p = 1, 2, 4$.

If $p = 2$, then $|N_S(x)| = 4$, $|X'| \leq 3$ and for all $v \in N(x), \sigma_q(v) \geq \Delta - d(x) + 1 = \Delta - 3$, so $|\varphi_{bad}(v)| \leq 3$. If one vertex in $N_S(x)$ has no neighbors other than $x$ with degree less than $q$, then we have $M'(x) \geq (\Delta - q + (\frac{\Delta - q}{2}) + (\frac{\Delta - q}{3})) + 4(\frac{\Delta - q}{6}) > q$. So we may assume that each vertex in $N_S(x)$ has exactly 1 neighbor other than $x$ with degree less than $q$. Moreover, if one vertex in $N_B(x)$ has less than 2 neighbors other than $x$ with degree less than $q$, then we have $M'(x) \geq 3(\frac{\Delta - q}{2}) + 3(\frac{\Delta - q}{3}) + \frac{\Delta - q}{2} > q$. Therefore we may assume that $|N_S(x)| = 3, |X'| = 3$, each vertex in $N_S(x)$ has exactly 1 neighbor other than $x$ with degree less than $q$ and each vertex in $N_B(x)$ has exactly 2 neighbors other than $x$ with degree less than $q$. Let $S_\varphi(q) \cap \varphi(x) = \{\xi_1, \xi_2, \xi_3\}$ and $X' = B_\varphi(q) = \{\beta_1, \beta_2, \beta_3\}$. Notice that $\varphi(y) \subseteq S_\varphi(q) \cap \varphi(x)$ and $|\varphi(y)| \geq 1$, we may assume $\xi_1 \in \varphi(y)$. Without loss of generality, let’s say $x_{\xi_1}$ has degree less than $q$. By Lemma 13, the two neighbors of $x_{\beta_1}$ with degree less than $q$ are $x_{\beta_1}\beta_2, x_{\beta_1}\beta_3$; The two neighbors of $x_{\beta_2}$ with degree less than $q$ are $x_{\beta_2}\beta_1, x_{\beta_2}\beta_3$. We don’t have to consider $x_{\beta_3}$. Now, recolor $xy$ by $\xi_1$ and leave $xx_{\xi_1}$ uncolored to obtain a new coloring $\varphi_1$. Then in the new coloring, $\beta_1 \in S_{\varphi_1}(x, x_{\xi_1}, q), x_{\beta_1}\beta_2$ has degree less than $q$, by Lemma 13, $x_{\beta_2}\beta_1$ should have degree at least $q$, giving a contradiction. This proves Case 4.

**Case 5.** $d(x) = 8$.

In this case, $p \leq d(x) - 2 = 6$. Actually all the cases can be verified by the same methods we used in Case 3.

If $p = 3$, then $|N_S(x)| = 4$, $\sum v_{N_S(x)}(|\varphi_{bad}(v)| - 1) = |X'| \leq 3$. Thus

$$M'(x) \geq \sum v_{N_S(x)}\left(\frac{\Delta - q}{|\varphi_{bad}(v)|}\right) + \sum v_{N_B(x) \cup \{y\}}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq (\Delta - q) + 3(\frac{\Delta - q}{2}) + 4(\frac{\Delta - q}{6}) > q.$$  

If $p = 4$, then $|N_S(x)| = 3$, $\sum v_{N_S(x)}(|\varphi_{bad}(v)| - 1) = |X'| \leq 4$. Thus

$$M'(x) \geq \sum v_{N_S(x)}\left(\frac{\Delta - q}{|\varphi_{bad}(v)|}\right) + \sum v_{N_B(x) \cup \{y\}}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq (\frac{\Delta - q}{3}) + 2(\frac{\Delta - q}{2}) + 5(\frac{\Delta - q}{3}) > q.$$  

If $p \geq 5$, then $M'(x) \geq \sum v_{N(x)}\left(\frac{\Delta - q}{d(x) - 1 - p}\right) \geq 8\left(\frac{\Delta - q}{2}\right) > q$.

This completes the proof of Claim 4.2. \qed
4.3 Part III: Vertices with degree two or three

4.3.1 Lemmas (Part III)

For the vertices with degree two or three, the neighbors of them are not enough to ensure what we want, hence we have to find more vertices other than the neighbors to help us attain the requirement. Fortunately, we do have some candidates for 2-vertices.

Lemma 20. Let $x$ be a 2-vertex in $V(G)$, the following statements are true:

1. For any $z \in N(x)$, $d(z) = \Delta$.
2. For any $z' \in N(N(x)) \setminus \{x\}$, $d(z') = \Delta$ and $d(v) \geq \Delta - 1$.
3. For any $z' \in N(N(x)) \setminus \{x\}$ and $v \in N(z') \setminus \{x\}$, $d(v) \geq \Delta - 1$.

Proof. Let $z \in N(x)$, $z' \in N(z) \setminus \{x\}$ and $v \in N(z') \setminus \{x\}$. We will prove that $d(z) = d(z') = \Delta$ and $d(v) \geq \Delta - 1$, which implies the lemma. Let $\phi \in \mathcal{C}^\Delta(G - xz)$. Since $d(x) = 2$, we have $|\phi(x)| = \Delta - 1$. Notice that $|\phi(z)| \geq 1$, $|\phi(z')| \geq 0$ and $\phi(x) \cap \phi(z) = \emptyset$. So $\Delta \geq |\phi(x) \cup \phi(z)| \geq |\phi(x)| + |\phi(z)| \geq \Delta$, the equality hold and $|\phi(x)| = 1$. Thus all colors are in $\phi(x) \cup \phi(z)$, $xzz'v$ form a Kierstead path when $v \neq z$. By Lemma 1 and 3, $\{x, z, z'\}$ is elementary and $|\phi(v) \cap (\phi(x) \cup \phi(z) \cup \phi(z'))| \leq 1$ if $v \neq z$. Therefore $|\phi(z')| \leq |\{1, \Delta\} \setminus (\phi(x) \cup \phi(z))| = 0$. Thus $d(z) = d(z') = \Delta$. Moreover, if $v \neq z$, we have $|\phi(v)| \leq 1$, i.e. $d(v) \geq \Delta - 1$, which is also true when $v = z$. \hfill \Box

Let $L(x) = N(N(x)) \setminus N[x]$. Then for any $x$ with $d(x) = 2$, each vertex in $L(x)$ has degree $\Delta$ and all its neighbors have degrees at least $\Delta - 1$ by Lemma 20. Thus all vertices in $L(x)$ are not used in the previous discharging rule. Now we consider all the vertices with degree 3. After applying the previous discharging rule, some 3-vertices may already have $M' \geq q$, we call these vertices type I 3-vertices and call the rest type II 3-vertices. Similar to the lemma above, for each type II 3-vertex $x$, we hope that many vertices in $L(x)$ have large degrees and no neighbors of them have small degrees. We will prove the following lemma by using some results in previous sections.

Lemma 21. If $x$ is a type II 3-vertex in $V(G)$ with $N(x) = \{x_1, x_2, x_3\}$. Then the following two statements hold.

1. Each $x_i$ has degree $\Delta$ and there is only one vertex in $L(x)$ with degree less than $q$.
2. There exists an $i \in \{1, 2, 3\}$ and a vertex $w \in N(x_i)$ such that for all $v \in N(x_i) \setminus (N[x] \cup \{w\}) \subseteq L(x)$, $d(v) \geq \Delta - 1$ and $d(v') \geq q$ for any $v' \in N(v)$.

Proof. We firstly prove (1). Choose $x_i \in N(x)$ such that $\sigma_q(x, x_i) - \Delta + d(x) - 1 = p_{\min}(x, q) = p$, without loss of generality, let’s say it is $x_1$. By the definition of $p$, we have $p \leq d(x) - 2 = 1$. Let $\phi \in \mathcal{C}^\Delta(G - xx_1)$ be a coloring such that $\phi(xx_2) = 2$, $\phi(xx_3) = 3$. If $p = 1$, then for all $x_i \in N(x)$, we have $\sigma_q(x, x_i) \geq \Delta - d(x) + 1 + p = \Delta - 1$, $d(x_i) = \Delta$ and $|\phi^\text{bad}(x_i)| = 1$. Hence $M'(x) = 3 + 3(\Delta - q) > q$, which contradict the fact that $x$ is of type II. Thus $p$ has to be 0. In this case, all the colors are contained in $S_\phi(q)$, hence by Lemma 12, $d(xx_2) = d(x_3) = \Delta$ and $x_2, x_3$ have no neighbors other than $x$ with degree less than $q$. Since $x$ is of type II, $x_1$ has a neighbor $w \neq x$ with degree less than $q$, thus $d(x_1) \geq \Delta - d(x_1) + 1 + |\{x\}| + |\{w\}| = \Delta$. This proves (1). Moreover, $\phi(x_1 w)$ cannot be a missing color of $x$ by Lemma 1, so it has to be a color in (2, 3), without loss of generality, let’s say $\phi(x_1 w) = 2$.

Now we prove (2). More specifically, we claim that under the setting in (1), $i = 1$ and the $w$ in (2) is just the $w$ we used above. Choose an arbitrary vertex in $N(x_1) \setminus \{w, x, x_2, x_3\} \subseteq L(x)$, assume the edge between this vertex and $x_1$ is colored by $\alpha \in \phi(x) \cup \phi(x_1) = \{1, 3, 4, \ldots, \Delta\}$ and
denote this vertex by \( v \). By Lemma 2, we have \( d(v) \geq \Delta - 1 \). For any \( v' \in N\left(N(x)\right) \setminus \{x, x_1, x_2, x_3\} \), if \( \varphi(vv') \in \{1, 3, 4, \ldots, \Delta\} \), then \( d(v') \geq \Delta - 2 > q \) by Lemma 3. (This also implies that \( w \) is not in \( N(v) \).) If \( \varphi(vv') = 2 \), choose \( \delta \neq \alpha \) such that \( \delta \in \varphi(x) \cap \varphi(w) \), this is possible because \( d(x) = 3 \) and \( d(w) < q \). Then we do a \( (2, 3, \delta) \)-switching in \( \varphi \) at \( x_1 \) to obtain a new coloring \( \varphi_1 \). This will not affect \( x_1v \) and \( vv' \) by Lemma 6. Now we have \( 2 \in \varphi_1(x_1) \), by Lemma 3 again, \( d(v') \geq \Delta - 2 > q \). This proves (2).

### 4.3.2 Proof of Claim 4.3 and Claim 4.4

In this subsection, \( q \) is still \( \frac{3}{4} \Delta - 8 \). We are finally able to complete our discharging rule as the following:

- **The First Step of Discharge:** Each vertex \( y \) with degree larger than \( q \) distributes \( d(y) - q \) equally among all neighbors of \( y \) with degree less than \( q \).

- **The Second Step of Discharge:** For all \( v \in V(G) \) with \( d(v) \geq \Delta - 1 \), if \( L(v) \) contains a 2-vertex or a 3-vertex of type II and all neighbors of \( v \) have degrees at least \( q \), then \( v \) sends \( \frac{1}{4} \) to each 2-vertex in \( L(v) \) and \( \frac{1}{8} \) to each type II 3-vertex in \( L(v) \).

We still denote by \( M' \) the resulting charge on each vertex \( x \). Now we prove the following claims.

**Claim 4.3.** Claim 4.1 and Claim 4.2 still hold under the new discharging rule.

*Proof.* Since the second step will only affect a vertex \( v \) when \( d(v) \geq \Delta - 1 \), \( L(v) \) contains a 2-vertex or a 3-vertex of type II and \( v \) has no neighbors with degree less than \( q \), we only need to prove that \( M'(v) \) is at least 3 for such \( v \). Assume that \( x \) is a 2-vertex or a 3-vertex of type II in \( L(v) \), then \( v \in L(x) \). By our assumption, we have that \( v \) has no neighbors with degree less than \( q \), thus the first step of discharge will not affect \( v \). Let \( z \) be a neighbor of \( v \), by Lemma 20 and Lemma 21 , if \( z \) is adjacent to a 2-vertex, then all other neighbors of \( z \) has degree \( \Delta \); If \( z \) is adjacent to a 3-vertex of type II, then \( z \) has at most 1 more neighbor with degree less than \( q \), this neighbor of \( z \) has degree at least 3 by Lemma 20. Thus in both cases, \( v \) sends at most \( \frac{1}{4} \) in total to the neighbors of \( z \) in the second step of discharge, \( M'(v) \geq \Delta - 1 - \frac{1}{4} \Delta > q \). This proves the claim. \( \square \)

**Claim 4.4.** For every \( x \in V(G) \) with \( d(x) = 2 \) or \( 3 \), \( M'(x) \geq q \).

*Proof.* If \( d(x) = 2 \), by Lemma 20, all vertices in \( N(N(x)) \setminus \{x\} \) are \( \Delta \)-vertices. So after the first step of charge, \( M'(x) \geq 2 + (\Delta - q) + (\Delta - q) \). By Lemma 20 again, there are at least \( \Delta - 1 \) distinct \( \Delta \)-vertices in \( L(x) \) which have no neighbors with degree less than \( q \), thus under the new discharging rule, \( M'(x) \geq 2 + (\Delta - q) + (\Delta - q) + \frac{1}{4}(\Delta - 1) > q \).

If \( x \) is a type I 3-vertex, we are done. If \( x \) is a type II 3-vertex, by Lemma 21, all its neighbors are \( \Delta \)-vertices and there is only one vertex in \( L(x) \) has degree less than \( q \). So after the first step of charge, \( M'(x) \geq 3 + (\Delta - q) + (\Delta - q) + \frac{1}{4}(\Delta - q) \). By Lemma 21 again, there are at least \( \Delta - 4 \) distinct vertices in \( L(x) \) with degree at least \( \Delta - 1 \) and have no neighbors with degree less than \( q \). (In Lemma 21, \( N(x_i) \setminus (N[x] \cup \{w\}) \) has size \( \Delta - 4 \).) Thus under the new discharging rule, \( M'(x) \geq 3 + (\Delta - q) + (\Delta - q) + \frac{1}{2}(\Delta - q) + \frac{1}{8}(\Delta - 4) > q \). This proves the claim. \( \square \)

Now combining Claim 4.1, Claim 4.2, Claim 4.3 and Claim 4.4, we proved Theorem 1.
5 Proofs of Lemma 7 and 8

In this section, let \( \epsilon \in (0, 1) \) be a given real number and \( c_0 = \left\lfloor \frac{1-\epsilon}{\epsilon} \right\rfloor \) as before. For any two positive integers \( m_1 \) and \( m_2 \), we define the functions \( f_1, f_2, f_3 \) and \( f \) as below.

\[
f_1(m_1, m_2, \epsilon) = \frac{1}{\epsilon^2}(m_1 + c_0m_2),
\]
\[
f_2(m_1, m_2, \epsilon) = \max\{f_1(m_1, m_2 + 1, \epsilon), \frac{1}{\epsilon}(m_2 + (c_0 + 1)m_1 + 1)\},
\]
\[
f_3(\epsilon) = (c_0 + 1)(c_0 + 3)(3c_0) + c_0 + 2,
\]
\[
f(\epsilon) = f_2(c_0 + 1, f_3(\epsilon), \epsilon).
\]

We also assume that \( G \) is a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( q = (1-\epsilon)\Delta \), \( xy \in E(G) \) and \( d(x) < \epsilon q \).

**Fact 1:** Let \( v_1, v_2, \ldots, v_{c_0+1} \) be \( c_0 + 1 \) vertices with degree less than \( q \) and \( \varphi \in C^\Delta(G - xy) \). If \( |\varphi(v_i) \cap \varphi(x)| \leq m_1 \) for all \( i = 1, 2, \ldots, c_0 + 1 \) and \( \Delta \geq f_1(m_1, m_2, \epsilon) \), then there exist two vertices \( v_i, v_j \) such that \( |(\varphi(v_i) \cap \varphi(v_j)) \setminus \varphi(x)| \geq m_2 \).

**Proof.** Notice that a vertex with degree less than \( q \) has more than \( \epsilon\Delta \) missing colors. If **Fact 1** fails, then for each \( v_i \), there are at least \( \epsilon \Delta - m_1 - c_0m_2 \) colors in \( \varphi(v_i) \) that are not in \( (\bigcup_{j\neq i} \varphi(v_j)) \cup \varphi(x) \). But then since \( c_0 + 1 \geq \frac{1}{\epsilon} \) and \( \Delta - d(x) > \epsilon \Delta \), there are at least \((c_0 + 1)(\epsilon \Delta - m_1 - c_0m_2) + \Delta - d(x) + 1 > (c_0 + 1)(\epsilon \Delta - \epsilon^2 \Delta) + \epsilon \Delta \geq \Delta \) distinct colors, which is a contradiction. \( \square \)

Given a coloring \( \varphi \in C^\Delta(G - xy) \), a color set \( C \) and a vertex set \( T \subset V(G) \), we call the Kempe change \( \varphi \rightarrow \varphi/P_v(\delta, \eta, \varphi) \) a \( S(C, T) \) if \( v \in T, \delta \in \varphi(x) \setminus (C \cup \varphi(T)), \eta \in \varphi(v) \setminus (\varphi(x) \cup C) \) and \( x \notin V(P_v(\delta, \eta, \varphi)) \).

**Fact 2:** Following the notation above, there exists a \( S(C, T) \) if the conditions below hold:

- \( |T| = c_0 + 1 \). For each \( v \in T \), \( d(v) < q \),
- \( |\bar{\varphi}(v) \cap \varphi(x)| \leq m_1 \) for each \( v \in T \) and \( \Delta \geq f_2(m_1, |C|, \epsilon) \).

**Proof.** Since \( \Delta > f_2(m_1, |C|, \epsilon) \geq f_1(m_1, |C| + 1, \epsilon) \), by **Fact 1**, there are two vertices \( u, v \in T \) such that \( |(\varphi(u) \cap \varphi(v)) \setminus \varphi(x)| \geq |C| + 1 \). So, there exists a color \( \eta \in (\varphi(u) \cap \varphi(v)) \setminus (\varphi(x) \cup C) \). Since \( \Delta \geq f_2(m_1, |C|, \epsilon) \geq (|C| + (c_0 + 1)m_1 + 1)/\epsilon, |\varphi(x)| \geq \Delta - \epsilon q = \epsilon \Delta \geq |C| + (c_0 + 1)m_1 + 1 \), we can find a color \( \delta \in C \cup (\varphi(T) \cap \varphi(x)) \). Now, one of \( u, v \) is not on \( P_v(\eta, \delta, \varphi) \), say \( v \). Then, \( \varphi' = \varphi/P_v(\eta, \delta, \varphi) \) is the desired switching. \( \square \)

The following observation follows directly from the definition of \( S(C, T) \).

**Fact 3.** If \( \varphi \rightarrow \varphi' \) is a \( S(C, T) \), then \( \sum_{v \in T} |\varphi'(v) \cap \varphi'(x)| \geq \sum_{v \in T} |\varphi(v) \cap \varphi(x)| + 1 \) with \( \delta \) accounting for the ‘+1’, and edges with colors in \( C \) are unchanged.

Recall that \( A_\varphi(q) = \{\alpha \in \varphi(x) \cap \varphi(y) : d(y_\alpha) < q\} \) and \( S_\varphi(q) = A_\varphi(q) \cup \varphi(x) \cup \varphi(y) \). We restate Lemma 7 as the following for the proof purpose.

**Lemma 7.** Let \( G, x \) and \( xy \) be defined as above. If \( \Delta \geq f(\epsilon) \), then for every \( \varphi \in C^\Delta(G - xy) \) there does not exist a vertex \( z \in N(x) \setminus \{y\} \) such that the following two statements hold.
A1: \( z \in N(x) \) and \( \varphi(xz) \in \bar{\varphi}(y) \).

A2: There exists \( c_0+1 \) colors \( \beta \in \varphi^{\text{bad}}(z) \cap \varphi(x) \setminus \{ \varphi(xz) \} \) such that \( |\varphi^{\text{bad}}(x_\beta) \cap S_\varphi(q) \setminus \{ \beta \}| \geq 3c_0 \).

**Proof.** Suppose on the contrary there exists a coloring \( \varphi \in \mathcal{C}^A(G - xy) \), a vertex \( z \in N(x) \) with \( \varphi(xz) \in \bar{\varphi}(y) \) and \( c_0 + 1 \) colors \( \beta_1, \beta_2, \ldots, \beta_{c_0+1} \) satisfies A2. Without loss of generality, we assume \( \varphi(xz) = 1 \), that is, \( z = x_1 \). We will show that the following statement holds.

(5.1) There exists a coloring \( \pi \in \mathcal{C}^A(G - xy) \) such that \( \pi(xz) = \varphi(xz) = 1 \in \bar{\pi}(y) \) and a color \( \beta \in \pi(x) \) such that \( |\bar{\pi}(x) \cap \pi^{\text{bad}}(x_\beta)| \) contains \( c_0 \) colors \( \xi_1, \xi_2, \ldots, \xi_{c_0} \). Additionally, we have either \( \beta \in \pi(z) \) or \( \beta \in \pi(z) \) and there exists a color \( \delta_1 \in (\pi(x_\beta) \cap \bar{\pi}(x)) \setminus \{ \xi_1, \xi_2, \ldots, \xi_{c_0} \} \).

We first prove Lemma 7 assuming (5.1) is true.

Let \( \pi_1 \in \mathcal{C}^A(G - xz) \) be obtained from \( \pi \) by assigning the color 1 to edge \( xy \) and leaving \( xz \) uncolored. We claim that we may assume \( \beta \in \pi_1(z) \). Otherwise there exists \( \delta_1 \in \pi_1(z_1) \cap \bar{\pi}_1(x) \) and \( \delta_1 \notin \{ \xi_1, \xi_2, \ldots, \xi_{c_0} \} \). Then we can do a \((\beta, 1, \delta_1)\)-switching at \( z \) in \( \pi_1 \), \( \beta \) is missing at \( z \) now. Notice that the color of \( xz_\beta \) will not change by Lemma 6. Also the edges with colors in \( \{ \xi_1, \xi_2, \ldots, \xi_{c_0} \} \) stay the same.

Now under \( \pi_1 \), \( \{ z, x, x_\beta \} \) is a multi-fan, so it is elementary, which in turn shows \( x_\beta, \xi_1, x_\beta, \xi_2, \ldots, x_\beta, \xi_{c_0} \) exist. For each \( i = 1, 2, \ldots, c_0 \), since \( \xi_i \in \pi_1^{\text{bad}}(x_\beta) \), \( d(x_\beta, \xi_i) < q \). Under \( \pi_1 \), \( \{ z, x, x_\beta, x_\beta, \xi_1, x_\beta, \xi_2, \ldots, x_\beta, \xi_{c_0} \} \) form a simple broom, so it is elementary by Lemma 4. Thus, \( \Delta > |\bar{\pi}_1(x)| + \sum_{i=1}^{c_0} |\bar{\pi}_1(x_\beta, \xi_i)| > (c_0 + 1)\epsilon \Delta \geq \Delta \), giving a contradiction.

Now we proceed with the proof of (5.1). For each color \( \beta_i \), let \( T_i \subseteq \varphi^{\text{bad}}(x_\beta_i) \cap S_\varphi(q) \setminus \{ \beta_i \} \) with \( |T_i| = 3c_0 \). Let \( W = \bigcup_{i=1}^{c_0+1} T_i \cup \{ 1, \beta_1, \beta_2, \ldots, \beta_{c_0+1} \} \). Clearly, \( |W| \leq (c_0 + 1)(3c_0 + 1) \). For any non-negative number \( m \), we introduce the following two subsets of \( A_\varphi(q) \).

\[
T_{A_\varphi}(m) = \{ \eta \in (\bigcup_{i=1}^{c_0+1} T_i) \cap A_\varphi(q) : |\bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| \leq m \};
\]
\[
T_{A_\varphi'}(m) = \{ \eta \in (\bigcup_{i=1}^{c_0+1} T_i) \cap A_\varphi(q) : |\bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| = m \}.
\]

Then we let

\[
R_\varphi(m) = \bigcup_{\eta \in T_{A_\varphi}(m+1)} \bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y)).
\]

Notice that \( |R_\varphi(m)| \leq (m + 1)|\bigcup_{i=1}^{c_0+1} T_i| \leq (m + 1)(c_0 + 1)(3c_0) \).

Claim 5.1. We may assume \( |T_{A_\varphi}(c_0 + 1)| \leq c_0 \).

**Proof.** Over all colorings satisfying A1 and A2, we assume \( \varphi \) satisfies the following additional conditions: (a) \( |T_{A_\varphi}(c_0 + 1)| \) is minimum; (b) subject to (a), \( \sum_{\eta \in T_{A_\varphi}(c_0 + 1)} |\bar{\varphi}(y_\eta) \cap \bar{\varphi}(x)| \) is maximum. We claim \( |T_{A_\varphi}(c_0 + 1)| \leq c_0 \). Otherwise, there exists a set \( T = \{ y_\eta : \eta \in T_{A_\varphi}(c_0 + 1) \} \) with \( |T| = c_0 + 1 \). Let \( C_\varphi = W \cup R_\varphi(c_0 + 1) \). Since \( |C_\varphi| \leq |W| + (c_0 + 1)|\bigcup_{i=1}^{c_0+1} T_i| \leq (c_0 + 1)(3c_0 + 1) + 1 + (c_0 + 2)(c_0 + 1)(3c_0) = f_3(\epsilon) \) and \( \Delta \geq f(\epsilon) \), we have that \( \Delta \geq f_2(c_0 + 1), |C_\varphi|, \epsilon \) Thus by Fact 2, there is a \( S(C_\varphi, T) : \varphi \rightarrow \varphi_1 \). For any \( \eta \in (\bigcup_{i=1}^{c_0+1} T_i) \cap A_\varphi(q) \), if \( |\bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| > c_0 + 2 \), then \( |\bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| > c_0 + 1 \). If \( |\bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y))| \leq c_0 + 2 \), then \( \bar{\varphi}(y_\eta) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y)) \subset R_\varphi(c_0 + 1) \), the colors in this set are unchanged. Thus, we have \( T_{A_\varphi}(c_0 + 1) \subseteq T_{A_\varphi}(c_0 + 1) \). By the minimality of \( |T_{A_\varphi}(c_0 + 1)| \), we have \( T_{A_\varphi}(c_0 + 1) = T_{A_\varphi}(c_0 + 1) \). Then \( \sum_{\eta \in T_{A_\varphi}(c_0 + 1)} |\bar{\varphi}(y_\eta) \cap \bar{\varphi}(x)| \geq \sum_{\eta \in T_{A_\varphi}(c_0 + 1)} |\bar{\varphi}(y_\eta) \cap \bar{\varphi}(x)| + 1 \) by Fact 3, which contradicts the maximality of \( \sum_{\eta \in T_{A_\varphi}(c_0 + 1)} |\bar{\varphi}(y_\eta) \cap \bar{\varphi}(x)| \). \( \Box \)

Claim 5.2. We may further assume that there exists a color in \( \{ \beta_1, \beta_2, \ldots, \beta_{c_0+1} \} \), say \( \beta_1 \), such that either \( \beta_1 \in \bar{\varphi}(z) \) or \( (\bar{\varphi}(x_{1\beta_1}) \cap \bar{\varphi}(x)) \setminus \bigcup_{i=1}^{c_0} T_i \neq \emptyset \).
Proof. If Claim 5.2 fails for $\varphi$, then $\beta_1, \beta_2, \ldots, \beta_{c_0 + 1} \in \varphi(z)$ and $\bigcup_{i=1}^{c_0+1}((\overline{\varphi}(x_{\beta_i}) \cap \varphi(x)) \setminus \bigcup_{i=1}^{c_0+1} T_i) = \emptyset$. Since $\beta_1, \beta_2, \ldots, \beta_{c_0 + 1}$ are in $\varphi^{\text{bad}}(z)$, $d(x_{\beta_i}) < q$ for $i = 1, 2, \ldots, c_0 + 1$. Let $T = \{x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_{c_0 + 1}}\}$ and $C_\varphi = (\bigcup_{i=1}^{c_0+1} T_i) \cup R_\varphi(c_0 + 1)$. Since $|C_\varphi| \leq (c_0 + 1)(3c_0) + (c_0 + 2)(c_0 + 1)(3c_0) < f_3(e)$ and $\Delta \geq f(e)$, we have that $\Delta \geq f_2(c_0 + 1, |C_\varphi|, e)$. Thus by Fact 2, let $\varphi \rightarrow \varphi_1$ be a $S(C_\varphi, T)$. For any $\eta \in (\bigcup_{i=1}^{c_0+1} T_i) \cap A_\varphi(q)$, if $|\overline{\varphi}(y_{\eta}) \cap (\overline{\varphi}(x) \cup \varphi(y))| > c_0 + 2$, then $|\overline{\varphi}(y_{\eta}) \cap (\overline{\varphi}(x) \cup \varphi(y))| > c_0 + 1$. If $|\overline{\varphi}(y_{\eta}) \cap (\overline{\varphi}(x) \cup \varphi(y))| = c_0 + 2$, then $\overline{\varphi}(y_{\eta}) \cap (\overline{\varphi}(x) \cup \varphi(y)) \subset R_\varphi(c_0 + 1)$, the colors in this set are unchanged. Thus, we have $TA_{\varphi}(c_0 + 1) \subseteq TA_\varphi(c_0 + 1)$, which in turn shows that Claim 5.1 holds for $\varphi_1$ as well.

Moreover, $\sum_{i=1}^{c_0+1} |\overline{\varphi}(x_{\beta_i}) \cup \varphi_1(x)| \geq \sum_{i=1}^{c_0+1} |\overline{\varphi}(x_{\beta_i}) \cup \varphi(x)| + 1$ and the additional color is not in $\bigcup_{i=1}^{c_0+1} T_i \subset C_\varphi$. Hence, there exists $\beta_1$ such that $(\overline{\varphi}(x_{\beta_1}) \cap \varphi_1(x)) \setminus \bigcup_{i=1}^{c_0+1} T_i \neq \emptyset$. □

We assume that $\beta_1$ is the color satisfies the claim above. Moreover, if $\beta_1 \in \varphi(z)$, we let $\delta_1 \in (\overline{\varphi}(x_{\beta_1}) \cap \varphi(x)) \setminus \bigcup_{i=1}^{c_0+1} T_i$.

For a sequence of colorings $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_i, \ldots$, with $\varphi_0 = \varphi$ such that $\varphi_{i+1} = \varphi_i / P_i$ for $i = 0, 1, \ldots$ where $P_i$’s are $(\alpha_i, \gamma_i)$-chains not passing through $x$. $(\alpha_i, \gamma_i \neq \beta_1)$. Let $T_1(\varphi_0) = T_1$, we define $T_1(\varphi_{i+1})$ for $i = 0, 1, \ldots$ as below:

$$T_1(\varphi_{i+1}) = \begin{cases} T_1(\varphi_i) \cup \{\gamma_i\} \setminus \{\alpha_i\} & x_{\beta_1} \in V(P_i), \alpha_i \in T_1(\varphi_i) \text{ and } \gamma_i \notin T_1(\varphi_i), \\ T_1(\varphi_i) \cup \{\alpha_i\} \setminus \{\gamma_i\} & x_{\beta_1} \in V(P_i), \gamma_i \in T_1(\varphi_i) \text{ and } \alpha_i \notin T_1(\varphi_i), \\ T_1(\varphi_i) & \text{Otherwise.} \end{cases}$$

Easy to see that each $T_1(\varphi_i)$ is still in $\varphi^{\text{bad}}(x_{\beta_1}) \cap S_\varphi(q) \setminus \{\beta_1\}$ if $\{\alpha_i, \gamma_i\} \subseteq S_{\varphi_{i-1}}(q)$ for $i = 1, 2, \ldots$. Let $S_1(\varphi_i) = T_1(\varphi_i) \cap \varphi(x)$. Then

Claim 5.3. There exists a coloring $\pi \in \mathcal{C}^\Delta(G - xy)$ obtained from $\varphi$ through Kempe changes such that $T_1(\pi) \in \pi^{\text{bad}}(x_{\beta_1}) \cap S_\pi(q) \setminus \{\beta_1\}$, colors on edges incident with $x$ unchanged and colors in $\{\beta_1, \delta_1\} \cup TA_\varphi(c_0 + 1)$ stay unchanged, and the following three statements hold.

B1. $\pi(xz) = \varphi(xz) = 1$ and $\pi(xz) \in \pi(x)$ or $\pi(xz) \in A_\pi(q) \setminus TA_\pi(c_0 + 1)$,

B2. $|S_1(\pi)| \geq c_0$, and

B3. $TA_\pi(c_0 + 1) \subseteq TA_\varphi(c_0 + 1)$.

We demonstrate how to use Claim 5.3 to prove (5.1) before proceed with its proof. Now assume $\pi$ satisfies Claim 5.3. Since $\varphi$ and $\pi$ agree on edges colored by $\beta_1$ or $\delta_1$, we have $\beta_1 \in \pi(x)$ and there exists $\delta_1 \in \pi(x_{\beta_1}) \cap \pi(x) \setminus T_1(\pi)$ if $\delta_1 \notin \pi(z)$. Assume that $S_1(\pi)$ contains $c_0$ colors $\xi_1, \xi_2, \ldots, \xi_{c_0}$. Clearly $\delta_1 \notin \{\xi_1, \xi_2, \ldots, \xi_{c_0}\}$ since $\delta_1 \notin T_1(\pi)$. $\{\xi_1, \xi_2, \ldots, \xi_{c_0}\} \subseteq T_1(\pi) \cap \pi(x) \subseteq \pi(x) \cap \pi^{\text{bad}}(x_{\beta_1})$ since $T_1(\pi) \in \pi^{\text{bad}}(x_{\beta_1})$.

We claim that we may assume $1 = \pi(xz) \in \pi(y)$. Otherwise, $1 = \pi(xz) \in A_\pi(q) \setminus TA_\pi(c_0 + 1)$, hence $|\pi(y_1) \cap (\pi(x) \cup \pi(y))| > c_0 + 1$. Choose $\theta \in \pi(y_1) \cap \pi(y)$ if $\pi(y_1) \cap \pi(y) \neq \emptyset$; otherwise let $\theta \in \pi(y)$ and choose $\delta \in \pi(y_1) \cap \pi(x)$ such that $\delta \notin \{\xi_1, \xi_2, \ldots, \xi_{c_0}\} \cup \{\delta_1\}$. This is possible since $|\pi(y_1) \cap (\pi(x) \cup \pi(y))| > c_0 + 1$ and $\pi(y_1) \cap \pi(y) = \emptyset$. Now we do a $(1, \theta, \delta)$-switching at $y$ in $\pi$. After the switching, $1 = \pi(xz)$ is missing at $y$.

Now assume $1 = \pi(xz) \in \pi(y)$. We point out that the colors in $\{\xi_1, \xi_2, \ldots, \xi_{c_0}\} \cup \{\delta_1\}$ are unchanged in the switching above. By Lemma 6, $x_{\beta_1}$ stays unchanged as well. So we still have $\delta_1 \in \pi(x_{\beta_1}) \cap \pi(x) \setminus T_1(\pi)$ and $\{\xi_1, \xi_2, \ldots, \xi_{c_0}\} \subseteq \pi^{\text{bad}}(x_{\beta_1}) \cap \pi(x) \setminus \{\beta_1\}$. Hence (5.1) holds. □

Finally we give the proof of Claim 5.3.

Proof of Claim 5.3
Proof. We firstly note that, besides the properties in the claim, using the Kempe changes we mentioned in the proof of this claim will keep the following properties (assuming \( \varphi \) is the resulting coloring for the Kempe change):

\[
\varphi_1(xz) = \varphi(xz) = 1; \text{ either } \beta_1 \in \bar{\varphi}_1(z) \text{ or } \beta_1 \in \varphi_1(z) \text{ and there exists } \delta_1 \in \bar{\varphi}_1(x_1\beta_1) \cap \bar{\varphi}_1(x) \setminus \bigcup_{i=1}^{c_0+1} T_i; \quad T_1(\varphi) \subseteq \bar{\varphi}_1^{bad}(x_\beta_1) \cap S_{\bar{\varphi}_1}(g) \setminus \{\beta_1\} \text{ and } |T_1(\varphi)| = 3c_0.
\]

Apparently \( \varphi \) satisfies all required conditions except B2. Hence we may let \( \pi \in C(G - \bar{x}y) \) satisfying all required conditions in the claim except B2 with maximum value of \( |S_1(\pi)| \). We will show that \( |S_1(\pi)| \geq c_0 \), which implies the claim.

Suppose otherwise \( |S_1(\pi)| \leq c_0 - 1 \). Since \( |T_1(\pi)| = |T_1(\varphi)| = 3c_0 \), by B3 and Claim 5.1, we can find \( c_0 + 1 \) colors \( \tau_1, \tau_2, \ldots, \tau_{c_0+1} \) in \( T_1(\pi) \setminus (S_1(\pi) \cup TA_\varphi(c_0 + 1)) = T_1(\pi) \setminus (\bar{\pi}_1(x) \cup TA_\varphi(c_0 + 1)) \). Recall that \( T_1(\pi) \subseteq \bar{\varphi}_1^{bad}(x_\beta_1) \cap S_{\bar{\varphi}_1}(g) \setminus \{\beta_1\} \subseteq S_{\bar{\varphi}_1}(g) \). We divide the proof into the following cases.

Case 1. One of \( \tau_1, \tau_2, \ldots, \tau_{c_0+1} \), say \( \tau_1 \), is in \( \bar{\pi}_1(x_{\beta_1}) \).

---

Choice a color \( \delta \in \bar{\pi}(x) \setminus (S_1(\pi) \cup \{\delta_1\} \cup R_\varphi(c_0 + 1)) \), it is possible since \( |\bar{\pi}(x)| > \epsilon \Delta \geq \epsilon f(\epsilon) \) and \( |S_1(\pi) \cup \{\delta_1\} \cup R_\varphi(c_0 + 1)| \leq c_0 + (c_0 + 2)(c_0 + 1)(3c_0) \). Since \( P_{\beta_1}(\tau_1, \delta, \pi) = P_{\beta_1}(\tau_1, \pi) \), \( P_{\beta_1}(\tau_1, \delta, \pi) \) is disjoint from \( x, y \). Let \( \pi_1 = \pi/P_{\beta_1}(\tau_1, \delta, \pi) \), then color \( \tau_1 \) in \( T_1(\pi) \) is replaced by \( \delta \). It is easy to see that under coloring \( \pi_1 \), all previous assumptions and B1 are satisfied. Since both \( \tau_1 \) and \( \delta \) are colors in \( \bar{\pi}(x) \cap \bar{\pi}(y), TA_\varphi(c_0 + 1) = TA_\varphi(c_0 + 1) \), so B3 holds. But \( S_1(\tau_1) = S_1(\pi_1) \cup \{\delta_1\} \) giving a contradiction to the maximality of \( |S_1(\pi)| \).

---

Case 2. \( \tau_1, \tau_2, \ldots, \tau_{c_0+1} \) are in \( \bar{\pi}_1(x_{\beta_1}) \).

As the previous method we used in Claim 5.1 for finding the coloring \( \varphi \), we firstly adjust \( \pi \) a little bit. Let \( T_1' = \{\tau_1, \tau_2, \ldots, \tau_{c_0+1}\} \). For a given number \( m \), let \( TA_\varphi(m) = \{\tau_1 \in T_1' \cap A_\varphi(g) : \bar{\pi}(x_{\beta_1}) \cap \bar{\pi}(x) \leq m\} \). Now let \( \pi' \) be a coloring satisfies all previous assumptions of \( \pi \) such that:

(a) \( |TA_\varphi(m) \cap \bar{\pi}(x_{\beta_1})| \) is minimum; (b) subject to (a), \( \sum_{\tau_1 \in TA_\varphi(c_0+1)} |\bar{\pi}(x_{\beta_1}) \cap \bar{\pi}(x')| \) is maximum.

We then claim that \( |TA_\varphi(c_0 + 1)| \leq c_0 \). If not, then \( T_1' = TA_\varphi(c_0 + 1) \). Let \( T' = \{\tau_1 : \tau_1 \in T_1' \cap A_\varphi(m) \} \), \( R_{\varphi}(m) = \bigcup_{\tau_1 \in TA_\varphi(m + 1)} (\bar{\pi}(x_{\beta_1}) \cap \bar{\pi}(x)) \) and \( C = T_1' \cup R_{\varphi}(m + 1) \cup A_\varphi(c_0 + 1) \). We have \( |C| \leq c_0 + 1 + (c_0 + 2)(c_0 + 1)(3c_0) \leq f_3(\epsilon) \) since \( c_0 \geq 1 \). Since \( \Delta \geq f(\epsilon) \), there exists a \( S(C, T') \) turns \( \pi' \) to \( \tau_1 \) by Fact 2. Easy to see that \( \tau_1 \) satisfies all previous assumptions and B1. Notice that the colors in \( R_{\varphi}(c_0 + 1) \) stay unchanged, so \( TA_\varphi(c_0 + 1) \subseteq TA_\varphi(c_0 + 1) \). Hence, B3 holds for \( \tau_1 \). Moreover, since the colors in \( R_{\varphi}(c_0 + 1) \) stay unchanged, \( TA_\varphi(c_0 + 1) \subseteq TA_\varphi(c_0 + 1) \). By the minimality of \( |TA_\varphi(c_0 + 1)| \), we have \( TA_\varphi(c_0 + 1) = TA_\varphi(c_0 + 1) \). Then \( \sum_{\tau_1 \in TA_\varphi(c_0 + 1)} |\bar{\pi}(x_{\beta_1}) \cap \bar{\pi}(x)| \) is unchangeable by Fact 3, which contradict the maximality of \( \sum_{\tau_1 \in TA_\varphi(c_0+1)} |\bar{\pi}(x_{\beta_1}) \cap \bar{\pi}(x)| \).
Now we have $|TA'(c_0 + 1)| \leq c_0$. Since $|T_1'| = c_0 + 1$, we may assume that there exists $\tau_1 \in T_1'$ such that $|\bar{\pi}'(x_{\beta_1, \tau_1}) \cap \bar{\pi}'(x)| > c_0 + 1$. Thus there exists a color $\delta \in \bar{\pi}'(x_{\beta_1, \tau_1}) \cap \bar{\pi}'(x) \setminus (S_1(\pi') \cup \{\delta_1\})$.

We first claim that we may assume $\pi_1 \in \bar{\pi}'(y)$. Suppose not, then $\tau_1 \in A_{\bar{\pi}'}(c_0 + 1)$, so $|\bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(y)| > c_0 + 1$. If $\bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(y) \neq \emptyset$, let $\theta$ be a color in it; If $\bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(y) = \emptyset$, let $\delta' \in \bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(x) \setminus (S_1(\pi') \cup \{\delta_1\})$. This is possible since $|\bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(x) \setminus \bar{\pi}'(y)| > c_0 + 1$, $\bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(y) = \emptyset$ and $|S_1(\pi')| \leq c_0 - 1$. Then in both cases, we can do a $(\tau_1, \theta, \delta')$-switching at $y$ in $\pi'$. (When $\delta \in \bar{\pi}'(y_{\tau_1}) \cap \bar{\pi}'(y)$, we do not need the color $\delta$ in the switching.) After the switching, $\tau_1$ is missing at $y$.

Now assume $\tau_1 \in \bar{\pi}'(y)$. Note that in the switching above, $S_1(\pi') \cup \{\delta, \delta_1\}$ stay unchanged. Since $\delta \in \bar{\pi}'(x)$ and $\tau_1 \in \bar{\pi}'(y)$, $P_\delta(\tau_1, \delta, \pi') = P_\delta(\tau_1, \delta, \pi')$ and $P_{x_{\beta_1}}(\tau_1, \delta, \pi')$ is disjoint from $x, y$. Let $\pi_1 = \pi'/P_{x_{\beta_1}}(\tau_1, \delta, \pi')$. Clearly, $\pi_1$ satisfies all previous assumptions and B1. Since both $\tau_1$ and $\delta$ are colors in $\bar{\pi}'(x) \cup \bar{\pi}'(y)$, $\pi_1$ satisfies B3. But $S_1(\pi_1) = S_1(\pi') \cup \{\tau_1\}$, giving a contradiction to the maximality of $|S_1(\pi')| = |S_1(\pi)|$. This completes the proof of Claim 5.3 and Lemma 7.

By using Lemma 7, we prove Lemma 8.

**Lemma 8.** Under the setting above, if $\Delta \geq f(\epsilon)$, then for any $c_0 + 2$ colors $\xi_1, \xi_2, \ldots, \xi_{c_0+2} \in S_\varphi(q) \cap \varphi(x)$, $|\bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}| < 3c_0 + 1$.

**Proof.** Suppose on the contrary, let $S = \{\xi_1, \xi_2, \ldots, \xi_{c_0+2}\} \subseteq S_\varphi(q) \cap \varphi(x)$ be a color set such that $|\bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}| \geq 3c_0 + 1$. Let $S' = \{\xi_1, \xi_2, \ldots, \xi_{c_0+1}\}$. Choose a color set $T' \subseteq \bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}$ with size $3c_0 + 1$.

**Claim 5.4.** We may assume that one of $\xi_1, \xi_2, \ldots, \xi_{c_0+1}$, say $\xi_1$, has the following property: either $\xi_1 \in \varphi(y)$ or $|\varphi(y_{\xi_1}) \cap (\varphi(x) \cup \varphi(y))| > 3c_0 + 1$.

**Proof.** Suppose the contrary, which implies $S' \cap \varphi(y) = \emptyset$. For a constant $m$, let $SA_\varphi(m) = \{\xi : \xi \in S'$ such that $|\varphi(y_{\xi}) \cap (\varphi(x) \cup \varphi(y))| \leq m\}$, $R(\varphi') = \bigcup_{\xi \in SA_\varphi(3c_0+2)} \varphi(y_{\xi}) \cap (\varphi(x) \cup \varphi(y))$. Now let $\varphi_1$ be a coloring such that $\xi_1, \xi_2, \ldots, \xi_{c_0+2} \in S_{\varphi_1}(q) \cap \varphi_1(x)$ and $T' \subseteq \bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}$ with the following additional conditions: (a) $|SA_\varphi(3c_0 + 1)|$ is minimum; (b) subject to (a), $\sum_{\xi \in SA_\varphi(3c_0 + 1)} |\varphi_1(y_{\xi}) \cap \varphi_1(x)|$ is maximum.

We then claim that under the coloring $\varphi_1$, $|SA_\varphi(3c_0 + 1)| \leq c_0$. Suppose not, then $SA_\varphi(3c_0 + 1) = S'$. Let $C = SU^\prime \cup R(\varphi_1)$ and $T = \{y_{\xi} : \xi \in S'\}$. Since $|S| = c_0 + 2$, $|T'| = 3c_0 + 1$ and $|R(\varphi_1)| \leq (c_0 + 1)(3c_0 + 2)$, it is easy to check that $|C| \leq f_3(\epsilon)$. Then since $\Delta \geq f(\epsilon)$, there exists a $S(C, T)$ turns $\varphi_1$ to $\varphi_2$ by Fact 2. Notice that the colors in $R(\varphi_1)$ stay unchanged, so $SA_\varphi(3c_0 + 1) \subseteq SA_\varphi(3c_0 + 1)$. By the maximality of $SA_\varphi(3c_0 + 1)$, we have $SA_\varphi(3c_0 + 1) = SA_\varphi(3c_0 + 1)$. Then $\sum_{\xi \in SA_\varphi(3c_0 + 1)} |\varphi_2(y_{\xi}) \cap \varphi_2(x)| \geq \sum_{\xi \in SA_\varphi(3c_0 + 1)} |\varphi_1(y_{\xi}) \cap \varphi_1(x)| + 1$ by Fact 3, which contradicts the maximality of $\sum_{\xi \in SA_\varphi(3c_0 + 1)} |\varphi_1(y_{\xi}) \cap \varphi_1(x)|$.

By the discussion above, we have $|SA_\varphi(3c_0 + 1)| \leq c_0$. Since $|S'| = c_0 + 1$, there exists a color in $S'$, say $\xi_1$, such that $|\varphi_1(y_{\xi_1}) \cap (\varphi_1(x) \cup \varphi_1(y))| > 3c_0 + 1$. This proves the claim. □

Now assume that under the coloring $\varphi_1$, color $\xi_1$ satisfies the claim above. We then claim that we can find a color $\xi \in S$ such that $\xi \in \varphi_1(y)$. Suppose not, we have $S \cap \varphi_1(y) = \emptyset$ and $|\varphi_1(y_{\xi_1}) \cap \varphi_1(y)| > 3c_0 + 1$. Now we choose $\delta \in \varphi_1(y_{\xi_1}) \cap \varphi_1(y) \setminus T'$, let $\theta \in \varphi_1(y)$, we can do a $(\xi_1, \theta, \delta)$-switching at $y$ in $\varphi_1$ to obtain $\varphi_2$. Under the new coloring we have $\xi_1 \in \varphi_2(y)$, which is what we want. Notice that after this switching, $T'$ may not be a subset of $\bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}$ if $\theta \in T'$. But we still have $T \setminus \{\theta\} \subseteq \bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}$.

By the discussion above, we assume that under $\varphi_1$, $\xi_1 \in \varphi_1(y)$ and $|\bigcap_{i=1}^{c_0+2} \varphi_{\text{bad}}(x_{\xi_i}) \setminus \{\xi_1\}| \geq |T'| + 1 = 3c_0$. Recolor $xy$ by $\xi_1$ and leave $xx_{\xi_1}$ uncolored to obtain a new coloring $\varphi_2$. Then
\( \varphi_2(xy) \in \vec{\varphi}_2(x_{\xi_1}); S \backslash \{\xi_1\} \) is a subset of \( \varphi_2^{\text{bad}}(y) \cap \varphi_2(x) \) with size \( c_0 + 1 \); For each \( \xi \in S \backslash \{\xi_1\}, |\varphi_2^{\text{bad}}(x_\xi) \cap S \varphi_2(x, x_{\xi_1}, q) \backslash \{s\}| \geq |(\varphi_2^{\text{bad}}(x_{\xi_1}) \cup \{s\}) \cap \varphi_2^{\text{bad}}(x_{\xi_1})| \geq |\cap_{i=1}^{c_0+2} \varphi_1^{\text{bad}}(x_{\xi_i}) \backslash \{\xi_i\}| \geq 3c_0.

But this cannot happen by Lemma 7, which is a contradiction. \qed

6 Proof of Lemma 16

Before the proof, we will discuss some basic properties first. Assume that \( \epsilon \in (0, 1) \) and \( G \) is a \( \Delta \)-critical graph with maximum degree \( \Delta \). Let \( x \in V(G) \) with \( d(x) < (1 - \Delta)q = q \) and \( y \in N(x) \). In this section we don’t assume \( \Delta \geq D(\epsilon) \).

A neighbor \( z \in N(x) \backslash \{y\} \) is called feasible if there exists a coloring \( \varphi \in C^\Delta(G - xy) \) such that \( \varphi(xz) \in \varphi(y) \), and such a coloring \( \varphi \) is called \( z \)-feasible. Denote by \( C_z \) the set of all \( z \)-feasible colorings. For each \( \varphi \in C_z \), let

\[
Y_Z(\varphi) = \{v \in N(y) \backslash \{x\} : \varphi(vy) \in \bar{\varphi}(x) \cup \bar{\varphi}(z)\},
R_Q(\varphi) = \{\varphi(vy) : v \in Y_Z(\varphi) \text{ and } d(v) < q\},
\]

\[
Z(\varphi) = \{v \in N(z) \backslash \{x\} : \varphi(vz) \in \bar{\varphi}(x) \cup \bar{\varphi}(y)\},
R_S(\varphi) = \{\varphi(vz) : v \in Z(\varphi) \text{ and } d(v) < q\}.
\]

In the remainder of the proof, we let \( R_z = R_z(\varphi) \) and \( R_y = R_y(\varphi) \) if the coloring \( \varphi \) is clear.

Recall that \( c_1 = c_0 - 1 = \left[ \frac{1 - \epsilon}{\epsilon} \right] - 1 \). When \( d(x) < q \), we have for any \( \varphi \in C^\Delta(G - xy) \), every elementary vertex subset \( X \) with \( x \in X \) contains at most \( c_1 \) other vertices with degree less than \( q \). Otherwise \( |\bigcup_{v \in X} \varphi(v)| \geq \epsilon \Delta + 1 + (c_1 + 1)(\epsilon \Delta) > \Delta \), giving a contradiction.

Now we claim that \( |R_y| \leq c_1 \) and \( |R_z| \leq c_1 \). Let \( z \in N(x) \backslash \{y\} \) be a feasible vertex and \( \varphi \in C_z \). By the definition of \( Z(\varphi) \), \( G[\{x, y, z\} \cup Z(\varphi)] \) forms a simple broom, so \( \{x, y, z\} \cup Z(\varphi) \) is elementary with respect to \( \varphi \). Consequently, it contains at most \( c_1 \) vertices other than \( x \) with degree less than \( q \). Thus, \( |R_z| \leq c_1 \). Similarly, if we color \( xy \) by \( \varphi(xz) \) and leave \( xz \) uncolored to obtain a coloring \( \varphi' \), then \( G[\{x, y\} \cup Y_Z(\varphi')] \) forms a simple broom, by the same reason, we have \( |R_y(\varphi)| = |R_y(\varphi')| \leq c_1 \). Let \( R = R_z \cup R_y \). A coloring \( \varphi \in C_z \) is called optimal if \( |R| \) is maximum over all feasible colorings. Obviously, we have \( |R| \leq 2c_1 \).

Recall that \( A_{\varphi}(q) \cup B_{\varphi}(q) = \varphi(x) \cap \varphi(y) \) and \( \varphi(x) \cap \varphi(y) = \emptyset \). So \( \{1, \Delta\} = \varphi(x) \cup \varphi(y) \cup (\varphi(x) \cap \varphi(y)) = S_{\varphi}(q) \cup B_{\varphi}(q) \). By Lemma 1, for a \( z \)-feasible coloring \( \varphi(x), \varphi(y) \) and \( \varphi(z) \) are disjoint (notice that \( \{x, y, z\} \) is elementary), thus

\[
[1, \Delta] = \varphi(x) \cup \varphi(y) \cup \varphi(z) \cup (A_{\varphi}(q) \cup B_{\varphi}(q) \backslash \varphi(z)), \quad \text{so}
\]

\[
\sigma_q(x, y) = |\varphi(y) \backslash (A_{\varphi}(q) \cup R_y)| = [1, \Delta] \backslash (\varphi(y) \cup A_{\varphi}(q) \cup R_y) = |\varphi(x)| + |\varphi(z)| + |B_{\varphi}(q) \backslash \varphi(z)| - |R_y|.
\]

By using this result, we further have

\[
|S_{\varphi}(q) \cap \varphi(z)| = |\varphi(z) \backslash B_{\varphi}(q)| = \Delta - |\varphi(z)| - |B_{\varphi}(q) \backslash \varphi(z)|
= \Delta - (\sigma_q(x, y) - |\varphi(x)|) - |R_y| \geq 2\Delta - d(x) - \sigma_q(x, y) + 1 - c_1 \quad \text{and}
\]

\[
|N_S(x)| = |\varphi(y) \cup A_{\varphi}(q)| = |\varphi(y) \cup A_{\varphi}(q) \cup R_y| - |R_y|
= \Delta - |\varphi(x, y) - |R_y| \geq \Delta - \sigma_q(x, y) - c_1.
\]

By the two inequalities above, we show that Lemma 17 is a corollary of Lemma 16.

**Lemma 17.** Let \( \epsilon \in (0, 1) \), \( q \) be a positive number, \( G \) be a \( \Delta \)-critical graph with maximum degree \( \Delta \), \( xy \in E(G) \) with \( d(x) < q \) and \( \varphi \in C^\Delta(G - xy) \). If \( q \leq \min\{(1 - \epsilon)\Delta - 2c_1, \)}
Δ − 6c₁}, then there are at least Δ − σₚ(x, y) − 2c₁ vertices z ∈ N(x) \ {y} such that σₚ(x, z) ≥ 2Δ − d(x) − σₚ(x, y) − 5c₁.

**Proof.** By Lemma 16, there are at least |Nₛ(x)| − c₁ many vertices z ∈ Nₛ(x) ⊆ N(x) \ {y} and a z-feasible coloring φ′ for each z such that σₚ(x, z) ≥ |Sₚ(φ′) ∩ φ′(z)| − 1 − 4c₁.

By the two inequalities we just proved, we have |Nₛ(x)| − c₁ ≥ Δ − σₚ(x, y) − 2c₁ and |Sₚ(φ′) ∩ φ′(z)| − 1 − 4c₁ ≥ 2Δ − d(x) − σₚ(x, y) − 5c₁. Thus Lemma 17 holds.

Now we give the proof of Lemma 16.

**Lemma 16.** Let ε ∈ (0, 1), q be a positive number, G be a Δ-critical graph with maximum degree Δ, and xy ∈ E(G) with d(x) < q and φ ∈ Cₐ(G − xy). If q ≤ min{(1 − ε)Δ − 2c₁, Δ − 6c₁}, then for each z ∈ Nₛ(x) except at most c₁ vertices, there exists a coloring φ′ ∈ Cₐ(G − xy) such that φ′(xz) ∈ ϕ(y) and for each ξ ∈ (Sₚ(φ′)) \ {φ′(xz)} except at most 4c₁ colors, d(x₁ξ) ≥ q where 1 = φ′(x).

**Proof.** When c₁ = 0, i.e. ε ≥ 1/3, this lemma is a corollary of Lemma 12. Thus we assume that c₁ ≥ 1. Lemma 16 follows the three statements below.

**I.** For any coloring φ ∈ Cₐ(G − xy) and any α ∈ Aₚ(q) except at most c₁ colors, xα is feasible. (Notice that by definition, xα is feasible if α ∈ (Sₚ(φ)) \ Aₚ(q) = ϕ(y).)

**II.** For any feasible vertex z ∈ N(x) and any z-feasible coloring φ, |Rₚ| ≤ c₁.

**III.** For any feasible vertex z ∈ N(x), there exists a z-feasible optimal coloring φ (assume φ(xz) = 1) such that for all α ∈ Aₚ(q) ∩ φ(z) except at most 3c₁ colors, d(x₁α) ≥ q. (Together with II, we have for all α ∈ Sₚ(φ) ∩ φ(z) except at most 4c₁ colors, d(x₁α) ≥ q.)

Since II has been proved at the beginning of this section, we give the proof of I and III.

**6.1 Proof of I.**
Suppose on the contrary that there exist φ ∈ Cₐ(G − xy) and c₁ + 1 colors α₁, α₂, ..., αₖ₊₁ in Aₚ(q) such that x₁α₁, x₁α₂, ..., x₁αₖ₊₁ are not feasible. We consider the following two cases:

**Case 1:** One of y₁α₁, y₂α₂, ..., yₖ₊₁ has a common missing color with x.

Assume that y₁α₁ has a common missing color, say δ, with x. Let θ ∈ ϕ(y). Then we can do an (α₁, θ, δ)-switching in ϕ at y to obtain a new coloring ϕ₁ such that ϕ₁(x₁α₁) = α₁ ∈ ϕ₁(y), which contradict the fact that x₁α₁ is not feasible.

**Case 2:** None of y₁α₁, y₂α₂, ..., yₖ₊₁ has a common missing color with x.

As we discussed before, every elementary vertex subset X with x ∈ X contains at most one other vertices with degree less than q, thus {x, y₁α₁, y₂α₂, ..., yₖ₊₁} is not elementary. Hence there exist two vertices of y₁α₁, y₂α₂, ..., yₖ₊₁ which have a common missing color. Assume without loss of generality that η ∈ ϕ(y₁α₁) \ ϕ(y₂α₂). Choose δ ∈ ϕ(x), obviously δ ≠ η. Then at least one of y₁α₁, y₂α₂ is not on the path Pₓ(δ, η, φ), say y₁α₁. Now, let φ₁ = φ \ Pₓ(δ, η, φ), then δ ∈ ϕ₁(y₁α₁) \ ϕ₁(x), we are back to Case 1.

**6.2 Proof of III.**
Let z ∈ N(x), φ be an optimal z-feasible coloring. Assume, without loss of generality, φ(xz) = 1 ∈ ϕ(y) and let T = {κ ∈ Aₚ(q) ∩ φ(z) : d(x₁κ) < q}. Suppose on the contrary that |T| ≥ 3c₁ + 1 under every optimal z-feasible colorings.

**Claim A.** For each η ∈ ϕ(x) \ R and κ ∈ T, Pₓ(η, κ, φ) contains both y and z.
Proof. We first show that \( z \in V(P_\eta(\eta, \kappa, \varphi)) \). Otherwise, \( P_\eta(\eta, \kappa, \varphi) \) is disjoint with \( P_\eta(\eta, \kappa, \varphi) \).
Let \( \varphi' = \varphi/P_\eta(\eta, \kappa, \varphi) \). Since \( 1 \neq \eta, \kappa, \varphi' \) is also feasible. Since colors in \( R \) are unchanged and \( d(x_{1\kappa}) < q, R_i(\varphi') = R_i \cup \{\eta\} \) and \( R_\eta(\varphi') \supseteq R_\eta \), giving a contradiction to the maximality of \( |R| \).
Similarly, if we color \( xy \) by \( \varphi(xz) \) and leave \( xz \) uncolored to obtain a coloring \( \varphi' \), then follow the same proof, we can easily verify that \( y \in V(P_\eta(\eta, \kappa, \varphi)) \) as well.

Since \( |T| \geq 3c_1 + 1 \), there are 3\( c_1 + 1 \) colors \( \kappa_1, \kappa_2, \ldots, \kappa_{3c_1+1} \in T \). We let

\[
V_T = V_T(\varphi) = \{x_{1\kappa_1}, \ldots, x_{1\kappa_{3c_1+1}}\} \cup \{y_{\kappa_1}, \ldots, y_{\kappa_{3c_1+1}}\},
\]
\[
W = W(\varphi) = \{u \in V_T(\varphi) : \varphi(u) \cap \varphi(x) \subseteq R\},
\]
\[
M = M(\varphi) = V_T(\varphi) - W(\varphi) = \{u \in V_T(\varphi) : \varphi(u) \cap \varphi(x) \cap \varphi(y) \neq \emptyset\},
\]
\[
E_T = E_T(\varphi) = \{zx_{1\kappa_1}, \ldots, zx_{1\kappa_{3c_1+1}}\},
\]
\[
E_W = E_W(\varphi) = \{e \in E_T(\varphi) : e \text{ is incident to a vertex in } W\}
\]
\[
E_M = E_M(\varphi) = E_T(\varphi) - E_W(\varphi) = \{e \in E_T(\varphi) : e \text{ is incident to a vertex in } M\}.
\]

We assume that \( |E_W| \) is minimum over all optimal z-feasible coloring \( \varphi \) and all sets of \( 3c_0 + 1 \) colors in \( T \). For each \( v \in M \), pick a color \( \alpha_v \in \varphi(v) \cap \varphi(x) \cap \varphi(y) \). Let \( C_M = \{\alpha_v : v \in M\} \). Clearly, \( |C_M| \leq |M| \). Since \( \{x_{1\kappa_1}, \ldots, x_{1\kappa_{3c_1+1}}\} \cap \{y_{\kappa_1}, \ldots, y_{\kappa_{3c_1+1}}\} \) may not be empty, so \( \frac{|E_{\varphi}|}{2} \leq |W| \leq |E_W| \) and \( \frac{|E_M|}{2} \leq |M| \leq |E_M| \).

**Claim B.** The following can not happen: \( \varphi(x) \cap (R \cup C_M) = \emptyset \) and there exist two vertices \( u, v \in W \) such that \( (\varphi(u) \cap \varphi(v)) \cap R \neq \emptyset \).

**Proof.** Suppose on the contrary. Let \( \beta \in \varphi(x) \cap (R \cup C_M), u, v \in W \) and \( \alpha \in (\varphi(u) \cap \varphi(v)) \cap R \). Then by the definition of \( W \), we have \( \alpha \in \varphi(x), \beta \in \varphi(u) \cap \varphi(v) \) and \( \alpha \neq \beta \). So both \( u \) and \( v \) are endpoints of \( (\alpha, \beta) \)-chains, at least one of the two paths does not pass through \( x \). Assume without loss of generality \( P_\alpha(\alpha, \beta, \varphi) \) is disjoint with \( P_\eta(\alpha, \beta, \varphi) \). We note that \( \beta \in \varphi(y) \cap \varphi(z) \) since \( \{x, y, z\} \) is an elementary set.

We first consider the case of \( \alpha = 1 \). In this case, \( P_\eta(\alpha, \beta, \varphi) = P_\eta(\alpha, \beta, \varphi) \) by Lemma 1. Since \( \varphi(xz) = 1, z \in P_\eta(\alpha, \beta, \varphi) \). So, \( P_\alpha(\alpha, \beta, \varphi) \cap \{x, y, z\} = \emptyset \). Hence, coloring \( \varphi' = \varphi/P_\alpha(\alpha, \beta, \varphi) \) is z-feasible, \( R_\eta(\varphi') = R_\eta \), \( R_\kappa(\varphi') = R_\kappa \), and \( T(\varphi') = T \). So, \( \varphi' \) is also optimal, \( u \in M(\varphi') \) and \( |E_W(\varphi')| < |E_W| \), contradict the fact that \( |E_W| \) is minimum.

We now suppose \( \alpha \in \varphi(x) \cap (R \cup \{1\}) \). So, both \( \alpha \) and \( \beta \) are not in \( R \cup \{1\} \). Notice that even if \( \alpha \) is a color in \( T \), \( P_\alpha(\alpha, \beta, \varphi) \) will not pass through \( yx_\alpha \) or \( yy_\alpha \) because of Claim A. Thus we may let \( \varphi' = \varphi/P_\alpha(\alpha, \beta, \varphi) \). Then, \( \varphi' \) is z-feasible, \( R_\eta(\varphi') = R_\eta \) and \( R_\kappa(\varphi') = R_\kappa \). Thus, \( \varphi' \) is still an optimal coloring and \( \beta \in \varphi'(u) \). We have \( u \in M(\varphi') \) and \( |E_W(\varphi')| < |E_W| \), giving a contradiction.

**Claim C.** There exist a color \( \kappa \in \{\kappa_1, \kappa_2, \ldots, \kappa_{3c_0+1}\} \) and three distinct colors \( \eta, \theta, \lambda \) such that \( \eta \in (\varphi(x) \cap R) \cap \varphi(x_{1\kappa}), \theta \in (\varphi(x) \cap R) \cap \varphi(y), \lambda \in (\varphi(x) \cap R) \cap \varphi(y) \).

Before we give the proof of Claim C, we first show III assuming Claim C is true.

Let \( \kappa, \eta, \theta, \lambda \) be as stated in Claim C. If \( \lambda \neq 1 \), we consider coloring obtained from \( \varphi \) by interchange colors 1 and \( \lambda \) for edges not on the path \( P_\eta(1, \lambda, \varphi) \), and rename it as \( \varphi \). So we may assume \( 1 \in \varphi(y) \).

By Claim A, the paths \( P_\eta(\eta, \kappa, \varphi) \) and \( P_\eta(\theta, \kappa, \varphi) \) both contain \( y, z \). Since \( \varphi(yy_\eta) = \varphi(x_{1\kappa}) = \kappa \), these two paths also contain \( y_\kappa, x_{1\kappa} \). Since \( \eta \in \varphi(x_{1\kappa}) \), we have \( x \) and \( x_{1\kappa} \) are the two endpoints of \( P_\eta(\eta, \kappa, \varphi) \). So, \( \eta \in \varphi(y) \cap \varphi(z) \cap \varphi(y_\eta) \). Similarly, we have \( \theta \in \varphi(y) \cap \varphi(z) \cap \varphi(x_{1\kappa}) \). We now consider the following sequence of colorings of \( G - xy \).

Let \( \varphi_1 \) be obtained from \( \varphi \) by assigning \( \varphi_1(yy_\eta) = 1 \). Since 1 is missing at both \( y \) and \( y_\kappa \), \( \varphi_1 \in C^\Delta(G - xy) \). Now \( \kappa \) is missing at \( y \) and \( y_\kappa \), \( \eta \) is still missing at \( x_{1\kappa} \). Since \( G \) is \( \Delta \)-critical,
Let \( P_x(\eta, \kappa, \varphi_1) = P_y(\eta, \kappa, \varphi_1) \); Furthermore, \( x_1, \kappa, y_\kappa \notin V(P_x(\eta, \kappa, \varphi_1)) \) since either \( \eta \) or \( \kappa \) is missing at these two vertices, which in turn shows that \( z \notin V(P_x(\eta, \kappa, \varphi_1)) \) since \( \varphi_1(z, x_1, \kappa) = \kappa \).

Let \( \varphi_2 = \varphi_1/P_x(\eta, \kappa, \varphi_1) \). We have \( \kappa \in \varphi_2(x, \eta) \in \varphi_2(y) \cap \varphi_2(x, \kappa) \) and \( \eta \in \varphi_2(x) \cap \varphi_2(y) \).

Then \( P_x(\eta, \theta, \varphi_2) = P_y(\eta, \theta, \varphi_2) \) contains neither \( \kappa \), nor \( x_1, \kappa \).

Let \( \varphi_3 = \varphi_2/P_x(\eta, \theta, \varphi_2) \). Then \( \kappa \in \varphi_3(x) \) and \( \theta \in \varphi_3(y) \cap \varphi_3(y) \).

Let \( \varphi_4 \) be obtained from \( \varphi_3 \) by recoloring \( yy_\kappa \) by \( \theta \). Then \( 1 \leq \varphi_4(\varphi_4) \leq 1, \kappa \in \varphi_4(x) \).

Now we start by showing that \( \varphi_4(\varphi_4(x)) \) contains three endpoints \( x, x_1, \kappa \), and \( y_\kappa \), a contradiction.

Suppose not. Then \( \left| E_M \right| \leq 4c_1+1 \) by the Pigeonhole Principle. Thus \( \left| E_W \right| \geq 2c_1+1 \), which in turn gives \( W \geq \left[ \frac{2c_1+1}{2} \right] = c_1 + 1 \) and \( M \leq \left| E_M \right| \leq 4c_1 + 1 \). Let \( w_1, w_2, \ldots, w_{c_1+1} \in W \).

By (4), \( |\varphi(x)| \geq 6c_1 + 2 > |R| + |M| \). So \( \varphi(x) \notin (R \cup C_M) \). Let \( |R| \leq 2c_1, d(x) < q \) and \( q \leq (1-\epsilon)\Delta - 2c_1 \), we have

\[
\sum_{i=1}^{c_1+1} |\varphi(w_i) \setminus R| + |\varphi(x)| > (c_1 + 1)(\Delta - q - 2c_1) + \Delta - d(x) + 1 > \Delta.
\]

So, there is a color \( \alpha \) shared by at least two of these sets. Since \( w_1, w_2, \ldots, w_{c_1+1} \in W \), we may assume that \( \alpha \in (\varphi(w_1) \setminus R) \cap (\varphi(w_2) \setminus R) \). Then by \textbf{Claim B}, this gives a contradiction. Thus we can assume that there exist \( \alpha \) colors \( \gamma_1, \gamma_2, \ldots, \gamma_{c_1+1} \in \{ \kappa_1, \kappa_2, \ldots, \kappa_{3c_1+1} \} \) such that \( \gamma_i \neq \gamma_j \) for \( i = 1, 2, \ldots, c_1+1 \).

Now we only need to show that additionally there exists another color \( \lambda \in \varphi(x) \setminus \{ \alpha \} \) such that \( \lambda \) in \( \varphi(y_\kappa) \) for some \( i \in \{ 1, 2, \ldots, c_1+1 \} \).

Suppose on the contrary that there is no such color \( \lambda \). Then \( \varphi(y_\kappa) \) contains \( \varphi(x) \) and \( \{ \theta_i \} \) for \( i = 1, 2, \ldots, c_1+1 \). Moreover, \( 1 \notin \varphi(y_\kappa) \).

Since \( |R| \leq 2c_1, d(x) < q \) and \( q \leq (1-\epsilon)\Delta - 2c_1 \), the following inequalities hold.

\[
\sum_{i=1}^{c_1+1} |\varphi(y_\kappa) \setminus (R \cup \{ \eta_i \})| + |\varphi(x) \setminus (R \cup \{ \alpha \})| > (c_1 + 1)(\Delta - q - |R| - 1) + \Delta - d(x) + 2 \geq \Delta
\]

So, there is a color \( \alpha \) in two of these sets. Since \( \varphi(y_\kappa) \cap \varphi(x) \setminus \{ \alpha \} \), we may assume that \( \alpha \in \varphi(y_\kappa) \cap \varphi(x) \setminus \{ \alpha \} \).

Since \( |\varphi(x)| \geq 6c_1 + 2 > |R| + 4c_1 + 2 \geq |R| + 6 \), \( \varphi(x) \setminus \{ \eta_i, \theta_1, \kappa_1, \theta_2, \kappa_2 \} \neq \emptyset \), there exists \( \lambda \in \varphi(x) \) and \( \eta_i, \theta_1, \kappa_1, \theta_2, \kappa_2 \). If \( \alpha = \lambda \) we are done. Assume \( \alpha \neq \lambda \). Then, both \( y_\gamma_1 \) and \( y_\gamma_2 \) are endpoints of \( \alpha, \lambda \)-chains. Assume without loss of generality \( P_{y_1}(\alpha, \lambda, \varphi) \) is disjoint with \( P_{y_2}(\alpha, \lambda, \varphi) \). Let \( \varphi' = \varphi/P_{y_1}(\alpha, \lambda, \varphi) \). Since \( \lambda \neq \eta_1, \theta_1 \), the previous assumptions still hold.

Now \( \varphi', \gamma_1, \eta_1, \theta_1 \) satisfy the requirements. \textbf{Claim C} holds.
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References


