Proof of the Goldberg-Seymour Conjecture on Edge-Colorings of Multigraphs

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Abstract

Given a multigraph \(G = (V, E)\), the edge-coloring problem (ECP) is to color the edges of \(G\) with the minimum number of colors so that no two adjacent edges have the same color. This problem can be naturally formulated as an integer program, and its linear programming relaxation is called the fractional edge-coloring problem (FECP). In the literature, the optimal value of ECP (resp. FECP) is called the chromatic index (resp. fractional chromatic index) of \(G\), denoted by \(\chi'(G)\) (resp. \(\chi'^*(G)\)). Let \(\Delta(G)\) be the maximum degree of \(G\) and let

\[
\Gamma(G) = \max \left\{ \frac{2|E(U)|}{|U| - 1} : U \subseteq V, |U| \geq 3 \text{ and odd} \right\},
\]

where \(E(U)\) is the set of all edges of \(G\) with both ends in \(U\). Clearly, \(\max\{\Delta(G), \lfloor \Gamma(G) \rfloor\}\) is a lower bound for \(\chi'(G)\). As shown by Seymour, \(\chi'(G) = \max\{\Delta(G), \Gamma(G)\}\). In the 1970s Goldberg and Seymour independently conjectured that \(\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\}\). Over the past four decades this conjecture, a cornerstone in modern edge-coloring, has been a subject of extensive research, and has stimulated a significant body of work. In this paper we present a proof of this conjecture. Our result implies that, first, there are only two possible values for \(\chi'(G)\), so an analogue to Vizing’s theorem on edge-colorings of simple graphs, a fundamental result in graph theory, holds for multigraphs; second, although it is \(NP\)-hard in general to determine \(\chi'(G)\), we can approximate it within one of its true value, and find it exactly in polynomial time when \(\Gamma(G) > \Delta(G)\); third, every multigraph \(G\) satisfies \(\chi'(G) - \chi'^*(G) \leq 1\), so FECP enjoys a fascinating integer rounding property.

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1 Introduction

Given a multigraph $G = (V, E)$, the edge-coloring problem (ECP) is to color the edges of $G$ with the minimum number of colors so that no two adjacent edges have the same color. Its optimal value is called the chromatic index of $G$, denoted by $\chi'(G)$. In addition to its great theoretical interest, ECP arises in a variety of applications, so it has attracted tremendous research efforts in several fields, such as combinatorial optimization, theoretical computer science, and graph theory. Holyer [15] proved that it is NP-hard in general to determine $\chi'(G)$, even when restricted to a simple cubic graph, so there is no efficient algorithm for solving ECP exactly unless NP = P, and hence the focus of extensive research has been on near-optimal solutions to ECP or good estimates of $\chi'(G)$.

Let $\Delta(G)$ be the maximum degree of $G$. Clearly, $\chi'(G) \geq \Delta(G)$. There are two classical upper bounds on the chromatic index: the first of these, $\chi'(G) \leq \lceil \frac{3\Delta(G)}{2} \rceil$, was established by Shannon [35] in 1949, and the second, $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of edges in $G$, was proved independently by Vizing [38] and Gupta [11] in the 1960s. This second result is widely known as Vizing’s theorem, which is particularly appealing when applied to a simple graph $G$, because it reveals that $\chi'(G)$ can take only two possible values: $\Delta(G)$ and $\Delta(G) + 1$. Nevertheless, in the presence of multiple edges, the gap between $\chi'(G)$ and these three bounds can be arbitrarily large. Therefore it is desirable to find some other graph theoretic parameters connected to the chromatic index.

Observe that each color class in an edge-coloring of $G$ is a matching, so it contains at most $(|U| - 1)/2$ edges in $E(U)$ for any $U \subseteq V$ with $|U|$ odd, where $E(U)$ is the set of all edges of $G$ with both ends in $U$. Hence the density of $G$, defined by

$$\Gamma(G) = \max \left\{ \frac{2|E(U)|}{|U| - 1} : U \subseteq V, |U| \geq 3 \text{ and odd} \right\},$$

provides another lower bound for $\chi'(G)$. It follows that $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$.

To facilitate better understanding of the parameter $\max\{\Delta(G), \Gamma(G)\}$, let $A$ be the edge-matching incidence matrix of $G$. Then ECP can be naturally formulated as an integer program, whose linear programming (LP) relaxation is exactly as given below:

$$\begin{align*}
\text{Minimize} & \quad 1^T x \\
\text{subject to} & \quad Ax = 1 \\
& \quad x \geq 0.
\end{align*}$$

In the literature, this linear program is called the fractional edge-coloring problem (FECP), and its optimal value is called the fractional chromatic index of $G$, denoted by $\chi^*(G)$. As shown by Seymour [34] using Edmonds’ matching polytope theorem [7], it is always true that $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$. Thus the preceding inequality actually states that $\chi'(G) \geq \chi^*(G)$.

As $\chi'(G)$ is integer-valued, we further obtain $\chi'(G) \geq \lceil \Delta(G) \rceil$. How good is this bound? In the 1970s Goldberg [9] and Seymour [34] independently made the following conjecture.

**Conjecture 1.1.** Every multigraph $G$ satisfies $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \Gamma(G) \rceil\}$.
Let \( r \) be a positive integer. A multigraph \( G = (V, E) \) is called an \( r \)-graph if \( G \) is regular of degree \( r \) and for every \( X \subseteq V \) with \( |X| \) odd, the number of edges between \( X \) and \( V - X \) is at least \( r \). Note that if \( G \) is an \( r \)-graph, then \( |V(G)| \) is even and \( \Gamma(G) \leq \Delta(G) \). Seymour [34] also proposed the following weaker version of Conjecture 1.1, which amounts to saying that \( \chi'(G) \leq \max\{\Delta(G), \left\lceil \Gamma(G) \right\rceil\} + 1 \) for any multigraph \( G \).

**Conjecture 1.2.** Every \( r \)-graph \( G \) satisfies \( \chi'(G) \leq r + 1 \).

The following conjecture was posed by Gupta [11] in 1967 and can be deduced from Conjecture 1.1, as verified by Scheide [30].

**Conjecture 1.3.** Let \( G \) be a multigraph with \( \mu(G) = \mu \), such that \( \Delta(G) \) cannot be expressed in the form \( 2pm - q \), where \( q \geq 0 \) and \( p > \lfloor (q + 1)/2 \rfloor \). Then \( \chi'(G) \leq \Delta(G) + \mu(G) - 1 \).

A multigraph \( G \) is called critical if \( \chi'(H) < \chi'(G) \) for any proper subgraph \( H \) of \( G \). In edge-coloring theory, critical multigraphs are of special interest, because they have much more structural properties than arbitrary multigraphs. The following two conjectures are due to Jakobsen [16, 17] and were proved by Andersen [1] to be weaker than Conjecture 1.1.

**Conjecture 1.4.** Let \( G \) be a critical multigraph with \( \chi'(G) \geq \Delta(G) + 2 \). Then \( G \) contains an odd number of vertices.

**Conjecture 1.5.** Let \( G \) be a critical multigraph with \( \chi'(G) > \frac{m\Delta(G) + (m - 3)}{m - 1} \) for an odd integer \( m \geq 3 \). Then \( G \) has at most \( m - 2 \) vertices.

Motivated by Conjecture 1.1, Hochbaum, Nishizeki, and Shmoys [14] formulated the following conjecture concerning the approximability of ECP.

**Conjecture 1.6.** There is a polynomial-time algorithm that colors the edges of any multigraph \( G \) using at most \( \chi'(G) + 1 \) colors.

Over the past four decades Conjecture 1.1 has been a subject of extensive research, and has stimulated a significant body of work, with contributions from many researchers; see McDonald [23] for a survey on this conjecture and Stiebitz et al. [36] for a comprehensive account of edge-colorings.

Several approximate results state that \( \chi'(G) \leq \max\{\Delta(G) + \rho(G), \left\lceil \Gamma(G) \right\rceil\} \), where \( \rho(G) \) is a positive number depending on \( G \). Asymptotically, Kahn [19] showed that \( \rho(G) = o(\Delta(G)) \). Scheide [31] and Chen, Yu, and Zang [5] independently proved that \( \rho(G) \leq \sqrt[4]{\Delta(G)}/2 \). Chen et al. [3] improved this to \( \rho(G) \leq \sqrt[4]{\Delta(G)}/2 \). Recently, Chen and Jing [4] further strengthened this as \( \rho(G) \leq \sqrt[4]{\Delta(G)/4} \).

There is another family of results, asserting that \( \chi'(G) \leq \max\{\frac{m\Delta(G) + (m - 3)}{m - 1}, \left\lceil \Gamma(G) \right\rceil\} \), for increasing values of \( m \). Such results have been obtained by Andersen [1] and Goldberg [9] for \( m = 5 \), Andersen [1] for \( m = 7 \), Goldberg [10] and Hochbaum, Nishizeki, and Shmoys [14] for \( m = 9 \), Nishizeki and Kashiwagi [25] and Tashkinov [37] for \( m = 11 \), Favrholdt, Stiebitz, and Toft [8] for \( m = 13 \), Scheide [31] for \( m = 15 \), Chen et al. [3] for \( m = 23 \), and Chen and Jing [4] for \( m = 39 \). It is worthwhile pointing out that, when \( \Delta(G) \leq 39 \), the validity of Conjecture 1.1 follows instantly from Chen and Jing’s result [4], because \( \frac{39\Delta(G) + 36}{38} < \Delta(G) + 2 \).
Haxell and McDonald [13] obtained a different sort of approximation to Conjecture 1.1: 
\[ \chi'(G) \leq \max\{\Delta(G) + 2\sqrt{\mu(G) \log \Delta(G)}, \lfloor \Gamma(G) \rfloor \}. \]
Another way to obtain approximations for Conjecture 1.1 is to incorporate the order \( n \) of \( G \) (that is, number of vertices) into a bound. In this direction, Plantholt [28] proved that 
\[ \chi'(G) = \max\{\Delta(G), \lfloor \Gamma(G) \rfloor \} + 1 + \sqrt{n \log(n/6)} \]
for any multigraph \( G \) with order \( n \geq 572 \). In [29], he established an improved result that is applicable to all multigraphs.

Marcotte [22] showed that 
\[ \chi'(G) = \max\{\Delta(G) + 1, \lfloor \Gamma(G) \rfloor \} \]
for any multigraph \( G \) with no \( K_5^- \)-minor, thereby confirming Conjecture 1.1 for this multigraph class. Recently, Haxell, Krivelevich, and Kronenberg [12] established Conjecture 1.1 for random multigraphs.

The purpose of this paper is to present a proof of the Goldberg-Seymour conjecture.

**Theorem 1.1.** Every multigraph \( G \) satisfies 
\[ \chi'(G) \leq \max\{\Delta(G) + 1, \lfloor \Gamma(G) \rfloor \}. \]

As stated before, Conjectures 1.2-1.5 are all weaker than the Goldberg-Seymour conjecture, so the truth of them follows from Theorem 1.1 as corollaries.

**Theorem 1.2.** Every \( r \)-graph \( G \) satisfies \( \chi'(G) \leq r + 1 \).

**Theorem 1.3.** Let \( G \) be a multigraph with \( \mu(G) = \mu \), such that \( \Delta(G) \) cannot be expressed in the form \( 2pq - q \), where \( q \geq 0 \) and \( p > \lfloor (q + 1)/2 \rfloor \). Then \( \chi'(G) \leq \Delta(G) + \mu(G) - 1 \).

**Theorem 1.4.** Let \( G \) be a critical multigraph with \( \chi'(G) \geq \Delta(G) + 2 \). Then \( G \) contains an odd number of vertices.

**Theorem 1.5.** Let \( G \) be a critical multigraph with \( \chi'(G) > \frac{m\Delta(G) + (m-3)}{m-1} \) for an odd integer \( m \geq 3 \). Then \( G \) has at most \( m - 2 \) vertices.

We have seen that FECP is intimately tied to ECP. For any multigraph \( G \), the fractional chromatic index \( \chi^*(G) = \max\{\Delta(G), \Gamma(G)\} \) can be determined in polynomial time by combining the Padberg-Rao separation algorithm for \( b \)-matching polyhedra [26] (see also [21, 27]) with binary search. In [6], Chen, Zang, and Zhao designed a combinatorial polynomial-time algorithm for finding the density \( \Gamma(G) \) of any multigraph \( G \), thereby resolving a problem posed in both Stiebitz et al. [36] and Jensen and Toft [18]. Nemhauser and Park [24] observed that FECP can be solved in polynomial time by an ellipsoid algorithm, because the separation problem of its LP dual is exactly the maximum-weight matching problem (see also Schrijver [33], Theorem 28.6 on page 477). In [6], Chen, Zang, and Zhao devised a combinatorial polynomial-time algorithm for FECP as well.

We believe that our proof of Theorem 1.1 can be adapted to yield a polynomial-time algorithm for finding an edge-coloring of any multigraph \( G \) with at most \( \max\{\Delta(G) + 1, \lfloor \Gamma(G) \rfloor \} \) colors, and we are working on the design of this algorithm. A successful implementation would lead to an affirmative answer to Conjecture 1.6 as well.

Some remarks may help to put Theorem 1.1 in proper perspective.

First, by Theorem 1.1, there are only two possible values for the chromatic index of a multigraph \( G \): \( \max\{\Delta(G), \lfloor \Gamma(G) \rfloor \} \) and \( \max\{\Delta(G) + 1, \lfloor \Gamma(G) \rfloor \} \). Thus an analogue to Vizing’s theorem on edge-colorings of simple graphs, a fundamental result in graph theory, holds for multigraphs.
Second, Theorem 1.1 exhibits a dichotomy on edge-coloring: While Holyer’s theorem \cite{15} tells us that it is \textsc{NP}-hard to determine $\chi'(G)$, we can approximate it within one of its true value, because $\max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\} - \chi'(G) \leq 1$. Furthermore, if $\Gamma(G) > \Delta(G)$, then $\chi'(G) = \lceil\Gamma(G)\rceil$, so it can be found in polynomial time \cite{6, 26}.

Third, by Theorem 1.1 and aforementioned Seymour’s theorem, every multigraph $G = (V, E)$ satisfies $\chi'(G) - \chi^*(G) \leq 1$, which can be naturally extended to the weighted case. Let $w(e)$ be a nonnegative integral weight on each edge $e \in E$ and let $w = (w(e) : e \in E)$. The \textit{chromatic index} of $(G, w)$, denoted by $\chi'_w(G)$, is the minimum number of matchings in $G$ such that each edge $e$ is covered exactly $w(e)$ times by these matchings, and the \textit{fractional chromatic index} of $(G, w)$, denoted by $\chi^*_w(G)$, is the optimal value of the following linear program:

$$\begin{align*}
\text{Minimize} & \quad 1^T x \\
\text{subject to} & \quad Ax = w \\
& \quad x \geq 0,
\end{align*}$$

where $A$ is again the edge-matching incidence matrix of $G$. Clearly, $\chi'_w(G)$ is the optimal value of the corresponding integer program. Let $G_w$ be obtained from $G$ by replacing each edge $e$ with $w(e)$ parallel edges between the same ends. It is then routine to check that $\chi'_w(G) = \chi'(G_w)$ and $\chi^*_w(G) = \chi^*(G_w)$. So the inequality $\chi'_w(G) - \chi^*_w(G) \leq 1$ holds for all nonnegative integral weight functions $w$, and hence FECP enjoys a fascinating integer rounding property (see Schrijver \cite{32, 33}).

So far the most powerful and sophisticated technique for multigraph edge-coloring is the method of Tashkinov trees \cite{37}, which generalizes the earlier methods of Vizing fans \cite{38} and Kierstead paths \cite{20}. (These methods are named after the authors who invented them, respectively.) Most approximate results described above Theorem 1.1 were obtained by using the method of Tashkinov trees. As remarked by McDonald \cite{23}, the Goldberg-Seymour conjecture and ideas culminating in this method are two cornerstones in modern edge-coloring. Nevertheless, this method suffers some theoretical limitation when applied to prove the conjecture; the reader is referred to Asplund and McDonald \cite{2} for detailed information. Despite various attempts to extend the Tashkinov trees (see, for instance, \cite{3, 4, 5, 31, 36}), the difficulty encountered by the method remains unresolved. Even worse, new problem emerges: it becomes very difficult to preserve the structure of an extended Tashkinov tree under Kempe changes (the most useful tool in edge-coloring theory). In this paper we introduce a new type of extended Tashkinov trees and develop an effective control mechanism over Kempe changes, which can overcome all the aforementioned difficulties.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic concepts and techniques of edge-coloring theory, and exhibit some important properties of the so-called stable colorings. In Section 3, we define the extended Tashkinov trees to be employed in subsequent proof, and give an outline of our proof strategy. In Section 4, we establish some auxiliary results concerning the extended Tashkinov trees and stable colorings, which ensure that this type of trees is preserved under some restricted Kempe changes. In Section 5, we develop an effective control mechanism over Kempe changes, the so-called good hierarchies of extended Tashkinov trees; our proof relies heavily on this novel recoloring technique. In Section 6, we derive some properties enjoyed by the good hierarchies introduced in the preceding section. In Section 7, we present the last step of our proof based on these good hierarchies.
2 Preliminaries

This section presents some basic definitions, terminology, and notations used in our paper, along with some important properties and results.

2.1 Terminology and Notations

Let \( G = (V, E) \) be a multigraph. For each \( X \subseteq V \), let \( G[X] \) denote the subgraph of \( G \) induced by \( X \), and let \( G - X \) denote \( G[V - X] \); we write \( G - x \) for \( G - \{x\} \). Moreover, we use \( \partial(X) \) to denote the set of all edges with precisely one end in \( X \), and write \( \partial(x) \) for \( \partial(X) \) if \( X = \{x\} \). For each pair \( x, y \in V \), let \( E(x, y) \) denote the set of all edges between \( x \) and \( y \). Since it is no longer appropriate to represent an edge \( f \) between \( x \) and \( y \) by \( xy \), we write \( f \in E(x, y) \) instead.

For each subgraph \( H \) of \( G \), let \( V(H) \) and \( E(H) \) denote the vertex set and edge set of \( H \), respectively, let \( |H| = |V(H)| \), and let \( G[H] = G[V(H)] \) and \( \partial(H) = \partial(V(H)) \).

Let \( e \) be an edge of \( G \). A tree sequence with respect to \( G \) and \( e \) is a sequence \( T = (y_0, e_1, y_1, \ldots, e_p, y_p) \) with \( p \geq 1 \), consisting of distinct edges \( e_1, e_2, \ldots, e_p \) and distinct vertices \( y_0, y_1, \ldots, y_p \), such that \( e_1 = e \) and each edge \( e_j \) with \( 1 \leq j \leq p \) is between \( y_j \) and some \( y_i \) with \( 0 \leq i < j \). Given a tree sequence \( T = (y_0, e_1, y_1, \ldots, e_p, y_p) \), we can naturally associate a linear order \( \prec \) with its vertices, such that \( y_i \prec y_j \) if \( i < j \). We write \( y_i \preceq y_j \) if \( i \leq j \). This linear order will be used repeatedly in subsequent sections. For each vertex \( y_j \) of \( T \) with \( j \geq 1 \), let \( T(y_j) \) denote \( (y_0, e_1, y_1, \ldots, e_j, y_j) \). Clearly, \( T(y_j) \) is also a tree sequence with respect to \( G \) and \( e \). We call \( T(y_j) \) the segment of \( T \) induced by \( y_j \). Let \( T_1 \) and \( T_2 \) be two tree sequences with respect to \( G \) and \( e \). We write \( T_2 - T_1 \) for \( E[T_2] - E[T_1] \), write \( T_1 \subseteq T_2 \) if \( T_1 \) is a segment of \( T_2 \), and write \( T_1 \subset T_2 \) if \( T_1 \) is a proper segment of \( T_2 \); that is, \( T_1 \subseteq T_2 \) and \( T_1 \neq T_2 \).

A \( k \)-edge-coloring of \( G \) is an assignment of \( k \) colors, \( 1, 2, \ldots, k \), to the edges of \( G \) so that no two adjacent edges have the same color. By definition, the chromatic index \( \chi'(G) \) of \( G \) is the minimum \( k \) for which \( G \) has a \( k \)-edge-coloring. We use \([k]\) to denote the color set \( \{1, 2, \ldots, k\} \), and use \( \mathcal{C}^k(G) \) to denote the set of all \( k \)-edge-colorings of \( G \). Note that every \( k \)-edge-coloring of \( G \) is a mapping from \( E \) to \([k]\).

Let \( \varphi \) be a \( k \)-edge-coloring of \( G \). For each \( \alpha \in [k] \), the edge set \( E_{\varphi, \alpha} = \{ e \in E : \varphi(e) = \alpha \} \) is called a color class, which is a matching in \( G \). For any two distinct colors \( \alpha \) and \( \beta \) in \([k]\), let \( H \) be the spanning subgraph of \( G \) with \( E(H) = E_{\varphi, \alpha} \cup E_{\varphi, \beta} \). Then each component of \( H \) is either a path or an even cycle; we refer to such a component as an \((\alpha, \beta)\)-path with respect to \( \varphi \), and also call it an \((\alpha, \beta)\)-path (resp. \((\alpha, \beta)\)-cycle) if it is a path (resp. cycle). We use \( P_\alpha(\alpha, \beta, \varphi) \) to denote the unique \((\alpha, \beta)\)-chain containing each vertex \( v \). Clearly, for any two distinct vertices \( u \) and \( v \), either \( P_\alpha(\alpha, \beta, \varphi) \) and \( P_\alpha(\alpha, \beta, \varphi) \) are identical or are vertex-disjoint. Let \( C \) be an \((\alpha, \beta)\)-chain with respect to \( \varphi \), and let \( \varphi' \) be the \( k \)-edge-coloring arising from \( \varphi \) by interchanging \( \alpha \) and \( \beta \) on \( C \). We say that \( \varphi' \) is obtained from \( \varphi \) by recoloring \( C \), and write \( \varphi' = \varphi/C \). This operation is called a Kempe change.

Let \( F \) be an edge subset of \( G \). As usual, \( G - F \) stands for the multigraph obtained from \( G \) by deleting all edges in \( F \); we write \( G - f \) for \( G - F \) if \( F = \{f\} \). Let \( \pi \in \mathcal{C}^k(G - F) \). For each \( K \subseteq E \), define \( \pi(K) = \cup_{e \in K - F} \pi(e) \). For each \( v \in V \), define

\[
\pi(v) = \pi(\partial(v)) \quad \text{and} \quad \pi(v) = |k| - \pi(v).
\]
We call \( \pi(v) \) the set of colors present at \( v \) and call \( \overline{\pi}(v) \) the set of colors missing at \( v \). For each \( X \subseteq V \), define
\[
\pi(X) = \bigcup_{v \in X} \pi(v).
\]
We call \( X \) elementary with respect to \( \pi \) if \( \pi(u) \cap \pi(v) = \emptyset \) for any two distinct vertices \( u, v \in X \). We call \( X \) closed with respect to \( \pi \) if \( \pi(\partial(X)) \cap \pi(X) = \emptyset \); that is, no missing coloring of \( X \) appears on the edges in \( \partial(X) \). Furthermore, we call \( X \) strongly closed with respect to \( \pi \) if \( X \) is closed with respect to \( \pi \) and \( \pi(e) \neq \pi(f) \) for any two distinct colored edges \( e, f \in \partial(X) \). For each subgraph \( H \) of \( G \), write \( \pi(H) \) for \( \pi(V(H)) \), and write \( \pi(H) \) for \( \pi(E(H)) \). Moreover, define
\[
\partial_{\pi, \alpha}(H) = \{ e \in \partial(H) : \pi(e) = \alpha \},
\]
and define
\[
I[\partial_{\pi, \alpha}(H)] = \{ v \in V(H) : v \text{ is incident with an edge in } \partial_{\pi, \alpha}(H) \}.
\]
For an edge \( e \in \partial(H) \), we call its end in (resp. outside) \( H \) the in-end (resp. out-end) relative to \( H \). Thus \( I[\partial_{\pi, \alpha}(H)] \) consists of all in-ends (relative to \( H \)) of edges in \( \partial_{\pi, \alpha}(H) \). A color \( \alpha \) is called a defective color of \( H \) with respect to \( \pi \) if \( |\partial_{\pi, \alpha}(H)| \geq 2 \). A color \( \alpha \in \pi(H) \) is called closed in \( H \) under \( \pi \) if \( \partial_{\pi, \alpha}(H) = \emptyset \). For convenience, we say that \( H \) is closed (resp. strongly closed) with respect to \( \pi \) if \( V(H) \) is closed (resp. strongly closed) with respect to \( \pi \). Let \( \alpha \) and \( \beta \) be two colors that are not assigned to \( \partial(H) \) under \( \pi \). We use \( \pi/(G - H, \alpha, \beta) \) to denote the coloring \( \pi' \) obtained from \( \pi \) by interchanging \( \alpha \) and \( \beta \) in \( G - V(H) \); that is, for any edge \( f \) in \( G - V(H) \), if \( \pi(f) = \alpha \) then \( \pi'(f) = \beta \), and if \( \pi(f) = \beta \) then \( \pi'(f) = \alpha \). Obviously, \( \pi' \in C^k(G - F) \).

### 2.2 Elementary Multigraphs

Let \( G = (V, E) \) be a multigraph. We call \( G \) an elementary multigraph if \( \chi'(G) = [\Gamma(G)] \). With this notion, Conjecture 1.1 can be rephrased as follows.

**Conjecture 2.1.** Every multigraph \( G \) with \( \chi'(G) \geq \Delta(G) + 2 \) is elementary.

Recall that \( G \) is critical if \( \chi'(H) < \chi'(G) \) for any proper subgraph \( H \) of \( G \). As pointed out by Stiebitz et al. [36] (see page 7), to prove Conjecture 2.1, it suffices to consider critical multigraphs. To see this, let \( G \) be an arbitrary multigraph with \( \chi'(G) \geq \Delta(G) + 2 \). Then \( G \) contains a critical multigraph \( H \) with \( \chi'(H) = \chi'(G) \), which implies that \( \chi'(H) \geq \Delta(H) + 2 \). Note that if \( H \) is elementary, then so is \( G \), because \( |\Gamma(G)| \leq \chi'(G) = \chi'(H) = |\Gamma(H)| \leq |\Gamma(G)| \). Thus both inequalities hold with equalities, and hence \( \chi'(G) = |\Gamma(G)| \).

To prove Conjecture 1.1, we shall actually establish the following statement.

**Theorem 2.1.** Every critical multigraph \( G \) with \( \chi'(G) \geq \Delta(G) + 2 \) is elementary.

In our proof we shall appeal to the following theorem, which reveals some intimate connection between elementary multigraphs and elementary sets. This result is implicitly contained in Andersen [1] and Goldberg [10], and explicitly proved in Stiebitz et al. [36] (see Theorem 1.4 on page 8).

**Theorem 2.2.** Let \( G = (V, E) \) be a multigraph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \). If \( G \) is critical, then the following conditions are equivalent:
(i) $G$ is an elementary multigraph.

(ii) For each edge $e \in E$ and each coloring $\varphi \in \mathcal{C}^k(G - e)$, the vertex set $V$ is elementary with respect to $\varphi$.

(iii) There exist an edge $e \in E$ and a coloring $\varphi \in \mathcal{C}^k(G - e)$, such that the vertex set $V$ is elementary with respect to $\varphi$.

(iv) There exist an edge $e \in E$, a coloring $\varphi \in \mathcal{C}^k(G - e)$, and a subset $X$ of $V$, such that $X$ contains both ends of $e$, and $X$ is elementary as well as strongly closed with respect to $\varphi$.

2.3 Stable Colorings

In this subsection, we assume that $T$ is a tree sequence with respect to a multigraph $G = (V, E)$ and an edge $e$, $C$ is a subset of $[k]$, and $\varphi$ is a coloring in $\mathcal{C}^k(G - e)$, where $k \geq \Delta(G) + 1$.

A coloring $\pi \in \mathcal{C}^k(G - e)$ is called a $(T, C, \varphi)$-stable coloring if the following two conditions are satisfied:

1. $\pi(f) = \varphi(f)$ for any $f \in E$ incident to $T$ with $\varphi(f) \in \overline{\varphi}(T) \cup C$;
2. $\pi(v) = \overline{\varphi}(v)$ for any $v \in V(T)$.

In our proof we shall perform a sequence of Kempe changes so that the resulting colorings are stable in some sense. The following lemma gives an equivalent definition of stable colorings.

Lemma 2.3. A coloring $\pi \in \mathcal{C}^k(G - e)$ is $(T, C, \varphi)$-stable iff the following two conditions are satisfied:

1. For any $f \in E$ incident to $T$, color $\pi(f) \in \overline{\varphi}(T) \cup C$ iff $\varphi(f) \in \overline{\varphi}(T) \cup C$. Furthermore, $\pi(f) = \varphi(f)$ in this case;
2. $\pi(v) = \overline{\varphi}(v)$ for any $v \in V(T)$.

Proof. Note that condition (ii) described here is exactly the same as given in the definition and that (i') implies (i), so the “if” part is trivial. To establish the “only if” part, let $f \in E$ be an arbitrary edge incident to $T$ with $\pi(f) \in \overline{\varphi}(T) \cup C$. We claim that $\varphi(f) = \pi(f)$, for otherwise, let $v \in V(T)$ be an end of $f$. By (ii), we have $\pi(v) = \overline{\varphi}(v)$. So $\varphi(v) = \varphi(f)$ and hence there exists an edge $g \in \partial(v) - \{f\}$ with $\varphi(g) = \pi(f)$. It follows that $\varphi(g) \in \overline{\varphi}(T) \cup C$. By (i), we obtain $\pi(g) = \varphi(g)$, which implies $\pi(f) = \pi(g)$, contradicting the hypothesis that $\pi \in \mathcal{C}^k(G - e)$. Our claim asserts that $\varphi(f) = \pi(f)$ for any $f \in E$ incident to $T$ with $\pi(f) \in \overline{\varphi}(T) \cup C$. Combining this with (i), we see that (i') holds.

Let us exhibit some properties enjoyed by stable colorings.

Lemma 2.4. Being $(T, C, \cdot)$-stable is an equivalence relation on $\mathcal{C}^k(G - e)$.

Proof. From Lemma 2.3 it is clear that being $(T, C, \cdot)$-stable is reflexive, symmetric, and transitive. So it defines an equivalence relation on $\mathcal{C}^k(G - e)$.
to \( \varphi \) if \( V(T) \cap V(P) = \{v\} \) and \( \overline{\varphi}(u) \cap \{\alpha, \beta\} \neq \emptyset \); in this case, \( v \) is called a \((T, \varphi, \{\alpha, \beta\})\)-exit and \( P \) is also called a \((T, \varphi, \{\alpha, \beta\})\)-exit path. Note that possibly \( \overline{\varphi}(v) \cap \{\alpha, \beta\} = \emptyset \).

Let \( f \in E(u, v) \) be an edge in \( \partial(T) \) with \( v \in V(T) \). We say that \( f \) is \( T \lor C \)-nonextendable with respect to \( \varphi \) if there exist a \((T, C \cup \{\varphi(f)\}, \varphi)\)-stable coloring \( \pi \) and a color \( \alpha \in \pi(v) \), such that \( v \) is a \((T, \pi, \{\alpha, \varphi(f)\})\)-exit. Otherwise, we say that \( f \) is \( T \lor C \)-extendable with respect to \( \varphi \).

**Lemma 2.5.** Suppose \( T \) is closed with respect to \( \varphi \), and \( f \in E(u, v) \) is an edge in \( \partial(T) \) with \( v \in V(T) \). If there exists a \((T, C \cup \{\varphi(f)\}, \varphi)\)-stable coloring \( \pi \), such that \( \pi(u) \cap \pi(T) \neq \emptyset \), then \( f \) is \( T \lor C \)-nonextendable with respect to \( \varphi \).

**Proof.** Let \( \alpha \in \pi(u) \cap \pi(T) \) and \( \beta \in \overline{\pi}(v) \). By the definition of stable colorings, we have \( \alpha \in \overline{\pi}(T) \) and \( \beta \in \overline{\pi}(v) \). Since both \( \alpha \) and \( \beta \) are closed in \( T \) under \( \varphi \), they are also closed in \( T \) under \( \pi \) by Lemma 2.3. Define \( \pi' = \pi/(G - T, \alpha, \beta) \). Clearly, \( \pi' \) is a \((T, C \cup \{\varphi(f)\}), \pi)\)-stable coloring. By Lemma 2.4, \( \pi' \) is also a \((T, C \cup \{\varphi(f)\}), \varphi)\)-stable coloring. Since \( P_e(\beta, \varphi(f), \pi') \) consists of a single edge \( f \), it is a \( T \)-exit path with respect to \( \pi' \). Hence \( f \) is \( T \lor C \)-nonextendable with respect to \( \varphi \).

**Lemma 2.6.** Suppose \( T \) is closed with respect to \( \varphi \), and \( f \in E(u, v) \) is an edge in \( \partial(T) \) with \( v \in V(T) \). If \( f \) is \( T \lor C \)-nonextendable with respect to \( \varphi \), then for any \( \alpha \in \overline{\pi}(v) \) there exists a \((T, C \cup \{\varphi(f)\}, \varphi)\)-stable coloring \( \pi \), such that \( v \) is a \((T, \pi, \{\alpha, \varphi(f)\})\)-exit.

**Proof.** Since \( f \) is \( T \lor C \)-nonextendable, by definition, there exist a \((T, C \cup \{\varphi(f)\}), \varphi)\)-stable coloring \( \varphi' \) and a color \( \beta \in \overline{\pi}(v) \), such that \( v \) is a \((T, \varphi', \{\beta, \varphi(f)\})\)-exit. Since both \( \alpha \) and \( \beta \) are closed in \( T \) under \( \varphi' \), they are also closed in \( T \) under \( \varphi' \) by Lemma 2.3. Define \( \pi = \varphi'/(G - T, \alpha, \beta) \). Clearly, \( \pi \) is a \((T, C \cup \{\varphi(f)\}), \varphi')\)-stable coloring. By Lemma 2.4, \( \pi \) is also a \((T, C \cup \{\varphi(f)\}), \varphi)\)-stable coloring. Note that \( P_e(\alpha, \varphi(f), \pi) = P_e(\beta, \varphi(f), \varphi') \), so \( P_e(\alpha, \varphi(f), \pi) \) is a \( T \)-exit path with respect to \( \pi \), and hence \( v \) is a \((T, \pi, \{\alpha, \varphi(f)\})\)-exit.

**Lemma 2.7.** Suppose \( T \) is closed but not strongly closed with respect to \( \varphi \) with \( |V(T)| \) odd, and suppose \( \pi \) is a \((T, C, \varphi)\)-stable coloring. Then \( T \) is also closed but not strongly closed with respect to \( \pi \).

**Proof.** Let \( X = V(T) \) and let \( t \) be the size of the set \([k] - \overline{\pi}(X)\). From Lemma 2.3 we deduce that \( \overline{\pi}(X) = \overline{\pi}(X) \) and that \( T \) is closed with respect to \( \pi \). Hence \([k] - \overline{\pi}(X)\) is also of size \( t \). Since \([V(T)]\) is odd, under the coloring \( \varphi \) each color in \([k] - \overline{\pi}(X)\) is assigned to at least one edge in \( \partial(T) \), and some color in \([k] - \overline{\pi}(X)\) is assigned to at least two edges in \( \partial(T) \). It follows that \( |\partial(T)| \geq t + 1 \). Note that under the coloring \( \pi \) only colors in \([k] - \pi(X)\) can be assigned to edges in \( \partial(T) \), so some of these colors is used at least twice by the Pigeonhole Principle. Hence \( T \) is not strongly closed with respect to \( \pi \).

### 2.4 Tashkinov Trees

A multigraph \( G \) is called \( k \)-critical if it is critical and \( \chi'(G) = k + 1 \). Throughout this paper, by a \( k \)-triple we mean a \( k \)-critical multigraph \( G = (V, E) \), where \( k \geq \Delta(G) + 1 \), together with an uncolored edge \( e \in E \) and a coloring \( \varphi \in C^k(G - e) \); we denote it by \((G, e, \varphi)\).
Let \((G,e,\varphi)\) be a \(k\)-triple. A Tashkinov tree with respect to \(e\) and \(\varphi\) is a tree sequence \(T = (y_0, e_1, y_1, \ldots, e_p, y_p)\) with respect to \(G\) and \(e\), such that for each edge \(e_j\) with \(2 \leq j \leq p\), there is a vertex \(y_i\) with \(0 \leq i < j\) satisfying \(\varphi(e_j) \in \overline{\varphi}(y_i)\).

The following theorem is due to Tashkinov [37]; its proof can also be found in Stiebitz et al. [36] (see Theorem 5.1 on page 116).

**Theorem 2.8.** Let \((G,e,\varphi)\) be a \(k\)-triple and let \(T\) be a Tashkinov tree with respect to \(e\) and \(\varphi\). Then \(V(T)\) is elementary with respect to \(\varphi\).

Let \(G = (V,E)\) be a critical multigraph \(G\) with \(\chi'(G) \geq \Delta(G) + 2\). For each edge \(e \in E\) and each coloring \(\varphi \in \mathcal{C}^k(G - e)\), there is a Tashkinov tree \(T\) with respect to \(e\) and \(\varphi\). The Tashkinov order of \(G\), denoted by \(t(G)\), is the largest number of vertices contained in such a Tashkinov tree. Scheide [31] (see Proposition 4.5) has established the following result, which will be employed in our proof.

**Theorem 2.9.** Let \(G\) be a critical multigraph \(G\) with \(\chi'(G) \geq \Delta(G) + 2\). If \(t(G) < 11\), then \(G\) is an elementary multigraph.

The method of Tashkinov trees consists of modifying a given partial edge-coloring with sequences of Kempe changes and resulting extensions (that is, coloring an edge \(e\) with a color \(\alpha\), which is missing at both ends of \(e\)). When applied to prove Conjecture 1.1, the crux of this method is to capture the density \(\Gamma(G)\) by exploring a sufficiently large Tashkinov tree (see Theorem 2.8). However, this target may become unreachable when \(\chi'(G)\) gets close to \(\Delta(G)\), even if we allow for an unlimited number of Kempe changes; such an example has been found by Asplund and McDonald [2]. To circumvent this difficulty and to make this method work, we shall introduce a new type of extended Tashkinov trees in this paper by using the procedure described below.

Given a \(k\)-triple \((G,e,\varphi)\) and a tree sequence \(T\) with respect to \(G\) and \(e\), we may construct a tree sequence \(T' = (T, e_1, y_1, \ldots, e_p, y_p)\) from \(T\) by recursively adding edges \(e_1, e_2, \ldots, e_p\) and vertices \(y_1, y_2, \ldots, y_p\) outside \(T\), such that

- \(e_1\) is incident to \(T\) and each edge \(e_i\) with \(1 \leq i \leq p\) is between \(y_i\) and \(V(T) \cup \{y_1, y_2, \ldots, y_{i-1}\}\);
- for each edge \(e_i\) with \(1 \leq i \leq p\), there is a vertex \(x_i\) in \(V(T) \cup \{y_1, y_2, \ldots, y_{i-1}\}\), satisfying \(\varphi(e_i) \in \overline{\varphi}(x_i)\).

Such a procedure is referred to as Tashkinov’s augmentation algorithm (TAA). We call \(T'\) a closure of \(T\) under \(\varphi\) if it cannot grow further by using TAA (equivalently, \(T'\) becomes closed).

We point out that, although there might be several ways to construct a closure of \(T\) under \(\varphi\), the vertex set of these closures is unique.

### 3 Extended Tashkinov Trees

The purpose of this section is to present extended Tashkinov trees to be used in our proof and to give an outline of our proof strategy.

Given a \(k\)-triple \((G,e,\varphi)\), we first propose an algorithm for constructing a Tashkinov series, which is a series of tuples \((T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}, \Theta_{n-1})\) for \(n = 1, 2, \ldots\), where

- \(\varphi_{n-1}\) is the \(k\)-edge-coloring of \(G - e\) exhibited in iteration \(n - 1\),
Let $T_n$ be a tree sequence with respect to $G$ and $e$, and $\varphi_{n-1}$ constructed in iteration $n-1$,

- $S_{n-1}$ consists of the connecting colors used in iteration $n-1$ with $|S_{n-1}| \leq 2$,
- $F_{n-1}$ consists of the connecting edge used in iteration $n-1$ if $n \geq 2$ and $F_0 = \emptyset$, and
- $\Theta_{n-1} \in \{RE, SE, PE\}$ if $n \geq 2$, which stands for the extension type used in iteration $n-1$; we set $\Theta_0 = \emptyset$.

For ease of description, we make some preparations. Since each $T_n$ is a tree sequence with respect to $G$ and $e$, the linear order $\prec$ defined in Subsection 2.1 is valid for $T_n$. By $T_n + f_n$ we mean the tree sequence augmented from $T_n$ by adding an edge $f_n$. By a **segment** of a cycle we mean a path contained in it.

Let $D_{n-1}$ be a certain subset of $[k]$ and let $\pi$ be a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring. We use $v_{\pi, \alpha}$ to denote the maximum vertex in $I[\partial_{\pi, \alpha}(T_n)]$ in the order $\prec$ for each defective color $\alpha$ of $T_n$ with respect to $\pi$, and use $v_\pi$ to denote the maximum vertex in the order $\prec$ among all these vertices $v_{\pi, \alpha}$. We reserve the symbol $v_n$ for the maximum vertex in the order $\prec$ among all these vertices $v_\pi$, where $\pi$ ranges over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings. We also reserve the symbol $\pi_n$ for the corresponding $\pi$ (that is, $v_n = v_{\pi_n}$), and reserve $f_n \in E(u_n, v_n)$ for an edge in $\partial(T_n)$ such that $\pi_n(f_n)$ is a defective color with respect to $\pi_n$. We call $v_n$ the maximum defective vertex with respect to $(T_n, D_{n-1}, \varphi_{n-1})$.

(3.1) In our algorithm, there are three types of augmentations: revisiting extension (RE), series extension (SE), and parallel extension (PE). Each iteration $(n \geq 1)$ involves a special vertex $v_n$, which is called an *extension* vertex if $\Theta_n = SE$ and a supporting vertex if $\Theta_n = PE$.

### Algorithm 3.1

**Step 0.** Let $\varphi_0 = \varphi$ and let $T_1$ be a closure of $e$ under $\varphi_0$, which is a closed Tashkinov tree with respect to $e$ and $\varphi_0$. Set $S_0 = F_0 = \Theta_0 = \emptyset$ and set $n = 1$.

**Step 1.** If $T_n$ is strongly closed with respect to $\varphi_{n-1}$, stop. Else, if there exists a subscript $h \leq n - 1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that $\Theta_i = RE$ for all $i$ with $h + 1 \leq i \leq n - 1$, if any, and such that some $(\gamma_h, \delta_h)$-cycle with respect to $\varphi_{n-1}$ contains both an edge $f_n \in \partial_{\pi_n-1, \gamma_h}(T_n)$ and a segment $L$ connecting $V(T_h)$ and $v_n$ with $V(L) \subseteq V(T_n)$, where $v_n$ is the end of $f_n$ in $T_n$, go to Step 2. Else, let $D_{n-1} = \cup_{1 \leq i \leq n-1} S_i - \pi_{n-1}(T_n-1)$, where $T_0 = \emptyset$. Let $v_n$, $\pi_n$, and $f_n \in E(u_n, v_n)$ be as defined above the algorithm, and let $\delta_n = \pi_{n-1}(f_n)$. If for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring $\sigma_{n-1}$, we have $\pi_{n-1}(T_n) \cap \pi_{n-1}(u_n) = \emptyset$, go to Step 3. Else, go to Step 4.

**Step 2.** Let $\varphi_n = \varphi_{n-1}$, let $T_{n+1}$ be a closure of $T_n + f_n$ under $\varphi_n$, and let $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = RE$. Set $n = n + 1$, return to Step 1. (We call this augmentation a **revisiting extension** (RE), and call $f_n$ an *RE connecting edge*. Note that $v_n$ is neither called an extension vertex nor called a supporting vertex (see (3.1)).

**Step 3.** Let $\varphi_n = \pi_{n-1}$, let $T_{n+1}$ be a closure of $T_n + f_n$ under $\varphi_n$, and let $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$. Set $n = n + 1$, return to Step 1. (We call this augmentation a **series extension** (SE), call $f_n$ an *SE connecting edge*, and call $v_n$ an extension vertex.)

**Step 4.** Let $A_{n-1}$ be the set of all subscripts $i$ with $1 \leq i \leq n - 1$ such that $\Theta_i = PE$ and $v_i = v_n$. Let $\gamma_n$ be a color in $\pi_{n-1}(v_n) \cap (\cup_{i \in A_{n-1}} S_i)$ if $A_{n-1} \neq \emptyset$ (see (3.5) below), and a color in $\pi_{n-1}(v_n)$.
otherwise. By Lemmas 2.5 and 2.6, there exists a \((T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})\)-stable coloring \(\pi'_{n-1}\), such that \(v_n\) is a \((T_n, \pi'_{n-1}, \{\gamma_n, \delta_n\})\)-exit. Let \(\varphi_n = \pi_{n-1}/P_{\pi_{n-1}}(\gamma_n, \delta_n, \pi'_{n-1})\), \(S_n = \{\delta_n, \gamma_n\}\), \(F_n = \{f_n\}\), and \(\Theta_n = PE\). Let \(T_{n+1}\) be a closure of \(T_n\) under \(\varphi_n\). Set \(n = n+1\), return to Step 1. (We call this augmentation a parallel extension (PE), call \(f_n\) a PE connecting edge, and call \(v_n\) a supporting vertex. Note that \(f_n\) is not necessarily contained in \(T_{n+1}\).

Throughout the remainder of this paper, we reserve all symbols used for the same usage as in this algorithm. In particular, \(D_n = \bigcup_{i \leq n} S_i - \varphi_n(T_n)\) (see Step 1) for \(n \geq 0\). So \(D_0 = \emptyset\).

To help understand the algorithm better, let us make a few remarks and offer some simple observations.

(3.2) In our proof we shall restrict our attention to the case when \(|T_n|\) is odd (as we shall see). Suppose \(T_n\) is not strongly closed with respect to \(\varphi_{n-1}\) (see Step 1). Then, by Lemma 2.7, \(T_n\) is closed but not strongly closed with respect to each \((T_n, D_{n-1}, \varphi_{n-1})\)-stable coloring. Thus \(v_n, \pi_{n-1}\), and \(f_n\) involved in Step 1 are all well defined. It follows that at least one of RE, SE and PE applies to each iteration, and hence the algorithm terminates only when \(T_n\) is strongly closed with respect to \(\varphi_{n-1}\), which contains the case when \(V(T_n) = V(G)\).

(3.3) As described in the algorithm, revisiting extension (RE) has priority over both series and parallel extensions (SE and PE). If \(\Theta_n = RE\), then from Algorithm 3.1 we see that the \((\gamma_h, \delta_h)\)-cycle with respect to \(\varphi_{n-1}\) displayed in Step 1 must contain at least one edge in \(G[T_h]\), at least two boundary edges of \(T_h\) colored with \(\gamma_h\), and at least two boundary edges of \(T_n\) colored with \(\gamma_h\), because \(\delta_h\) is a missing color in \(T_h\) under both \(\varphi_h\) and \(\varphi_{n-1}\).

(3.4) It is clear that \(\delta_n\) is a defective color of \(T_n\) with respect to \(\varphi_n\) when \(\Theta_n = SE\) or PE (as \(|T_n|\) is odd), while \(\gamma_n\) is a defective color of \(T_n\) with respect to \(\varphi_n\) when \(\Theta_n = RE\). Moreover, \(D_{n-1}\) is the set of all connecting colors in \(\cup_{h \leq n-1} S_h\) that are not missing in \(T_n\) with respect to \(\varphi_{n-1}\).

(3.5) As we shall prove in Lemma 3.3, if \(A_{n-1} \neq \emptyset\), then \(\gamma_n\) in Step 4 can be selected in a unique way. This property will play an important role in our proof.

**Lemma 3.2.** For \(n \geq 1\), the following statements hold:

(i) \(\varphi_{n-1}(T_n) \cup D_{n-1} \subseteq \varphi_n(T_n) \cup D_n \subseteq \varphi_n(T_{n+1}) \cup D_n\).

(ii) For any edge \(f\) incident to \(T_n\), if \(\varphi_{n-1}(f) \in \varphi_{n-1}(T_n) \cup D_{n-1}\), then \(\varphi_n(f) = \varphi_{n-1}(f)\), unless \(\Theta_n = PE\) and \(f = f_n\). So \(\varphi_n(f) \in \varphi_n(T_n) \cup D_n\), provided that \(\varphi_{n-1}(f) \in \varphi_{n-1}(T_n) \cup D_{n-1}\).

(iii) \(\varphi_{n-1}(T_n) \subseteq \varphi_{n-1}(T_n) \cup D_{n-1}\) and \(\varphi_n(T_n) \subseteq \varphi_n(T_n) \cup D_n\). So \(\sigma_n(f) = \varphi_n(f)\) for any \((T_n, D_n, \varphi_n)\)-stable coloring \(\sigma_n\) and any edge \(f\) on \(T_n\).

(iv) If \(\Theta_n = PE\), then \(\partial_{\varphi_n, \gamma_n}(T_n) = \{f_n\}\), and edges in \(\partial_{\varphi_n, \delta_n}(T_n)\) are all incident to \(V(T_n(v_n) - v_n)\). Furthermore, each color in \(\varphi_n(T_n) - \{\delta_n\}\) is closed in \(T_n\) under \(\varphi_n\).

**Proof.** By definition, \(D_{n-1} = \bigcup_{i \leq n-1} S_i - \varphi_{n-1}(T_{n-1})\). So \(\varphi_{n-1}(T_n) \cup D_{n-1} = \varphi_{n-1}(T_n) \cup [\bigcup_{i \leq n-1} S_i - \varphi_{n-1}(T_{n-1})]\). Since \(\varphi_{n-1}(T_{n-1}) \subseteq \varphi_{n-1}(T_n)\), we obtain

1. \(\varphi_{n-1}(T_n) \cup D_{n-1} = \varphi_{n-1}(T_n) \cup (\bigcup_{i \leq n-1} S_i)\).

Similarly, we can prove that

2. \(\varphi_n(T_n) \cup D_n = \varphi_n(T_n) \cup (\bigcup_{i \leq n} S_i)\).
(i) For any $\alpha \in \mathcal{S}_{n-1}(T_n)$, from Algorithm 3.1 and definition of stable colorings we see that $\alpha \in \mathcal{S}_n(T_n)$, unless $\Theta_n = PE$ and $\alpha = \gamma_n$: in this exceptional case, $\alpha \in S_n$. So $\mathcal{S}_{n-1}(T_n) \subseteq \mathcal{S}_n(T_n) \cup S_n$ and hence $\mathcal{S}_{n-1}(T_n) \cup (\cup_{i \leq n-1} S_i) \subseteq \mathcal{S}_n(T_n) \cup (\cup_{i \leq n} S_i)$. It follows from (1) and (2) that $\mathcal{S}_{n-1}(T_n) \cup D_{n-1} \subseteq \mathcal{S}_n(T_n) \cup D_n$. Clearly, $\mathcal{S}_n(T_n) \cup D_n \subseteq \mathcal{S}_{n-1}(T_n+1) \cup D_n$.

(ii) Let $f$ be an edge incident to $T_n$ with $\varphi_{n-1}(f) \in \mathcal{S}_{n-1}(T_n) \cup D_{n-1}$. If $\Theta_n = RE$, then $\varphi_n = \varphi_{n-1}$ by Step 1 of Algorithm 3.1, which implies $\varphi_n(f) = \varphi_{n-1}(f)$. So we may assume that $\Theta_n \neq RE$. Let $\pi_{n-1}$ be the $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring as specified in Step 1 of Algorithm 3.1. By the definition of stable colorings, we obtain $\pi_{n-1}(f) = \varphi_{n-1}(f)$. If $\Theta_n = SE$, then $\varphi_n(f) = \pi_{n-1}(f)$ by Step 3 of Algorithm 3.1. Hence $\varphi_n(f) = \varphi_{n-1}(f)$. It remains to consider the case when $\Theta_n = PE$. Let $\pi_{n-1}'$ be the $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring as specified in Step 4 of Algorithm 3.1. By Lemma 2.4, $\pi_{n-1}'$ is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. Hence $\pi_{n-1}'(f) = \varphi_{n-1}(f)$. Since $\varphi_n = \pi_{n-1}'/P_{n_1}(\delta_n, \gamma_n, \pi_{n-1})$ and $P_{n_1}(\delta_n, \gamma_n, \pi_{n-1})$ contains only one edge $f_n$ incident to $T_n$ (see Step 4 of Algorithm 3.1), we have $\varphi_n(f) = \pi_{n-1}'(f)$, unless $f = f_n$. It follows that $\varphi_n(f) = \varphi_{n-1}(f)$, unless $f = f_n$; in this exceptional case, $\varphi_{n-1}(f) = \delta_n$ and $\varphi_n(f) = \gamma_n \in S_n$. Hence $\varphi_n(f) \in \mathcal{S}_{n-1}(T_n) \cup D_{n-1} \cup S_n \subseteq \mathcal{S}_n(T_n) \cup D_n \cup S_n = \mathcal{S}_n(T_n) \cup D_n$ by (i) and (2), as desired.

(iii) Let us first prove the statement $\varphi_{n-1}(T_n) \subseteq \mathcal{S}_{n-1}(T_n) \cup D_{n-1}$ by induction on $n$. As the statement holds trivially when $n = 1$, we proceed to the induction step and assume that the statement has been established for $n - 1$; that is,

(3) $\varphi_{n-1}(T_{n-1}) \subseteq \mathcal{S}_{n-1}(T_{n-1}) \cup D_{n-2}$.

By (3) and (ii) (with $n - 1$ in place of $n$), for each edge $f$ on $T_{n-1}$ we have $\varphi_{n-1}(f) \in \mathcal{S}_{n-1}(T_{n-1}) \cup D_{n-1} \subseteq \mathcal{S}_{n-1}(T_n) \cup D_{n-1}$. For each edge $f \in T_n - T_{n-1}$, from Algorithm 3.1 and TAA we see that $\varphi_{n-1}(f) \in D_n$ if $f$ is a connecting edge and $\varphi_{n-1}(f) \in \mathcal{S}_{n-1}(T_n)$ otherwise. Combining these observations, we obtain $\varphi_{n-1}(f) \in \mathcal{S}_{n-1}(T_n) \cup D_{n-1}$. Hence $\varphi_{n-1}(T_n) \subseteq \mathcal{S}_{n-1}(T_n) \cup D_{n-1}$, which together with (ii) implies $\varphi_n(T_n) \subseteq \mathcal{S}_n(T_n) \cup D_n$.

It follows that for any edge $f$ on $T_n$, we have $\varphi_n(f) \in \mathcal{S}_n(T_n) \cup D_n$. Thus $\sigma_n(f) = \varphi_n(f)$ for any $(T_n, D_n, \varphi_n)$-stable coloring $\sigma_n$.

(iv) From the definitions of $\pi_{n-1}$ and stable colorings, we see that edges in $\partial \pi_{n-1}, \delta_n(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\pi_{n-1}(T_n)$ is closed in $T_n$ under $\pi_{n-1}$. So, by the definitions of $\pi_{n-1}$ and stable colorings, edges in $\partial \pi_{n-1}, \delta_n(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\pi_{n-1}(T_n)$ is closed in $T_n$ under $\pi_{n-1}$. Thus the desired statements follow instantly from the definition of $\varphi_n$ in Step 4.

Lemma 3.3. Suppose $A_{n-1}$ in Step 4 is the set $\{i_1, i_2, \ldots, i_p\}$ with $i_1 < i_2 < \ldots < i_p \leq n - 1$. Then the following statements hold:

(i) $\pi_{n-1}(v_n) \cap (\cup_{j \in A_{n-1}} S_j) = \mathcal{S}_{n-1}(v_n) \cap (\cup_{j \in A_{n-1}} S_j) = \mathcal{S}_p(v_n) \cap (\cup_{j \in A_{n-1}} S_j) = \{\delta_p\}$;

(ii) $\gamma_{i_2} = \delta_{i_1}, \gamma_{i_3} = \delta_{i_2}, \ldots, \gamma_{i_p} = \delta_{i_{p-1}}$; and

(iii) $\varphi_{i_1}(v_n) = (\varphi_{i_p}(v_n) \setminus \{\delta_p\}) \cup \{\gamma_{i_1}\}$.

Proof. Since $\pi_{n-1}$ is a $(T_n, D_n, \varphi_{n-1})$-stable coloring, we have $\mathcal{S}_{n-1}(v_n) = \mathcal{S}_{n-1}(v_n)$ by definition. As $\Theta_j = RE$ or $SE$ for $i_p + 1 \leq j \leq n - 1$, if any, from Algorithm 3.1 and the definition of stable colorings we see that $\mathcal{S}_{n-1}(v_n) = \mathcal{S}_{i_p}(v_n)$. So to prove (i), we only need to show that $\mathcal{S}_{i_p}(v_n) \cap (\cup_{j \in A_{n-1}} S_j) = \{\delta_p\}$. 

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Set $B_h = \{i_1, i_2, \ldots, i_h\}$ for $1 \leq h \leq p$. Then $B_p = A_{n-1}$. Thus (i) is equivalent to saying that
\[(i') \quad \varphi_{ip}(v_n) \cap (\cup_{j \in B_p} S_j) = \{\delta_{ip}\}.\]
Let us prove statements (i'), (ii'), and (iii) simultaneously by induction on $p$.

In view of Step 4, we have $\gamma_i \in \pi_{i-1}(v_n) = \varphi_{i-1}(v_n)$, $\delta_i \in \varphi_{i}(v_n)$, and $\gamma_i \notin \varphi_{i}(v_n)$.
So $\varphi_{i}(v_n) \cap (\cup_{j \in B_p} S_j) = \varphi_{i}(v_n) \cap \{\gamma_i, \delta_i\} = \{\delta_i\}$ and $\varphi_{i-1}(v_n) = \varphi_{i}(v_n) \setminus \{\delta_i\} \cup \{\gamma_i\}$. Thus both (i') and (iii) hold for $p = 1$. For (ii'), there is nothing to prove now.

Suppose we have established these statements for $p - 1$. Let us proceed to the induction step for $p$.

By the induction hypotheses on (i') and (iii), we obtain $\varphi_{ip-1}(v_n) \cap (\cup_{j \in B_{p-1}} S_j) = \{\delta_{ip-1}\}$ and $\varphi_{i-1}(v_n) = (\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cup \{\gamma_i\}$. As $\Theta_j = RE$ or $SE$ for $i_{p-1} + 1 \leq j \leq i_{p-1}$, if any, from Algorithm 3.1 and the definition of stable colorings, we see that $\varphi_{ip-1}(v_n) = \varphi_{ip-1}(v_n) = \varphi_{ip-1}(v_n)$. According to Step 4,
\begin{enumerate}
  \item $\gamma_{ip} = \delta_{ip-1} \in \varphi_{ip-1}(v_n)$ while $\delta_{ip} \notin \varphi_{ip-1}(v_n)$, and $\varphi_{ip}(v_n) = (\varphi_{ip-1}(v_n) \setminus \{\gamma_{ip}\}) \cup \{\delta_{ip}\}$. Thus (1) implies (ii') and the following equality.
  \item $\varphi_{ip}(v_n) = (\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cup \{\delta_{ip}\}$.
\end{enumerate}

Using the induction hypothesis on (iii), we obtain
\begin{enumerate}
  \item $\varphi_{i-1}(v_n) = (\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cup \{\gamma_i\}$.
\end{enumerate}

Combining (1)-(3) yields $\varphi_{i-1}(v_n) = (\varphi_{ip}(v_n) \setminus \{\delta_{ip}\}) \cup \{\gamma_i\}$, thereby proving (iii).

By the induction hypothesis on (i'), we have $\varphi_{ip-1}(v_n) \cap (\cup_{j \in B_{p-1}} S_j) = \{\delta_{ip-1}\}$. So $(\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cap (\cup_{j \in B_{p-1}} S_j) = \emptyset$. Hence $(\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cap (\cup_{j \in B_{p-1}} S_j) = (\varphi_{ip-1}(v_n) \setminus \{\delta_{ip-1}\}) \cap S_{ip} = (\varphi_{ip-1}(v_n) \setminus \{\gamma_i\}) \cap \{\gamma_{ip}, \delta_{ip}\} = \emptyset$ by (1). Thus, by (2), we obtain $\varphi_{ip}(v_n) \cap (\cup_{j \in B_p} S_j) = \{\delta_{ip}\} \cup (\cup_{j \in B_p} S_j) = \{\delta_{ip}\}$. So (i') is established.

Lemma 3.4. $|D_n| \leq n$.

**Proof.** Recall that $S_i = \{\delta_i\}$ if $\Theta_i = SE$ and $S_i = \{\delta_i, \gamma_i\}$ otherwise. Moreover, $\delta_i \in \varphi_i(T_i)$ if $\Theta_i = PE$ or $RE$ for $1 \leq i \leq n$.

We apply induction on $n$. Trivially, the statement holds when $n = 0, 1$. So we proceed to the induction step, and assume that $|D_{n-1}| \leq n - 1$ for some $n \geq 2$.

If $\Theta_n = RE$, then $\varphi_n = \varphi_n - 1$ and $S_n = S_{n-1}$ by Step 2. So $D_n \subseteq D_{n-1}$ and hence $|D_n| \leq n - 1$. If $\Theta_n = SE$, then $S_n = \{\delta_n\}$ and $\varphi_n(T_n) = \varphi_n(T_n)$ by Algorithm 3.1 and the definition of stable coloring. It follows that $D_n \subseteq D_{n-1} \cup \{\delta_n\}$. Hence $|D_n| \leq |D_{n-1}| + 1 \leq n$.

It remains to consider the case when $\Theta_n = PE$. From Step 4 and definition of stable coloring, we see that $S_n = \{\delta_n, \gamma_n\}$ and $(\varphi_{n-1}(T_n) \setminus \{\gamma_n\}) \cup \{\delta_n\} \subseteq \varphi_n(T_n)$. So
\[
D_n = \cup_{i \leq n} S_i - \varphi_n(T_n)
\]
\[
\subseteq \cup_{i \leq n} S_i \cup \{\delta_n, \gamma_n\} - (\varphi_{n-1}(T_n) \setminus \{\gamma_n\}) \cup \{\delta_n\}
\]
\[
\subseteq \cup_{i \leq n} S_i \cup \{\gamma_n\} - (\varphi_{n-1}(T_n) \setminus \{\gamma_n\})
\]
\[
\subseteq [\cup_{i \leq n} S_i - \varphi_{n-1}(T_n)] \cup \{\gamma_n\}
\]
\[
\subseteq D_{n-1} \cup \{\gamma_n\}.
\]

Hence $|D_n| \leq |D_{n-1}| + 1 \leq n$. □
Lemma 3.5. Suppose $\Theta_n = PE$ (see Step 4). Let $\sigma_n$ be a $(T_n, D_n, \varphi_n)$-stable coloring and let $\sigma_{n-1} = \sigma_n/P_v_n(\gamma, \delta_n, \sigma_n)$. If $P_v_n(\gamma, \delta_n, \sigma_n) \cap T_n = \{v_n\}$, then $\sigma_{n-1}$ is $(T_n, D_n-1, \varphi_n-1)$-stable.

Proof. Let $\pi_{n-1}$ and $\pi'_{n-1}$ be as specified in Step 4 of Algorithm 3.1. Recall that $\pi_{n-1}$ is $(T_n, D_n-1, \varphi_n-1)$-stable and $\pi'_{n-1}$ is $(T_n, D_n-1 \cup \{\delta_n\}, \pi_{n-1})$-stable. By definition, $\pi'_{n-1}$ is also $(T_n, D_n-1, \pi_{n-1})$-stable. It follows from Lemma 2.4 that

1. $\pi'_{n-1}$ is $(T_n, D_n-1, \varphi_n-1)$-stable.

By definition, $\varphi_n = \pi'_{n-1}/P_v_n(\gamma, \delta_n, \pi_{n-1})$. So

(2) $\pi_{n-1} = \varphi_n/P_v_n(\gamma, \delta_n, \varphi_n)$.

We propose to show that

(3) $\sigma_{n-1}$ is $(T_n, D_n-1, \pi'_{n-1})$-stable.

To justify this, note that $\sigma_n(v) = \varphi_n(v)$ for all $v \in V(T_n)$, because $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring. Thus, by the definition of $\sigma_{n-1}$ and (2), we obtain

(4) $\sigma_{n-1}(v) = \pi'_{n-1}(v)$ for all $v \in V(T_n)$.

Let $f$ be an edge incident to $T_n$ with $\pi'_{n-1}(f) \in \pi'_{n-1}(T_n) \cup D_n-1$. Then $\pi'_{n-1}(f) \in \varphi_n(T_n) \cup D_n$ by the definition of stable colorings and Algorithm 3.1. So $\pi'_{n-1}(f) \in \varphi_n(T_n) \cup D_n$ by Lemma 3.2(i). From Step 4 we see that $\varphi_n(f) = \pi'_{n-1}(f)$, unless $f = f_n$; in this exceptional case, $\varphi_n(f) = \gamma_n \in D_n$. So $\varphi_n(f) \in \varphi_n(T_n) \cup D_n$. Hence $\sigma_n(f) = \varphi_n(f)$, because $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring. Again, from the definition of $\sigma_{n-1}$ and (2), we deduce that if $f \neq f_n$ (see Step 4), then $\sigma_{n-1}(f) = \sigma_n(f)$ and $\pi'_{n-1}(f) = \varphi_n(f)$, which implies $\sigma_{n-1}(f) = \pi'_{n-1}(f)$; if $f = f_n$, then $\sigma_{n-1}(f) = \delta_n = \pi'_{n-1}(f)$. Hence

(5) $\sigma_{n-1}(f) = \pi'_{n-1}(f)$ for any edge $f$ incident to $T_n$ with $\pi'_{n-1}(f) \in \pi'_{n-1}(T_n) \cup D_n-1$.

Thus (3) follows instantly from (4) and (5). Using (1), (3) and Lemma 2.4, we conclude that $\sigma_{n-1}$ is $(T_n, D_n-1, \varphi_n-1)$-stable.

Let us now present the generalized version of Tashkinov trees to be used in our proof.

Definition 3.6. Let $(G, e, \varphi)$ be a $k$-triple. A tree sequence $T$ with respect to $G$ and $e$ is called an extended Tashkinov tree (ETT) if there exists a Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$ constructed from $(G, e, \varphi)$ by using Algorithm 3.1, such that $T_n \subset T \subset T_{n+1}$, where $T_0 = \emptyset$.

As introduced in Subsection 2.1, by $T_n \subset T \subset T_{n+1}$ we mean that $T_n$ is a proper segment of $T$, and $T$ is a segment of $T_{n+1}$.

Observe that the extended Tashkinov tree $T$ has a built-in ladder-like structure. So we propose to call the sequence $T_1 \subset T_2 \subset \ldots \subset T_n \subset T$ the ladder of $T$, and call $n$ the rung number of $T$ and denote it by $r(T)$. Moreover, we call $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ the coloring sequence of $T$, call $\varphi_n$ the generating coloring of $T$, and call $T$ the Tashkinov series corresponding to $T$.

In our proof we shall frequently work with stable colorings; the following concept can help to keep track of the structures of ETTs we consider.

Definition 3.7. Let $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$ be a Tashkinov series constructed from a $k$-triple $(G, e, \varphi)$ by using Algorithm 3.1. A coloring $\sigma_n \in C^k(G - e)$ is called $\varphi_n$ mod $T_n$ if there exists an ETT $T^*$ with corresponding Tashkinov series $T^* = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$, satisfying $\sigma_0 \in C^k(G - e)$ and the following conditions for all $i$ with $1 \leq i \leq n$:
\begin{itemize}
    \item $T^*_i = T_i$ and
    \item $\sigma_i$ is a $(T_i, D_i, \varphi_i)$-stable coloring in $\mathcal{C}^k(G - e)$, where $D_i = \cup_{h \leq i} S_h - \overline{\varphi}_i(T_i)$.
\end{itemize}

We call $T^*$ an ETT corresponding to $(\sigma_n, T_n)$ (or simply corresponding to $\sigma_n$ if no ambiguity arises).

**Remark.** By definition, $T^*_{i+1}$ in $T^*$ is obtained from $T_i$ by using the same connecting edge, connecting color, and extension type as $T_{i+1}$ in $T$ for $1 \leq i \leq n$. Furthermore, $T_1 \subset T_2 \subset \ldots \subset T_n \subset T^*$ is the ladder of $T^*$ and $r(T^*) = n$. Since $\sigma_i$ is a $(T_i, D_i, \varphi_i)$-stable coloring, by Lemma 3.2(iii), we have $\sigma_i(f) = \varphi_i(f)$ for any edge $f$ on $T_i$ and $1 \leq i \leq n$. This fact will be used repeatedly in our paper.

To ensure that the structures of ETTs are preserved under taking stable colorings, we need to impose some restrictions on these trees.

**Definition 3.8.** Let $T$ be an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. We say that $T$ has the maximum property (MP) under $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ (or simply under $\varphi_n$ if no ambiguity arises), if $|T_1|$ is maximum over all Tashkinov trees $T'_i$ with respect to an edge $e' \in E$ and a coloring $\varphi'_0 \in \mathcal{C}^k(G - e')$, and $|T_{i+1}|$ is maximum over all $(T_i, D_i, \varphi_i)$-stable colorings for any $i$ with $1 \leq i \leq n - 1$; that is, $|T_{i+1}|$ is maximum over all tree sequences $T'_{i+1}$, which is a closure of $T_i + f_i$ (resp. $T_i$) under a $(T_i, D_i, \varphi_i)$-stable coloring $\varphi'_i$ if $\Theta_i = RE \text{ or } SE$ (resp. if $\Theta_i = PE$), where $f_i$ is the connecting edge in $F_i$.

**Remark.** Recall that, in the construction of a Tashkinov series by using Algorithm 3.1, RE has priority over both SE and PE (see Step 1), and PE at iteration $n$ is based on the $T_n$-exit path $P_n(\gamma_n, \delta_n, \pi'_n)$ (see Step 4). So at this point a natural question is to ask whether such operations can be preserved under taking stable colorings, and whether an ETT with sufficiently large size and satisfying the maximum property can be constructed to fulfill our needs. We shall demonstrate (see Lemma 3.11) that it is indeed the case.

The statement below follows instantly from the above two definitions and Lemma 2.4.

**Lemma 3.9.** Let $T$ be an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, let $\sigma_n$ be a $\varphi_n$ mod $T_n$ coloring, and let $T^*$ be an ETT corresponding to $(\sigma_n, T_n)$ (see Definition 3.7). If $T$ satisfies MP under $\varphi_n$, then $T^*$ satisfies MP under $\sigma_n$.

The importance of the maximum property is revealed by the following statement to be established: If $T$ enjoys the maximum property under $\varphi_n$, then $V(T)$ is elementary with respect to $\varphi_n$. From Theorem 2.2 we see that Theorem 2.1 follows from it as a corollary. We shall prove this statement by induction on the rung number $r(T)$. To carry out the induction step, we need several auxiliary results concerning ETTs with the maximum property. Thus what we are going to establish is a stronger version.

Let us define a few terms before presenting our theorem. For each $v \in V(T)$, we use $m(v)$ to denote the minimum subscript $i$ such that $v \in V(T_i)$. Let $\alpha$ and $\beta$ be two colors in $[k]$. We say that $\alpha$ and $\beta$ are $T$-interchangeable under $\varphi_n$ if there is at most one $(\alpha, \beta)$-path with respect to $\varphi_n$ intersecting $T$. When $T$ is closed (that is, $T = T_{n+1}$), we also say that $T$ has
the interchangeability property with respect to \( \varphi_n \) if under any \((T, D_n, \varphi_n)\)-stable coloring \( \sigma_n \), any two colors \( \alpha \) and \( \beta \) are \( T \)-interchangeable, provided that \( \overline{\sigma_n}(T) \cap \{\alpha, \beta\} \neq \emptyset \) (equivalently \( \overline{\varphi_n}(T) \cap \{\alpha, \beta\} \neq \emptyset \)).

The undefined symbols and notations in the theorem below can all be found in Algorithm 3.1.

**Theorem 3.10.** Let \( T \) be an ETT constructed from a \( k \)-triple \((G, e, \varphi)\) by using the Tashkinov series \( T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \). If \( T \) enjoys the maximum property under \( \varphi_n \), then the following statements hold:

(i) \( V(T) \) is elementary with respect to \( \varphi_n \).

(ii) \( T_{n+1} \) has the interchangeability property with respect to \( \varphi_n \).

(iii) For any \( i \leq n \), if \( v_i \) is a supporting vertex and \( m(v_i) = j \), then every \((T_i, D_i, \varphi_i)\)-stable coloring \( \sigma_i \) is \((T(v_i) - v_i, D_{j-1}, \varphi_{j-1})\)-stable. In particular, \( \sigma_i \) is \((T_{j-1}, D_{j-1}, \varphi_{j-1})\)-stable. Furthermore, for any two distinct supporting vertices \( v_i \) and \( v_j \) with \( i, j \leq n \), if \( m(v_i) = m(v_j) \), then \( S_i \cap S_j = \emptyset \).

(iv) If \( \Theta_n = PE \), then \( P_{n+1}(\gamma_n, \delta_n, \sigma_n) \) contains precisely one vertex, \( v_n \), from \( T_n \) for any \((T_n, D_n, \varphi_n)\)-stable coloring \( \sigma_n \).

(v) For any \((T_n, D_n, \varphi_n)\)-stable coloring \( \sigma_n \) and any defective color \( \delta \) of \( T_n \) with respect to \( \sigma_n \), if \( v \) is a vertex but not the smallest one (in the order \( \prec \)) in \( \left[\Theta_{n-1} \delta(T_n)\right] \), then \( v \preceq v_i \) for any supporting or extension vertex \( v_i \) with \( i \geq m(v) \).

(vi) Every \((T_n, D_n, \varphi_n)\)-stable coloring \( \sigma_n \) is a \( \varphi_n \) mod \( T_n \) coloring. (So every ETT \( T^* \) corresponding to \( (\sigma_n, T_n) \) (see Definition 3.7) satisfies MP under \( \sigma_n \) by Lemma 3.9.)

Let us show that Theorem 2.1 can be deduced easily from this theorem.

**Lemma 3.11.** Let \( T \) be an ETT constructed from a \( k \)-triple \((G, e, \varphi)\) by using the Tashkinov series \( T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \). If \( T \) enjoys MP under \( \varphi_n \), then there exists a closed ETT \( T'_{n+1} \) corresponding to a \( \varphi_n \) mod \( T_n \) coloring \( \sigma_n \) (see Definition 3.7), with ladder \( T_1 \subset T_2 \subset \ldots \subset T_n \subset T'_{n+1} \), such that \( |T'_{n+1}| \) is maximum over all \((T_n, D_n, \varphi_n)\)-stable colorings (see Definition 3.8). Furthermore, if \( T'_{n+1} \) is not strongly closed, then it can be extended further to a larger ETT \( T_{n+2} \) satisfying MP.

**Proof.** Let \( T'_{n+1} \) be a closure of \( T_n + f_n \) (resp. \( T_n \)) under a \((T_n, D_n, \varphi_n)\)-stable coloring \( \sigma_n \) if \( \Theta_n = RE \) or \( SE \) (resp. if \( \Theta_n = PE \)), such that \( |T'_{n+1}| \) is maximum over all \((T_n, D_n, \varphi_n)\)-stable colorings (see Definition 3.8). Clearly, \( T'_{n+1} \) is closed with respect to \( \sigma_n \). By Theorem 3.10(vi), \( \sigma_n \) is a \( \varphi_n \) mod \( T \) coloring. So \( T'_{n+1} \) is an ETT corresponding to \( (\sigma_n, T_n) \).

Suppose \( T'_{n+1} \) is not strongly closed. Then we can further grow \( T'_{n+2} \) to get a larger ETT \( T_{n+2} \) by using Algorithm 3.1: let \( \sigma_{n+1} \) be the corresponding coloring in \( C^k(G - e) \). By Definition 3.8 and Lemma 3.9, \( T'_{n+2} \) satisfies MP under \( \sigma_{n+1} \).

**Proof of Theorem 2.1.** Let \( T \) be an ETT constructed from a \( k \)-triple \((G, e, \varphi)\) satisfying MP and having the largest \( |T| \), and let \((\varphi_0, \varphi_1, \ldots, \varphi_n)\) be the coloring sequence of \( T \). If \( T \) is strongly closed with respect to \( \varphi_n \), then \( G \) is an elementary multigraph by Theorem 2.2(i) and
(iv). In the opposite case, by Lemma 3.11, we have \( V(T) = V(G) \). From Theorem 3.10(i) and Theorem 2.2(i) and (iii), we can also conclude that \( G \) is an elementary multigraph.

The proof of Theorem 3.10 will take up the entire remainder of this paper.

4 Auxiliary Results

We prove Theorem 3.10 by induction on the rung number \( r(T) = n \). The present section is devoted to a proof of statement (ii) in Theorem 3.10 in the base case and proofs of statements (iii)-(iv) in the general case.

For \( n = 0 \), statement (i) follows from Theorem 2.8, statements (iii)-(iv) hold trivially, and statement (ii) is a corollary of the following more general lemma.

**Lemma 4.1.** Let \((G, e, \varphi)\) be a k-triple, let \( T \) be a closed Tashkinov tree with respect to \( e \) and \( \varphi \), and let \( \alpha \) and \( \beta \) be two colors in \([k]\) with \( \varphi(T) \cap \{\alpha, \beta\} \neq \emptyset \). Then there is at most one \((\alpha, \beta)\)-path with respect to \( \varphi \) intersecting \( T \).

**Proof.** Assume the contrary: there are at least two \((\alpha, \beta)\)-paths \( Q_1 \) and \( Q_2 \) with respect to \( \varphi \) intersecting \( T \). By Theorem 2.8, \( V(T) \) is elementary with respect to \( \varphi \). By hypothesis, \( T \) is closed with respect to \( \varphi \). So \( |V(T)| \) is odd and precisely one of \( \alpha \) and \( \beta \), say \( \alpha \), is in \( \varphi(T) \). Thus, using \( Q_1 \) and \( Q_2 \), we deduce that \( G \) contains at least three \( (T, \varphi, \{\alpha, \beta\}) \)-exit paths \( P_1, P_2, P_3 \). We call the tuple \((\varphi, T, \alpha, \beta, P_1, P_2, P_3)\) a counterexample and use \( K \) to denote the set of all such counterexamples.

With a slight abuse of notation, let \((\varphi, T, \alpha, \beta, P_1, P_2, P_3)\) be a counterexample in \( K \) with the minimum \(|P_1| + |P_2| + |P_3|\). For \( i = 1, 2, 3 \), let \( a_i \) and \( b_i \) be the ends of \( P_i \) with \( b_i \in V(T) \), and \( f_i \) be the edge of \( P_i \) incident to \( b_i \). Renaming subscripts if necessary, we may assume that \( b_1 < b_2 < b_3 \). Let \( \gamma \in \varphi(b_3) \) and let \( \sigma_1 = \varphi/(G - T, \alpha, \gamma) \). Clearly, \( \sigma_1 \in C^k(G - e) \) and \( T \) is also a Tashkinov tree with respect to \( e \) and \( \sigma_1 \). Furthermore, \( f_i \) is colored by \( \beta \) under both \( \varphi \) and \( \sigma_1 \) for \( i = 1, 2, 3 \).

Consider \( \sigma_2 = \sigma_1/P_{b_3}(\gamma, \beta, \sigma_1) \). Note that \( \beta \in \sigma_2(b_3) \). Let \( T' \) be obtained from \( T(b_3) \) by adding \( f_1 \) and \( f_2 \) and let \( T'' \) be a closure of \( T' \) under \( \sigma_2 \). Obviously, both \( T' \) and \( T'' \) are Tashkinov trees with respect to \( e \) and \( \sigma_2 \). By Theorem 2.8, \( V(T'') \) is elementary with respect to \( \sigma_2 \).

Observe that none of \( a_1, a_2, a_3 \) is contained in \( T'' \), for otherwise, let \( a_i \in V(T'') \) for some \( i \) with \( 1 \leq i \leq 3 \). Since \( \{\beta, \gamma\} \cap \sigma_2(a_i) \neq \emptyset \) and \( \beta \in \sigma_2(b_3) \), we obtain \( \gamma \in \sigma_2(a_i) \). Hence from TAA we see that \( P_1, P_2, P_3 \) are all entirely contained in \( G[T''] \), which in turn implies \( \gamma \in \sigma_2(a_j) \) for \( j = 1, 2, 3 \). So \( V(T'') \) is not elementary with respect to \( \sigma_2 \), a contradiction. Each \( P_i \) contains a subpath \( Q_i \), which is a \( T_2 \)-exit path with respect to \( \sigma_2 \). Since \( f_1 \) is not contained in \( Q_1 \), we obtain \(|Q_1| + |Q_2| + |Q_3| < |P_1| + |P_2| + |P_3| \). Thus the existence of the counterexample \((\sigma_2, T'', \gamma, \beta, Q_1, Q_2, Q_3)\) violates the minimality assumption on \((\varphi, T, \alpha, \beta, P_1, P_2, P_3)\).

So Theorem 3.10 is true in the base case. Suppose we have established that

(4.1) Theorem 3.10 holds for all ETT’s with at most \( n - 1 \) rungs and satisfying MP, for some \( n \geq 1 \).

Let us proceed to the induction step. We postpone the proof of Theorem 3.10(i) and (ii) to Section 7, and present a proof of Theorem 3.10(iii)-(vi) in this section. In our proof of the \((i + 2)\)th statement in Theorem 3.10 for \( 2 \leq i \leq 4 \), we further assume that
(4.1) the jth statement in Theorem 3.10 holds for all ETTs with at most n rungs and satisfying MP, for all $j$ with $3 \leq j \leq i + 1$.

We break the proof of the induction step into a series of lemmas. The following lemma generalizes Lemma 3.2(ii), and will be used in the proofs of Theorem 3.10(iii) and (iv).

**Lemma 4.2.** Let $T$ be an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$. For any $s \leq n$ and any edge $f$ incident to $T_s$, if $\varphi_{s-1}(f) \in \varphi_{s-1}(T_s) \cup D_{s-1}$, then $\varphi_i(f) = \varphi_{s-1}(f)$ for any $t$ with $s \leq t \leq n$, unless $f = f_p \in F_p$ for some $p$ with $s \leq p \leq t$ and $\Theta_p = PE$. In particular, if $f$ is an edge in $G[T_s]$ with $\varphi_{s-1}(f) \in \varphi_{s-1}(T_s) \cup D_{s-1}$, then $\varphi_i(f) = \varphi_{s-1}(f)$ for any $t$ with $s \leq t \leq n$.

**Proof.** By Lemma 3.2(i), we have $\varphi_{i-1}(T_i) \cup D_{i-1} \subseteq \varphi_i(T_{i+1}) \cup D_i$ for all $i \geq 1$. So to establish the first half, it suffices to prove the statement for $t = s$, which is exactly the same as Lemma 3.2(ii).

Note that if $f$ is an edge in $G[T_s]$, then $f \notin \partial(T_p)$ for any $p$ with $s \leq p \leq t$. Hence $f \neq f_p \in F_p$ for any $p$ with $s \leq p \leq t$ and $\Theta_p = PE$. Thus the second half also holds. \qed

**Lemma 4.3.** (Assuming (4.1)) Theorem 3.10(iii) holds for all ETTs with $n$ rungs and satisfying MP; that is, for any $i \leq n$, if $v_i$ is a supporting vertex and $m(v_i) = j$, then every $(T_i, D_i, \varphi_i)$-stable coloring $\sigma_i$ is $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable. In particular, $\sigma_i$ is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. Furthermore, for any two distinct supporting vertices $v_i$ and $v_j$ with $i, j \leq n$, if $m(v_i) = m(v_j)$, then $S_i \cap S_j = \emptyset$.

**Proof.** By hypothesis, $T$ is an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$, and $T$ satisfies MP under $\varphi_n$.

We prove the first half by contradiction. Assume the contrary: there exists a subscript $i \leq n$, such that $v_i$ is a supporting vertex, $m(v_i) = j$, and some $(T_i, D_i, \varphi_i)$-stable coloring $\sigma_i$ is not $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable. By definition, there exists an edge $f$ incident to $T(v_i) - v_i$, with $\varphi_{j-1}(f) \in \varphi_{j-1}(T(v_i) - v_i) \cup D_{j-1}$, such that $\sigma_i(f) \neq \varphi_{j-1}(f)$, or there exists a vertex $v$ of $T(v_i) - v_i$ such that $\sigma_i(v) \neq \varphi_{j-1}(v)$. In the former case, since $j \leq i$, repeated application of Lemma 3.2(i) and (ii) yields $\varphi_{j-1}(T(v_i) - v_i) \cup D_{j-1} \subseteq \varphi_{j-1}(T_i) \cup D_i \subseteq \varphi_{j-1}(T_{i+1}) \cup D_{i+1}$ and $\varphi_i(f) = \varphi_i(f)$, which implies $\varphi_i(f) \neq \varphi_{j-1}(f)$. In the latter case, since $\sigma_i(v) = \varphi_i(v)$, we have $\varphi_i(v) \neq \varphi_{j-1}(v)$. From Lemma 2.3 we deduce that $\varphi_i$ is not $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable in either case.

Set $V^- = V'(T(v_i) - v_i)$. Then there exists an edge $f$ incident to $V^-$ with $\varphi_{j-1}(f) \in \varphi_{j-1}(V^-) \cup D_{j-1}$ such that $\varphi_{j-1}(f) \neq \varphi_i(f)$, or there exists a vertex $v$ in $V^-$ such that $\varphi_i(v) \neq \varphi_{j-1}(v)$. In either case, by Lemma 4.2 and Algorithm 3.1, there exists a supporting vertex $v_k \in V_i$ with $j \leq k < i$. Thus $j \leq i - 1$ and $v_k \prec v_i$.

Since $v_i \in V(T_j)$, we have $v_i \in V(T_{i-1})$. Let $\pi_{i-1}$ be the $(T_i, D_{i-1}, \varphi_{i-1})$-stable coloring as specified in Steps 1 and 4 of Algorithm 3.1. Recall that $\delta_i = \pi_{i-1}(f_i)$. Since $v_i$ is the maximum vertex in $I[\partial_{\delta_{i-1}}, \delta_i(T_i)]$, we see that $\delta_i$ is a defective color of $T_{i-1}$ with respect to $\pi_{i-1}$, and $v_i$ is not the smallest vertex in $I[\partial_{\delta_{i-1}}, \delta_i(T_{i-1})]$. As $\pi_{i-1}$ is $(T_{i-1}, D_{i-1}, \varphi_{i-1})$-stable, applying (4.1) and Theorem 3.10(v) to $v = v_i$ and $\pi_{i-1}$, we obtain $v_i \preceq v_k$; this contradiction establishes the first half of the assertion. Since $m(v_i) = j$, we have $v_i \notin V(T_{j-1})$. So $T_{j-1}$ is entirely contained in $T(v_i) - v_i$, and hence $\pi_i$ is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable.
To establish the second half, let \(v_i\) and \(v_j\) be two distinct supporting vertices with \(i < j \leq n\) and \(m(v_i) = m(v_j)\). We aim to show that \(S_i \cap S_j = \emptyset\).

For \(k = i, j\), let \(\pi_{k-1}\) be the \((T_k, D_{k-1}, \phi_{k-1})\)-stable coloring as specified in Steps 1 and 4 of Algorithm 3.1. Recall that \(\delta_k = \pi_{k-1}(f_k)\) is a defective color of \(T_k\) with respect to \(\pi_{k-1}\), and \(v_k\) is the maximum vertex in \(I[\partial_{\pi_{k-1}}, \delta_k(T_k)]\). Let \(r = m(v_i) = m(v_j)\). Since \(r \leq i < j\) and \(v_j \in V(T_j)\), we have \(v_j \in V(T_j)\). As \(\pi_{j-1}\) is also \((T_{j-1}, D_{j-1}, \phi_{j-1})\)-stable, applying Theorem 3.10(v) to \(\pi_{j-1}\), \(T_j\) and \(v = v_j\), we obtain \(v_j < v_j\). By definition, \(S_i = \{\delta_i, \gamma_i\}\). Observe that

\(1\) \(\gamma_i \notin S_j\). Indeed, since \(\gamma_i \in \mathcal{P}_{j-1}(v_i)\) and \(V(T_i)\) is elementary with respect to \(\phi_i\) by \((4.1)\) and Theorem 3.10(i), we have \(\gamma_i \notin \mathcal{P}_{j-1}(v_j)\). Let \(f\) be the edge incident to \(v_j\) with \(\phi_{j-1}(f) = \gamma_i\). Then \(f\) is an edge in \(G[T_i]\), because \(T_i\) is closed with respect to \(\phi_i\). By Lemma 4.2, we have \(\phi_{j-1}(f) = \phi_i(f) = \gamma_i\). So \(\gamma_i \notin \mathcal{P}_{j-1}(v_j)\) and \(f \notin \partial(T_{j-1})\). Let \(\pi'_{j-1}\) be as specified in Step 4 in Algorithm 3.1. Since \(\pi'_{j-1}\) is \((T_j, D_{j-1}, \phi_{j-1})\)-stable, we have \(\gamma_i \notin \mathcal{P}_{j-1}(v_j)\) and \(\phi_{j-1}(f) = \gamma_i\), which implies \(\gamma_i \notin S_j\).

\(2\) \(\delta_i \notin S_j\). To justify this, note that \(V(T_{i+1})\) is elementary with respect to \(\phi_i\) by \((4.1)\) and Theorem 3.10(i). Since \(\delta_i \in \mathcal{P}_i(v_i)\), we have \(\delta_i \notin \mathcal{P}_i(v_j)\). Let \(f\) be the edge incident to \(v_j\) with \(\phi_i(f) = \delta_i\). Since \(T_{i+1}\) is closed with respect to \(\phi_i\), edge \(f\) is contained in \(G[T_{i+1}]\). Since \(j < i\) and \(\phi_i(f) \in \mathcal{P}_i(T_i) \cup D_i\), we have \(\phi_{j-1}(f) = \phi_i(f) = \delta_i\) and \(f \notin \partial(T_{j-1})\) by Lemma 4.2. Let \(\pi'_{j-1}\) be as specified in Step 4 of Algorithm 3.1. Then \(\pi'_{j-1}\) is \((T_j, D_{j-1}, \phi_{j-1})\)-stable. By definition, \(\delta_i \notin \mathcal{P}_{j-1}(v_j)\) and \(\phi_{j-1}(f) = \delta_i\). Hence \(\delta_i \notin S_j\).

Combining \((1)\) and \((2)\), we conclude that \(S_i \cap S_j = \emptyset\), as desired.

The following lemma asserts that parallel extensions used in Algorithm 3.1 are preserved under taking certain stable colorings.

**Lemma 4.4.** *(Assuming \((4.1)\) and \((4.2)\))* Theorem 3.10(iv) holds for all ETTs with \(n\) runs and satisfying MP; that is, if \(\Theta_n = PE\), then \(P_{\Theta_n}(\gamma_n, \delta_n, \sigma_n)\) contains precisely one vertex, \(v_n\), from \(T_n\) for any \((T_n, D_n, \phi_n)\)-stable coloring \(\sigma_n\).

**Proof.** Assume the contrary: \(P_{\Theta_n}(\gamma_n, \delta_n, \sigma_n)\) contains at least two vertices from \(T_n\) for some \((T_n, D_n, \phi_n)\)-stable coloring \(\sigma_n\). Let \(j = m(v_n)\). We aim to find an ETT \(T'\) under a certain generating coloring \(\rho \in C^k(G - e)\), with \(r(T') = j - 1\) and \(T_{j-1} \subseteq T'\), such that either \(|V(T')| > |V(T_j)|\) or \(V(T')\) is not elementary with respect to \(\rho\), which contradicts either the maximum property enjoyed by \(T\) or the induction hypothesis on Theorem 3.10(i).

In our proof we shall repeatedly use the following statement.

\(1\) Let \(\sigma\) be a \((T_j(v_n) - v_n, D_{j-1}, \phi_{j-1})\)-stable coloring. Then \(\sigma(f) = \phi_{j-1}(f)\) for any edge \(f\) on \(T_j(v_n)\).

To justify this, note that each edge \(f\) on \(T_j(v_n)\) is contained in \(T_j\). By Lemma 3.2(iii), we have \(\phi_{j-1}(f) \in \mathcal{P}_{j-1}(T_j) \cup D_{j-1}\). From TAA we further deduce that \(\phi_{j-1}(f) \in \mathcal{P}_{j-1}(T_j(v_n) - v_n) \cup D_{j-1}\). Since \(\sigma\) is \((T_j(v_n) - v_n, D_{j-1}, \phi_{j-1})\)-stable, we obtain \(\sigma(f) = \phi_{j-1}(f)\). So \((1)\) holds.

Let \(L\) denote the set of all subscripts \(i\) with \(i \geq j\), such that \(\Theta_i = PE\) and \(m(v_i) = j\), where \(v_i\) is the supporting vertex involved in iteration \(i\). We partition \(L\) into disjoint subsets \(L_1, L_2, \ldots, L_k\), such that two subscripts \(s, t \in L\) are in the same subset if and only if \(v_s = v_t\). For \(1 \leq i \leq k\), write \(L_i = \{i_1, i_2, \ldots, i_{k(i)}\}\), where \(i_1 < i_2 < \ldots < i_{k(i)}\), and let \(w_i\) denote the common supporting vertex corresponding to \(L_i\). Renaming subscripts if necessary, we may
assume that \( w_1 \prec w_2 \prec \ldots \prec w_\kappa \). For each \( L_i \), we define \( P_i \) to be the graph with \( V(P_i) = \cup_{t \in L_i} S_t = \cup_{t \in L_i} \{ \delta_t, \gamma_t \} \) and \( E(P_i) = \{ \delta_t \gamma_t : t \in L_i \} \). By Lemma 3.3(ii), we have

\[
(2) \quad \gamma_{i_2} = \delta_{i_1}, \gamma_{i_3} = \delta_{i_2}, \ldots, \gamma_{i_c(i)} = \delta_{i_{c(i)}-1}. \quad \text{So } P_i \text{ is the walk: } \gamma_{i_1} \to \delta_{i_1} = \gamma_{i_2} \to \delta_{i_2} = \gamma_{i_3} \to \cdots \to \delta_{i_{c(i)}-1} = \gamma_{i_{c(i)}} \to \delta_{i_{c(i)}} \text{ where } \gamma_{i_1} \in \mathcal{P}_{i_1-1}(w_i) \text{ and } \delta_{i_{c(i)}} \in \mathcal{P}_{i_{c(i)}}(w_i).
\]

(3) \( P_1, P_2, \ldots, P_n \) are pairwise vertex-disjoint paths. In particular, for any \( 1 \leq i \leq \kappa \) and any \( 1 \leq s < t \leq c(i) \), we have \( \gamma_{i_s} \neq \delta_{i_t} \).

To justify this, note that \( S_p \cap S_q = \emptyset \) whenever \( p \) and \( q \) are contained in different \( L_i \)'s by Theorem 3.10(iii). So \( P_1, P_2, \ldots, P_n \) are pairwise vertex-disjoint. It remains to prove that each \( P_i \) is a path.

Assume on the contrary that \( P_i \) contains a cycle. Then \( \gamma_{i_s} = \delta_{i_t} \) for some subscripts \( s \) and \( t \) with \( s < t \) by (2). Let \( v \in V(T) \) be an arbitrary vertex with \( v \prec w_i \). Since \( \gamma_{i_s} \in \mathcal{P}_{i_s-1}(w_i) \), we have \( \gamma_{i_s} \notin \mathcal{P}_{i_t-1}(v) \) by (4.1) and Theorem 3.10(i). Let \( f \) be the edge incident with \( v \) with \( \varphi_{i_s-1}(f) = \gamma_{i_t} \). Since \( T_{i_s} \) is closed with respect to \( \varphi_{i_s-1} \), edge \( f \) is contained \( G[T_{i_s}] \). By Lemma 4.2, we have \( \varphi_{i_s-1}(f) = \varphi_{i_t-1}(f) \). From the definitions of \( \pi_{i_s-1} \) and \( \pi'_{i_t-1} \) in Step 4 of Algorithm 3.1, it follows that \( f \notin [\partial \varphi_{i_s-1}-\gamma_{i_t}(T_{i_s})] = [\partial \varphi_{i_t-1}-\delta_{i_s}(T_{i_s})] \). Therefore \( w_i \) cannot be the supporting vertex of \( T_{i_s} \) with respect to \( \varphi_{i_s} \); this contradiction proves (3).

(4) \( v_0 = w_1. \)

Assume the contrary: \( v_n \neq w_1. \) Then \( w_1 \prec v_n. \) By (3), \( P_i \) is a path and \( \gamma_{i_1} \neq \delta_{i_{c(i)}} \). Thus, from Lemma 3.3(iii) and Algorithm 3.1, we deduce that \( \mathcal{P}_{j-1}(w_1) = \mathcal{P}_{j-1}(w_1) = (\mathcal{P}_{i_{c(i)}}(w_1) - \{ \delta_{i_{c(i)}} \}) \cup \{ \gamma_{i_1} \} \neq \mathcal{P}_{i_{c(i)}}(w_1) = \mathcal{P}_{i_{c(i)}}(w_1). \) On the other hand, by Theorem 3.10(iii), coloring \( \varphi_n \) is \( (T(v_n) - v_n, D_{j-1}, \varphi_{j-1}) \)-stable, which implies \( \mathcal{P}_{j-1}(w_1) = \mathcal{P}_{i_{c(i)}}(w_1) \), a contradiction.

(5) Let \( v_s \) be a supporting vertex with \( s < n \) and let \( \sigma_s \) be a \( (T_s, D_s, \varphi_s) \)-stable coloring. Then \( P_v(\delta_s, \gamma_s, \sigma_s) \) is a cycle for any \( v \in V(T_s - v_s). \)

To justify this, assume the contrary: \( Q = P_v(\delta_s, \gamma_s, \sigma_s) \) is a path for some vertex \( v \in V(T_s - v_s). \) By (4.1) and Theorem 3.10(iv), we have \( P_v(\delta_s, \gamma_s, \sigma_s) \cap T_s = \{ v_s \} \), so \( Q \) and \( P_v(\delta_s, \gamma_s, \sigma_s) \) are vertex-disjoint. Define \( \sigma_{s-1} = \sigma_s / P_v(\delta_s, \gamma_s, \sigma_s) \). Then \( P_v(\delta_s, \gamma_s, \sigma_{s-1}) = Q \) and \( P_v(\delta_s, \gamma_s, \sigma_{s-1}) = P_v(\delta_s, \gamma_s, \sigma_s) \). By Lemma 3.5, \( \sigma_{s-1} \) is \( (T_s, D_{s-1}, \varphi_{s-1}) \)-stable. By (4.1) and Theorem 3.10(ii), there is at most one \( \{ \delta_s, \gamma_s \} \)-path with respect to \( \sigma_{s-1} \). Intersecting \( T_s \), contradicting the existence of both \( P_v(\delta_s, \gamma_s, \sigma_{s-1}) \) and \( Q \). So (5) holds.

(6) Let \( v_s \) be a supporting vertex with \( s < n \), and let \( t > s \) be the smallest subscript with \( \Theta_t \neq RE \). Then, for any \( v \in V(T_t - v_s) \), the chain \( P_v(\delta_s, \gamma_s, \varphi_t-1) \) is a cycle entirely contained in \( G[T_t] \). Furthermore, \( \varphi_{t-1}(f) = \varphi_t(f) = \ldots = \varphi_n(f) = \sigma_n(f) \) for any edge \( f \) in \( P_v(\delta_s, \gamma_s, \varphi_{t-1}) \), so \( P_v(\delta_s, \gamma_s, \sigma_n) = P_v(\delta_s, \gamma_s, \varphi_t-1). \)

To justify this, note that \( \varphi_{t-1} = \varphi_{t-2} = \ldots = \varphi_s \) by Algorithm 3.1, because \( \Theta_{s+1} = \Theta_{s+2} = \ldots = \Theta_{t-1} = RE \), if any. In view of (5), \( C = P_v(\delta_s, \gamma_s, \varphi_t-1) \) is a cycle for all \( v \in V(T_t - v_s) \). If \( C \) is not fully contained in \( G[T_t] \), then \( C \) contains a segment \( L \) and an edge \( \varphi_{t-1}^{-1} = \partial \varphi_{t-1}(T_t) \) which have all the properties as described in Steps 1 and 2 of Algorithm 3.1 (with \( f_t \) in place of \( f_t \)). Thus we can further grow \( T_t \) by TAA using RE, contradicting the hypothesis that \( \Theta_t \neq RE \), because RE has priority over both SE and PE. This proves the first half of (6).

Let \( f \) be an edge contained in \( P_v(\delta_s, \gamma_s, \varphi_{t-1}) \). Then \( f \) is an edge in \( G[T_t] \). Since \( \varphi_{t-1}(f) \in \{ \delta_s, \gamma_s \} = S_{t-1} \subseteq \mathcal{P}_{i_{t-1}}(T_t) \cup D_{t-1} \), we have \( \varphi_{t-1}(f) = \varphi_t(f) = \ldots = \varphi_n(f) \) by Lemma 4.2. From the definition of \( D_n \), we see that \( \{ \delta_s, \gamma_s \} \subseteq \mathcal{P}_n(T_n) \cup D_n \). So \( \varphi_n(f) \in \mathcal{P}_n(T_n) \cup D_n \). Since \( \sigma_n \) is a \( (T_n, D_n, \varphi_n) \)-stable coloring, the equality \( \varphi_n(f) = \sigma_n(f) \) holds. Hence (6) is established.
Claim 4.1. There exists a coloring $\varphi_1 \in \mathcal{C}^k(G - c)$ with the following properties:

(a1) $\varphi_1$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;

(a2) $\overline{\varphi}(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$, $\overline{\varphi}(v_n) = \overline{\varphi}_n(v_n)$, and $\varphi_1(f) = \varphi_{j-1}(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$;

(a3) $\varphi_1(f) = \sigma_n(f)$ for all $f \in E(G)$ with $\sigma_n(f) \in \bigcup_{i \in L_1} S_i$; and

(a4) for any $i \in L_1 - \{1\}$ so $v_i = v_n$ and any $(\delta_i, \gamma_i)$-cycle $Q$ with respect to $\varphi_n$ intersecting $T_i - v_i$, we have $\varphi_1(f) = \varphi_n(f)$ for all $f \in E(Q)$.

The proof idea of this claim is very simple: We have proved in (2) and (3) that each $P_i$ is a path. What we are going to do is to perform a sequence of Kempe changes starting from $\sigma_n$, so that the direction of each $P_i$, for $i = \kappa, \kappa - 1, \ldots, 2$, is reversed (see (8) below). The resulting coloring is the desired $\varphi_1$. Despite this, to ensure correctness we have to construct $\varphi_1$ in a recursive way.

To justify this claim, note that, by (6), every $(\delta_i, \gamma_i)$-cycle $Q$ with respect to $\varphi_n$ intersecting $T_i - v_i$ is contained in $G[T_n]$ for all $i \in L_1 - \{n\}$ and $\varphi_n(f) = \sigma_n(f)$ for any edge $f \in E(Q)$. So (a3) implies (a4); we state (a4) here just for ease of reference. Hence we only need to find a coloring $\varphi_1 \in \mathcal{C}^k(G - c)$ with properties (a1) – (a3). Moreover, by Lemma 4.2, we have

(7) $\varphi_n(f) = \varphi_{j-1}(f)$ for all $f \in T_j - T_j(v_n)$. So the condition $\varphi_1(f) = \varphi_{j-1}(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$ in (a2) is equivalent to saying that $\varphi_1(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$.

To find the desired coloring $\varphi_1$, we define a linear order $\prec_v$ on $\cup_{2 \leq g \leq \kappa} L_g \cup \{1_{c(1)}\}$, such that $i_t \prec_v h_x$ if $h < i$ or $h = i$ but $s < t$. The following is the ordering of all subscripts in $\cup_{2 \leq g \leq \kappa} L_g \cup \{1_{c(1)}\}$ according to $\prec_v$:

$$
\begin{array}{ccccccccc}
\prec_v & \prec_v & \prec_v & \prec_v & \prec_v & \prec_v & \prec_v & \prec_v & \prec_v \\
\kappa(\kappa) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1) & \kappa(\kappa-1)
\end{array}
$$

For simplicity, we write $P_{v_i}(\delta_i, \gamma_i, \cdots)$ for the chain $P_{v_i}(\delta_i, \gamma_i, \cdots)$. Let us recursively define a sequence of colorings along the above ordering as follows:

(8) $\mu_{\kappa(\kappa)} = \sigma_n$ and $\mu_{h_s} = P_{v_t}(\delta_i, \gamma_i, \mu_{i_t})$ for all $h_s \in \cup_{2 \leq g \leq \kappa} L_g \cup \{1_{c(1)}\} - \{\kappa(\kappa)\}$, where $i_t$ is the largest subscript in $\cup_{2 \leq g \leq \kappa} L_g \cup \{1_{c(1)}\}$ smaller than $h_s$ in the order $\prec_v$. For example, if $h_s = \kappa(\kappa+1)$, then $i_t = \kappa(\kappa)$. If $h_s = 2_{c(2)}$, then $i_t = 3_1$.

For the subscript $i_t$, set $U_i = \{w_1, w_2, \ldots, w_i\}$. Recall that $w_i$ is the common supporting vertex corresponding to $L_i$, so $w_i = v_i$ for each $i$ and $w_1 = v_n = v_{c(1)}$ by (4).

(9) Each coloring $\mu_{h_s}$, with $h_s \in \cup_{2 \leq g \leq \kappa} L_g \cup \{1_{c(1)}\}$, satisfies the following properties:

(b1) $\mu_{h_s}$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;

(b2) $\overline{\mu}_{h_s}(v) = \overline{\mu}_{j-1}(v)$ for all $v \in V(T_j) - U_i$, $\overline{\mu}_{h_s}(v) = \overline{\mu}_n(v)$ for $v \in U_i - \{v_i\}$, $\overline{\mu}_{h_s}(v_n) = \overline{\mu}_s(v_n)$ if $i = h$ and $\overline{\mu}_{h_s}(v_i) = \overline{\mu}_{j-1}(v_i)$ if $i = h + 1$, and $\mu_{h_s}(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$, where $U_i = U_n$ and $\{v_i\} = \emptyset$ if $h_s = \kappa(\kappa)$;

(b3) $\mu_{h_s}(f) = \sigma_n(f)$ for all $f \in E(G)$ with $\sigma_n(f) \in \cup_{2 \leq g \leq \kappa} \Omega_n$, where $\Omega_n = \cup_{2 \leq g \leq \kappa} L_g$ if $s = c(h)$ and $\Omega_n = \cup_{1 \leq g \leq h-1} L_g \cup \{h_1, h_2, \ldots, h_{s-1}\}$ otherwise; and

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(b4) \( \mu_{h_{s}}(f) = \varphi_{n}(f) \) for all \( f \in E(Q) \), where \( Q \) is an arbitrary \((\delta_{h_{s}}, \gamma_{h_{s}})\)-cycle with respect to \( \varphi_{n} \) intersecting \( T(v_{h_{s}}) - v_{h_{s}} \), provided that \( h_{s} \preceq v_{1}(e_{1}) \).

Let us prove (9) by induction on the linear order \( \prec_{v} \), starting from \( \mu_{e_{c}(e)} = \sigma_{n} \) (see (8)).

By Theorem 3.10(iii), coloring \( \sigma_{n} \) is \((T_{j}(v_{n}) - v_{n}, D_{j-1}, \varphi_{j-1})\)-stable. So (b1) holds for \( \sigma_{n} \).

For each \( v \in V(T_{j}) - U_{n} \), note that \( v \) is not a supporting vertex during any iteration \( p \) with \( j \leq p \leq n \). Hence \( \varphi_{n}(v) = \varphi_{j-1}(v) \) by Algorithm 3.1 and the definition of stable colorings. Since \( \sigma_{n} \) is a \((T_{n}, D_{n}, \varphi_{n})\)-stable coloring, we have \( \sigma_{n}(f) = \varphi_{n}(f) \) for any edge \( f \) on \( T_{n} \) by Lemma 3.2(iii) and \( \sigma_{n}(u) = \varphi_{n}(u) \) for each vertex \( u \) of \( T_{n} \). Thus (b2) is true for \( \sigma_{n} \). Trivially, (b3) holds for \( \sigma_{n} \). To prove (b4) for \( \sigma_{n} \) when \( e_{c}(e(k)) \prec_{v} 1(e_{1}) = n \) (equivalently, \( \kappa \geq 2 \)), note that \( \Theta_{n} = PE \neq RE \). So statement (b4) follows instantly from (6) (with \( e_{c}(e) \) in place of \( s \)). Therefore (9) is established in the base case.

Suppose we have established \((b1) - (b4)\) for \( \mu_{i_{t}} \). Let us proceed to the induction step for \( \mu_{h_{s}} \).

As \( n > i_{t} \) and \( \Theta_{n} = PE \), by (6) (with \( i_{t} \) in place of \( s \)), the vertices in \( T_{i_{t}} - v_{i_{t}} \) are all contained in \((\delta_{i_{t}}, \gamma_{i_{t}})\)-cycles with respect to both \( \varphi_{n} \) and \( \sigma_{n} \), which are also \((\delta_{i_{t}}, \gamma_{i_{t}})\)-cycles with respect to \( \mu_{i_{t}} \) by (b4) for \( \mu_{i_{t}} \). It follows that

\[
(10) \quad P_{i_{t}}(\delta_{i_{t}}, \gamma_{i_{t}}, \mu_{i_{t}}) \cap T_{i_{t}} = \{v_{i_{t}}\}.
\]

Since (b1) holds for \( \mu_{i_{t}} \), it also holds for \( \mu_{h_{s}} \) by (8) and (10). Let us prove (b2) for \( \mu_{h_{s}} \).

Now the nontrivial part is to verify that \( P_{h_{s}}(v_{i_{t}}) = \varphi_{h_{s}}(v_{i_{t}}) \) if \( i = h \) and \( P_{h_{s}}(v_{i_{t}}) = \varphi_{j-1}(v_{i_{t}}) \) if \( i = h + 1 \). For this purpose, let \( p_{k} \) be the largest subscript in \( \cup_{2 \leq g \leq k} E_{g} \cup \{e_{c}(1)\} \) smaller than \( i_{t} \) in the order \( \prec_{v} \). Then there are two possible values for \( p_{k} \): \( i_{t} + 1 \) and \( (i_{t} + 1) \). If \( p_{k} = i_{t} + 1 \), then \( v_{i_{t}} = v_{i_{t}+1} = v_{i_{t}} \). By induction hypothesis for \( \mu_{i_{t}} \), we have \( P_{i_{t}}(w_{i}) = \varphi_{i_{t}}(w_{i}) \). If \( p_{k} = (i_{t} + 1) \), then \( v_{i_{t}} = v_{i_{t}} \) and \( v_{i_{t}+1} = v_{i_{t}+1} \). By induction hypothesis for \( \mu_{i_{t}} \), we have \( P_{i_{t}}(w_{i}) = \varphi_{i_{t}}(w_{i}) = \varphi_{h_{s}}(w_{i}) \), where the last equality holds because \( w_{i} \) does not serve as a supporting vertex in any iteration \( m \) with \( i_{t} + 1 \leq m \leq n \) (see Algorithm 3.1). So the equality \( P_{h_{s}}(w_{i}) = \varphi_{h_{s}}(w_{i}) \) holds in either case. According to Algorithm 3.1, \( \delta_{i_{t}} \in \varphi_{i_{t}}(w_{i}) \) and \( \gamma_{i_{t}} \notin \varphi_{i_{t}}(w_{i}) \). It follows from (8) that \( P_{h_{s}}(w_{i}) = (P_{i_{t}}(w_{i}) - \{\delta_{i_{t}}\}) \cup \{\gamma_{i_{t}}\} = (\varphi_{i_{t}}(w_{i}) - \{\delta_{i_{t}}\}) \cup \{\gamma_{i_{t}}\} \).

If \( i = h \), then \( (\varphi_{i_{t}}(w_{i}) - \{\delta_{i_{t}}\}) \cup \{\gamma_{i_{t}}\} = \varphi_{h_{s}}(w_{i}) \), where the last equality holds because \( w_{i} \) does not serve as a supporting vertex in any iteration \( m \) with \( h_{s} + 1 \leq m \leq i_{t} - 1 \) (see Algorithm 3.1). If \( i = h + 1 \), then \( P_{h_{s}}(w_{i}) = (\varphi_{i_{t}}(w_{i}) - \{\delta_{i_{t}}\}) \cup \{\gamma_{i_{t}}\} = \varphi_{i_{t}+1}(w_{i}) = \varphi_{j-1}(w_{i}) \), because \( w_{i} \) does not serve as a supporting vertex in any iteration \( m \) with \( j \leq m \leq i_{t} - 1 \). Hence \( P_{h_{s}}(v_{i_{t}}) \) is exactly as described in (b2).

Using (10) and the induction hypothesis on (b2) for \( \mu_{i_{t}} \), it is routine to check that the remaining statements in (b2) for \( \mu_{h_{s}} \) are true. Since \( \Omega_{h_{s}} \subseteq \Omega_{i_{t}} \) and \( (\cup_{g \in \Omega_{h_{s}}} S_{g}) \cap S_{i_{t}} = \emptyset \) by (2), (3) and Theorem 3.10(iii), statement (b3) for \( \mu_{h_{s}} \) follows instantly from that for \( \mu_{i_{t}} \).

It remains to verify (b4) for \( \mu_{h_{s}} \). Let \( Q \) be as specified in (b4). By (6) (with \( h_{s} \) in place of \( s \)) and (b3) for \( \varphi_{i_{t}} \), we have \( \varphi_{i_{t}}(f) = \sigma_{n}(f) = \mu_{i_{t}}(f) \) for all edges \( f \) on \( Q \). If \( i \neq h \), then \( S_{h_{s}} \subseteq S_{i_{t}} = \emptyset \) by Theorem 3.10(iii). It follows that \( Q \) and \( P_{i_{t}}(\delta_{i_{t}}, \gamma_{i_{t}}, \mu_{i_{t}}) \) are edge-disjoint under \( \mu_{i_{t}} \). Hence \( \mu_{h_{s}}(f) = \mu_{i_{t}}(f) = \varphi_{n}(f) \) for all \( f \in E(Q) \) by (8) and (10). So we assume that \( i = h \). Then \( v_{i_{t}} = v_{h_{s}} \). It follows from (6) that \( Q \) is fully contained in \( G[T_{i_{t}}] \). From (8) and (10) we deduce that \( \mu_{h_{s}}(f) = \mu_{i_{t}}(f) \) for all \( f \in E(Q) \). So (b4) and therefore (9) is established.

Finally, define \( q_{1} = \mu_{e_{c}(e)} \). From (9) for \( \mu_{e_{c}(e)} \) and (7), we see that \( q_{1} \) satisfies all the properties (a1) – (a3). This proves Claim 4.1 (recall the remark above (7)).

Consider the coloring \( q_{1} \in C^{k}(G - e) \) described in Claim 4.1. Let \( T'_{j} \) be a closure of \( T_{j}(v_{n}) \)
under $\varrho_1$. By (4.1) and Theorem 3.10(i), $V(T_n)$ is elementary with respect to $\varphi_{n-1}$, so $|V(T_n)|$ is odd. From Step 4 in Algorithm 3.1, we see that $|\partial_{\varphi_{n-1},\delta_n}(T_n)| \geq 3$. Hence $|\partial_{\varphi_n,\delta_n}(T_n)| \geq 2$. Since $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring, we have $|\partial_{\varphi_n,\delta_n}(T_n)| \geq 2$. By Lemma 3.2(iv), edges in $\partial_{\sigma_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. Furthermore, each color in $\varphi_n(T_n) - \{\delta_n\}$ is closed in $T_n$ under $\sigma_n$. It follows from (a3) and TAA that $T'_j - T_n \not= \emptyset$ and at least one edge in $T'_j - T_n$ is colored by $\delta_n$ under $\varrho_1$. By (a1), $\varrho_1$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, so it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1}$ mod $T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi). By (1), we have $\varrho_1(f) = \varphi_{j-1}(f)$ for any edge $f$ on $T_j(v_n)$. Note that $T_j(v_n)$ under $\varrho_1$ is obtained from $T_{j-1}$ by using the same connecting edge, connecting color, and extension type as $T_j$. By (4.1) and Theorem 3.10(vi), we obtain

(11) $T'_j$ is an ETT under coloring $\varrho_1$ and satisfies MP. So $V(T'_j)$ is elementary with respect to $\varrho_1$ by (4.1) and Theorem 3.10(i).

Depending on the intersection of $\varphi_1(T'_j - V(T_j(v_n)))$ and $\bigcup_{i \in L_1} S_i$, we consider two cases.

**Case 1.** $\varphi_1(T'_j - V(T_j(v_n))) \cap (V(T_j(v_n))) \not= \emptyset$.

Let $u$ be the minimum vertex (in the order $<\infty$) in $T'_j - V(T_j(v_n))$ such that $\varphi_1(u) \cap (V(T_j(v_n))) \not= \emptyset$. Clearly, $u \not= v_n$. By (11), $V(T'_j)$ is elementary with respect to $\varrho_1$. Since $\delta_n \in \varphi_n(v_n) = \varphi_1(v_n)$ by (a2), we have $\delta_n \not\in \varphi_1(T'_j - v_n)$; in particular, $\delta_n \not\in \varphi_1(u)$. Recall that $L_1 = \{1, 2, \ldots, 1_{c(1)}\}$ and that $n = 1_{c(1)}$. Since $\delta_1 = \gamma_{1_{c+1}}$ for any $1_{c+1} \in L_1$ with $1_{c+1} < n$ (see (2)), there exists a minimum member $r$ (as an integer) of $L_1$, such that $\gamma_r \in \varphi_1(u)$. Since $m(v_r) = j$, we have $r \geq j$.

Observe that

(12) $u \in V(T'_j) - V(T_j)$.

Otherwise, $u \in V(T_j)$. Since $\gamma_r \in \varphi_1(u)$, it follows from (a3) that $\gamma_r \in \varphi_n(u)$. Since $\sigma_n$ is $(T_n, D_n, \varphi_n)$-stable, $\gamma_r \in \varphi_n(u)$. On the other hand, by (4.1) and Theorem 3.10(i), $V(T_j)$ is elementary with respect to $\varphi_{j-1}$. From Step 4 in Algorithm 4.1, we see that $\varphi_r \in \varphi_{r-1}(v_n)$ as $v_r = v_n$, so $G[T_r]$ contains an edge $f$ incident to $u$ with $\varphi_{r-1}(f) = \gamma_r$. By Lemma 4.2, we obtain $\varphi_n(f) = \gamma_r$. Hence $\gamma_r \in \varphi_n(u)$; this contradiction justifies (12).

**Claim 4.2.** There exists a coloring $\varrho_2 \in C^k(G - e)$ with the following properties:

(c1) $\varrho_2$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;

(c2) $\varrho_2(v) = \varphi_1(v)$ for all $v \in V(T_j \cup T'_j(u) - u)$ and $\varrho_2(f) = \varphi_1(f)$ for all $f \in E(T_j \cup T'_j(u))$;

(c3) $\gamma_1 \in \varphi_2(u)$.

**(Assuming Claim 4.2)** By (c1) in Claim 4.2, $\varrho_2$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, so it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1}$ mod $T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi). By (1), we have $\varrho_2(f) = \varphi_{j-1}(f)$ for each edge $f$ on $T_j(v_n)$. Let $T'_j$ be a closure of $T_j(v_n)$ under $\varrho_2$. Then $T'_j$ is an ETT under the generating coloring $\varrho_2$, because it is obtained from $T_{j-1}$ by using the same connecting edge, connecting color, and extension type as $T_j$. It follows from Theorem 3.10(vi) that $T'_j$ satisfies MP. In view of (a2) and (c2), we have

- $\varrho_2(v) = \varphi_{j-1}(v)$ for all $v \in V(T_j - v_n)$;
- $\varrho_2(f) = \varphi_{j-1}(f)$ for all $f \in T_j - T_j(v_n)$;
- $\varrho_2(v) = \varphi_1(v)$ for all $v \in V(T'_j(u) - u)$;
- $\varrho_2(f) = \varphi_1(f)$ for all $f \in E(T'_j(u))$; and
- $\varrho_2(v_n) = \varphi_n(v_n)$.
Using (c3) and Lemma 3.3(iii), we obtain $\gamma_1 \in \overline{\varphi}_2(u)$ and $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1-1}(v_n) \subseteq \overline{\varphi}_{1+(i)}(v_n) \cup \{\gamma_1\} = \overline{\varphi}_n(v_n) \cup \{\gamma_1\} = \overline{\varphi}_2(v_n) \cup \{\gamma_1\}$. Thus from TAA it is clear that $V(T_j \cup T_j'(u)) \subseteq V(T_j')$, which contradicts the maximum property satisfied by $T$ because $u \notin V(T_j)$.

To justify Claim 4.2, let $r$ be the subscript as defined above (12). Then $r = 1_p$ for some $1 \leq p \leq c(1)$. By (2), we have $\gamma_r = \delta_{1-p-1}$ if $p > 1$. Let us recursively define a sequence of colorings along the linear order $<_u$ as follows:

(13) $\mu_1 = \varrho_1$ and $\mu_i = \mu_{i+1+1}/p_n(\delta_1, \gamma_1, \mu_{i+1})$ for $i = p = 1, p-2, \ldots, 1$. We aim to show that $\varrho_2 = \mu_1$ is as desired. Once again our proof proceeds in a recursive manner.

(14) Each coloring $\mu_{i-1}$, for $i = p, p - 1, \ldots, 1$, satisfies the following properties:

(d1) $\mu_{i-1}$ is $(T_j(v_n) - v_n, D_j, \varphi_{j-1})$-stable;

(d2) $\overline{\varphi}_1(v) = \overline{\varphi}_1(v)$ for all $v \in V(T_j \cup T_j'(u) - u)$ and $\mu_i(f) = \varrho_i(f)$ for all $f \in E(T_j \cup T_j'(u))$;

(d3) $\gamma_1 \in \overline{\varphi}_1(v)$;

(d4) $\mu_i(f) = \varphi_n(f)$ for all $f \in E(Q)$, where $Q$ is an arbitrary $(\delta_1, \gamma_1)$-cycle with respect to $\varphi_n$, intersecting $T_{1-i} - v_1$, with $1 \leq i \leq 1 - 1$ (recall that $v_1 = v_n$); and

(d5) $\overline{\varphi}_1(T_j'(u) - u) \cap (\cup_{1 \leq \ell \leq p-1} S_{i+1}) = \emptyset$ and $\mu_i(T_j'(u) - u) \cup (\cup_{1 \leq \ell \leq p-1} S_{i+1}) = \emptyset$.

We prove (14) by induction on the order $<_u$, starting from $\mu_p = \varrho_1$ (see (13)). For $\mu_p = \varrho_1$, properties (d2) and (d3) hold trivially, and (d1) and (d4) follow from (a1) and (a4), respectively. It remains to justify (d5). Set $S = \cup_{1 \leq \ell \leq p-1} S_{i+1}$. Let us first show that

(15) $\overline{\varphi}_1(T_j'(u) - u) \cap S = \emptyset$.

From the minimality assumption on $u$, we see that $S$ and $\overline{\varphi}_1(T_j'(u) - u)$ are disjoint. Assume on the contrary that some color $\gamma_1$ is contained in both $S$ and $\overline{\varphi}_1(T_j'(v_n))$. Let $v$ be the vertex in $T_j'(v_n) - v_n$ with $\gamma_1 \in \overline{\varphi}_1(v)$. Then $\gamma_1 \in \overline{\varphi}_1(v)$ by (a2). By Theorem 3.10(iii), coloring $\varphi_n$ is $(T_j(v_n) - v_n, D_j, \varphi_{j-1})$-stable. So $\gamma_1 \in \overline{\varphi}_n(v)$. On the other hand, since $\gamma_1 \in \overline{\varphi}_{1-1}(v_n)$ and $V(T_{1-i})$ is elementary with respect to $\varphi_{1-i}$ by (4.1) and Theorem 3.10(i), $G[T_{1-i}]$ contains an edge $e$ incident to $v$ with $\varphi_{1-i-1}(f) = \gamma_1$. It follows from Lemma 2.4 that $\varphi_n(f) = \gamma_1$, so $\gamma_1 \in \varphi_n(v)$, a contradiction. Hence $\overline{\varphi}_1(T_j'(v_n)) \cap S = \emptyset$. By (3) and Lemma 3.3(i), we have $\overline{\varphi}_n(v_n) \cap (\cup_{1 \leq \ell \leq p} S_1) = \{ \delta_n \}$. Since $\delta_n \notin S$ by (3), it follows that $\overline{\varphi}_n(v_n) \cap S = \emptyset$.

By (a2), we obtain $\overline{\varphi}_1(v_n) \cap S = \emptyset$. So $\overline{\varphi}_1(T_j'(v_n)) \cap S = \emptyset$. Therefore (15) is true.

Since $T_j'$ is a closure of $T_j'(v_n)$ under $\varrho_1$, from (15) we see immediately that no color in $S$ is used by any edge in $T_j'(u) - T_j'(v_n)$ under $\varrho_1$. So $\varrho_1(T_j'(u) - T_j'(v_n)) \cap S = \emptyset$, and hence (d5) holds for $\varrho_1$. This proves (14) in the base case.

Suppose we have established (d1) – (d5) for $\mu_{i+1}$. Let us proceed to the induction step for $\mu_{i+1}$.

By (6) (with $i$ in place of $s$ and $v_{1-i} = v_n$), the vertices in $T_{1-i} - v_{1-i}$ are all contained in $(\delta_1, \gamma_1)$-cycles with respect to both $\varphi_n$ and $\sigma_n$, which are also $(\delta_1, \gamma_1)$-cycles with respect to $\mu_{i+1}$ by (d4) for $\mu_{i+1}$. Hence $P_u(\delta_1, \gamma_1, \mu_{i+1}) \cap T_{1-i} = \{v_{1-i}\} \cup \emptyset$ (see (12)). By the induction hypothesis on (d2), we have $\overline{\varphi}_{i+1}(v_n) = \overline{\varphi}_1(v_n)$. Since $S_{1-i} = \{ \delta_1, \gamma_1 \} \subseteq S$, it follows from (15) that

(16) $v_n$ is not an end of $P_u(\delta_1, \gamma_1, \mu_{i+1})$. Furthermore, $P_u(\delta_1, \gamma_1, \mu_{i+1})$ has the second end (other than $u$) outside $T_j'(u)$ and contains no edge from $T_j'(u) - T_j'(v_n)$ by (d5) for $\mu_{i+1}$.

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By (2), we have \( \delta_1 = \gamma_{i+1} \). Since (d1), (d2), (d3), and (d5) hold for \( \mu_{i+1} \), from (13) and (16) it is clear that they also hold for \( \mu_{i+1} \), because \( T_j(v_n) \subset T'_j(u) \) and \( T_j \subset T'_i \).

By (6), every \((\delta_1, \gamma_{i+1})\)-cycle \( Q \) with respect to \( \varphi_n \) intersecting \( T_{i+1} - v_n \) with \( 1 \leq s \leq i - 1 \) is contained in \( G[T_{i+1}] \subseteq G[T_i] \). In view of (16), \( Q \) is colored the same under \( \mu_{i+1} \) and \( \mu_{i+1} \).

Thus (d4) holds for \( \mu_{i+1} \), hence (14) is established.

Finally, define \( \varphi_2 = \mu_{i+1} \). From (14) for \( \mu_{i+1} \) we see that \( \varphi_2 \) satisfies all the properties (c1)–(c3) and hence is as desired. This proves Claim 4.2.

Case 2. \( \overline{\varphi}_1(T'_j - V(T_j(v_n))) \cap (\cup_{i \in \mathbb{L}_s} S_i) = \emptyset \).

Recall that \( L_1 = \{1,1,2,...,1_{e(1)}\} \). Set \( S' = \cup_{i \in \mathbb{L}_s} S_i \). Let us make some simple observations about \( T_j \) and \( T'_j \).

(17) \( \overline{\varphi}_1(T'_j) \cap S' = \overline{\varphi}_1(v_n) \cap S' = \{\delta_n\} \) and \( \varphi_1(T'_j - T_j(v_n)) \cap S' = \{\delta_n\} \).

To justify this, note that \( V(T'_j) \) is elementary with respect to \( \varphi_1 \) by (11) and that \( \overline{\varphi}_n(v_n) = \overline{\varphi}_n(v_n) \) by (a2). In view of Lemma 3.3(i), we have \( \overline{\varphi}_n(v_n) \cap S' = \{\delta_n\} \). So \( \overline{\varphi}_1(v_n) \cap S' = \{\delta_n\} \) and hence \( \delta_n \notin \overline{\varphi}_1(T'_j - v_n) \).

Assume the contrary: \( \overline{\varphi}_1(T'_j) \cap S' \neq \{\delta_n\} \). Then the hypothesis of the present case and (2) would guarantee the existence of some \( \gamma_{i+1} \in \overline{\varphi}_1(T_j(v_n) - v_n) \), where \( 1 \leq s \leq c_1 \). Let \( u \) be a vertex in \( T_j(v_n) - v_n \) with \( \gamma_{i+1} \in \overline{\varphi}_1(u) \). By Theorem 3.10(iii), \( \varphi_n \) is \( (T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1}) \)-stable. By (a1), \( \varphi_1 \) is also \( (T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1}) \)-stable. Hence \( \gamma_{i+1} \in \overline{\varphi}_1(u) \). On the other hand, since \( \gamma_{i+1} \in \overline{\varphi}_1(v_n) \) and \( V(T_1) \) is elementary with respect to \( \varphi_{j-1} \) by (4.1) and Theorem 3.10(i), we have \( \gamma_{i+1} \notin \overline{\varphi}_1(u) \). So \( G[T_1] \) contains an edge \( f \) incident to \( u \) colored by \( \gamma_{i+1} \) under \( \varphi_{j-1} \). Thus \( \varphi_n(f) = \gamma_{i+1} \) by Lemma 4.2 and hence \( \gamma_{i+1} \in \varphi_n(u) \); this contradiction implies that \( \overline{\varphi}_1(T'_j) \cap S' = \{\delta_n\} \).

Again, since \( V(T'_j) \) is elementary with respect to \( \varphi_1 \) and since \( \overline{\varphi}_1(T'_j) \cap S' = \{\delta_n\} \), no edge in \( T'_j - T_j(v_n) \) is assigned to a color in \( S' \) under \( \varphi_1 \) except \( \delta_n \) by TAA (see the paragraph above (11)). Thus \( \varphi_1(T'_j - T_j(v_n)) \cap S' = \{\delta_n\} \). Hence (17) holds.

(18) \( \delta_n \notin \overline{\varphi}_1(T_j - V(T_j(v_n))) \) and \( \delta_n \notin \varphi_1(T_j - T_j(v_n)) \).

Assume on the contrary that \( \delta_n \in \overline{\varphi}_1(T_j - V(T_j(v_n))) \). Then \( \delta_n \in \overline{\varphi}_{j-1}(T_j - V(T_j(v_n))) \) by (a2) in Claim 4.1. By (4.1) and Theorem 3.10(vi), \( V(T_j) \) is elementary with respect to \( \varphi_{j-1} \). So \( \delta_n \notin \overline{\varphi}_{j-1}(v_n) \) and \( G[T_j] \) contains an edge \( f \) incident to \( v_n \) colored by \( \delta_n \) under \( \varphi_{j-1} \). By Lemma 4.2, \( \varphi_n(f) = \varphi_{j-1}(f) = \delta_n \), so \( \delta_n \in \varphi_n(v_n) \); this contradiction proves that \( \delta_n \notin \overline{\varphi}_1(T_j - V(T_j(v_n))) \).

By (3) and Lemma 3.3(iii), we have \( \overline{\varphi}_1(v_n) = \overline{\varphi}_{1-1}(v_n) = (\overline{\varphi}_1(v_n) - \{\delta_n\}) \cup \{\gamma_{i+1}\} = (\overline{\varphi}_1(v_n) - \{\delta_n\}) \cup \{\gamma_{i+1}\} \). So \( \delta_n \notin \overline{\varphi}_{j-1}(v_n) \). By (11) and (17), we obtain \( \delta_n \notin \overline{\varphi}_1(T_j(v_n) - v_n) \), which together with (a2) implies \( \delta_n \notin \overline{\varphi}_{j-1}(T_j(v_n) - v_n) \). As \( \delta_n \notin \overline{\varphi}_1(T_j - V(T_j(v_n))) \), we conclude that \( \delta_n \notin \overline{\varphi}_{j-1}(T_j(v_n) - v_n) \). Hence no edge in \( T_j - T_j(v_n) \) is colored by \( \delta_n \) under \( \varphi_{j-1} \) by TAA. It follows from (a2) that \( \delta_n \notin \overline{\varphi}_1(T_j - T_j(v_n)) \). So (18) is justified.

By Lemma 3.2(iv), \( \partial_{\varphi_n, \gamma_n}(T_n) = \{f_n\} \), and edges in \( \partial_{\varphi_n, \delta_n}(T_n) \) are all incident to \( V(T_n(v_n) - v_n) \). These two properties remain valid if we replace \( \varphi_n \) by \( \sigma_n \), because \( \sigma_n \) is \( (T_n, D_n, \varphi) \)-stable. Thus, by (a3) in Claim 4.1, they also hold true if we replace \( \varphi_n \) by \( \varphi_1 \). Since \( T'_j \) is a closure of \( T_j(v_n) \) under \( \varphi_1 \) and \( \delta_n \notin \overline{\varphi}_1(v_n) = \overline{\varphi}_1(v_n) \) by (a2), from TAA we see that no boundary edge of \( T_n \cup T'_j \) is colored by \( \delta_n \) under \( \varphi_1 \).

At the beginning of our proof, we assume that \( P_n(\gamma_n, \delta_n, \sigma_n) \) contains at least two vertices
from $T_n$. Let $P$ denote $P_{v_n}(\gamma_n, \delta_n, \varphi_1)$. Then $P = P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ by (a3) and hence $P \cap T_n \neq \{v_n\}$. Since $\partial_{v_1, \gamma_2}(T_n) = \{f_n\}$ and $\partial_{v_2, \delta_2}(T_n \cup T_j') = \emptyset$, from the hypothesis of the present case, we deduce that the other end $x$ of $P$ is outside $T_n \cup T_j'$. Furthermore, $P$ contains a subpath $P[w, x]$, which is a $T_n \cup T_j'$-exit path with respect to $\varphi_1$. Note that $w$ is contained in $T_j' - V(T_n)$, because the edge incident with $w$ on $P[w, x]$ is colored by $\gamma_n$. Let $\beta \in \varphi_1(w)$. By the hypothesis of the present case, we have

$$(19) \, \beta \notin S'. $$

Possibly $\beta \in \varphi_1(T_j - V(T_j(v_n)))$; in this situation, let $z$ be the first vertex in $T_j - V(T_j(v_n))$ in the order $\prec$ such that $\beta \in \varphi_3(z)$.

Claim 4.3. There exists a coloring $\varphi_3 \in C^k(G-c)$ with the following properties:

(e1) $\varphi_3$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;

(e2) if $\beta \notin \varphi_3(T_j)$, then $\varphi_3(v) = \varphi_1(v)$ for all $v \in V(T_j \cup T_j'(w) - w)$ and $\varphi_3(f) = \varphi_1(f)$ for all $f \in E(T_j \cup T_j'(w))$;

(e3) if $\beta \in \varphi_3(T_j - V(T_j(v_n)))$, then $\varphi_3(v) = \varphi_1(v)$ for all $v \in V(T_j(z) \cup T_j'(w))$ and $\varphi_3(f) = \varphi_1(f)$ for all $f \in E(T_j(z) \cup T_j'(w))$. Furthermore, $\delta_n \in \varphi_3(z)$; and

(e4) $\gamma_1 \in \varphi_3(w)$.

(Assuming Claim 4.3) By (e1) in Claim 4.3, $\varphi_3$ is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. So it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a $\varphi_{j-1}$ mod $T_{j-1}$ coloring by (4.1) and Theorem 3.10(vi). Since both $\varphi_1$ and $\varphi_3$ are $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, we have $\varphi_1(f) = \varphi_3(f) = \varphi_{j-1}(f)$ for each edge $f$ on $T_j(v_n)$ by (1). Let $T_j^2$ be a closure of $T_j'(w)$ under $\varphi_3$. Then $T_j^2$ is an ETT under the generating coloring $\varphi_3$, because it is obtained from $T_{j-1}$ by using the same connecting edge, connecting color, and extension type as $T_j$. It follows from Theorem 3.10(vi) that $T_j^2$ satisfies MP under $\varphi_3$. Hence, by (4.1) and Theorem 3.10(i), $V(T_j^2)$ is elementary with respect to $\varphi_3$. By (e4), we have $\gamma_1 \in \varphi_3(w)$. By Lemma 3.3(iii), we obtain $\varphi_{j-1}(v_n) = \varphi_{j-1} \{v_n\} \subseteq \varphi_1(v_n) \cup \{\gamma_1\} = \varphi_n(v_n) \cup \{\gamma_1\}$. So $\varphi_{j-1}(v_n) \subseteq \varphi_3(v_n) \cup \{\gamma_1\}$ by (a2), (e2) and (e3).

When $\beta \notin \varphi_3(T_j)$, by (a2) and (e2) we have

- $\varphi_3(v) = \varphi_{j-1}(v)$ for all $v \in V(T_j - v_n)$;
- $\varphi_3(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j)$;
- $\varphi_3(v) = \varphi_1(v)$ for all $v \in V(T_j'(w) - w)$; and
- $\varphi_3(f) = \varphi_1(f)$ for all $f \in E(T_j'(w))$.

From TAA we see that $V(T_j \cup T_j'(w)) \subseteq V(T_j^2)$, which contradicts the maximum property satisfied by $T$.

When $\beta \in \varphi_3(T_j - V(T_j(v_n)))$, by (a2) and (e3) we get

- $\varphi_3(v) = \varphi_{j-1}(v)$ for all $v \in V(T_j(z) - \{z, v_n\})$;
- $\varphi_3(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j(z))$;
- $\varphi_3(v) = \varphi_1(v)$ for all $v \in V(T_j'(w) - w)$; and
- $\varphi_3(f) = \varphi_1(f)$ for all $f \in E(T_j'(w))$.

From TAA we conclude that $V(T_j(z) \cup T_j'(w)) \subseteq V(T_j^2)$. As $\delta_n \in \varphi_3(z) \cap \varphi_3(v_n)$, $V(T_j^2)$ is not elementary with respect to $\varphi_3$, a contradiction again.
To justify Claim 4.3, consider the coloring \( q_0 = q_1/(G - T_j', \beta, \delta_n) \). Since \( T_j' \) is closed with respect to \( q_1 \) and \( \{v_n, w\} \subseteq V(T_j') \), no boundary edge of \( T_j' \) is colored by \( \beta \) or \( \delta_n \) under \( q_1 \). So \( q_0 \) is \( (T_j', D_{j-1, q_1}) \)-stable and hence is \( (T_j(v_n) - v_n, D_{j-1, q_1}) \)-stable. Since \( q_1 \) is \( (T_j(v_n) - v_n, D_{j-1, q_{j-1}}) \)-stable by (1), we deduce from Lemma 2.4 that

(20) \( q_0 \) is \( (T_j(v_n) - v_n, D_{j-1, q_{j-1}}) \)-stable and hence is \( (T_{j-1}, D_{j-1, q_{j-1}}) \)-stable.

Since \( w \notin V(T_j(v_n)) \), by (11) we have

(21) \( \beta \notin \overline{\pi}_1(T_j(v_n)) = \overline{\pi}_0(T_j(v_n)) \).

Let us recursively define a sequence of colorings along the linear order \( \prec_v \) as follows:

(22) \( \mu_{c(1)} = q_0/P_w(\beta, \gamma_{c(1)}, q_0) \) and \( \mu_{i+1} = \mu_{c(i)}/P_w(\delta_{i}, \gamma_{i+1}, \mu_{i+1}) \) for \( c(1) - 1 \geq i \geq 1 \).

Recall that \( 1_{c(1)} = n \) and that \( \gamma_{c(1)} = \delta_{c(1)-1} \) if \( c(1) \geq 2 \) (see (2)).

(23) Each coloring \( \mu_{c(i)} \), for \( i = c(1), c(1) - 1, \ldots, 1 \), satisfies the following properties:

(1) \( \mu_{c(i)} \) is \( (T_j(v_n) - v_n, D_{j-1, q_{j-1}}) \)-stable;

(2) if \( \beta \notin \overline{\pi}_1(T_j - V(T_j(v_n))) \), then \( \overline{\pi}_1(v) = \overline{\pi}_1(v) \) for all \( v \in V(T_j \cup T_j'(w) - v_n) \) and \( \mu_{c(i)}(f) = q_1(f) \) for all \( f \in E(T_j \cup T_j'(w)) \);

(3) if \( \beta \notin \overline{\pi}_1(T_j - V(T_j(v_n))) \), then \( \overline{\pi}_1(v) = \overline{\pi}_1(v) \) for all \( v \in V(T_j(z) \cup T_j'(w) - w, z) \) and \( \mu_{c(i)}(f) = q_1(f) \) for all \( f \in E(T_j(z) \cup T_j'(w)) \). Furthermore, \( \delta_{i} \notin \overline{\pi}_1(z) \);

(4) \( \gamma_{c(i)} \in \overline{\pi}_1(v) \);

(5) \( \mu_{c(i)}(f) = \varphi_n(f) \) for all \( f \in E(Q) \), where \( Q \) is an arbitrary \( (\delta_{i}, \gamma_{i}) \)-cycle with respect to \( \varphi_n \) intersecting \( T_{i-1} - v_{i-1} \) with \( 1 \leq s \leq c(1) - 1 \) (recall that \( v_{i-1} = v_n \)) and \( \delta_{i} \notin \overline{\pi}_1(v) \).

(6) \( \overline{\pi}_1(T_j' - w) \cap S' = \delta_n \) and \( \mu_{c(i)}(T_j' - T_j(v_n)) \cap S' = \delta_n \).

We prove (23) by induction on the order \( \prec_v \) starting from \( \mu_{c(1)} \) (see (22)).

From the definitions of the path \( P[w, x] \) and \( q_0 \), we see that

(24) \( P_0(\beta, \gamma_{c(i)}, q_0) \cap V(T_j \cup T_j') = \{w\} \).

It follows instantly from (20), (22) and (24) that (1) and (4) hold for \( \mu_{c(i)} \). By (6), each \( (\gamma_{i}, \delta_{i}) \) cycle \( Q \) with respect to \( \varphi_n \) intersecting \( T_{i-1} - v_{i-1} \) is contained in \( G[T_{n}] \) for \( 1 \leq s \leq c(1) - 1 \). Since \( \beta \notin S' \) (see (19)) and \( \delta_{i} \notin \gamma_{c(i)}, \delta_{i} \} \) for any \( 1 \leq s \leq c(1) - 1 \) (see (2) and (3)), cycle \( Q \) is colored the same under \( q_0 \) as under \( \varphi_n \), so it is still contained in \( G[T_{n}] \) under \( q_0 \). From (20) and (24) we deduce that (5) holds for \( \mu_{c(i)} \). By (17) and the definition of \( q_0 \), we obtain \( \overline{\pi}_0(T_j') \cap S' = \overline{\pi}_0(v_n) \cap S' = \delta_n \) and \( q_0(T_j' - T_j(v_n)) \cap S' = \delta_n \), which together with (24) yields (6) for \( \mu_{c(i)} \).

By (11), \( V(T_j') \) is elementary with respect to \( q_1 \). Since \( \beta \notin \overline{\pi}_1(w) \), we have \( \beta \notin \overline{\pi}_1(T_j' - w) \). By (a2) we obtain \( \beta \notin \overline{\pi}_{j-1}(T_j(v_n) - v_n) \) and \( \beta \notin \overline{\pi}_{j-1}(v_n) \). From Lemma 3.3(iii) we deduce that \( \overline{\pi}_{j-1}(v_n) = \overline{\pi}_{j-1}(v_n) \subseteq \overline{\pi}_{c(i)}(v_n) \cup \{\gamma_{i+1}\} \subseteq \overline{\pi}_{c(i)}(v_n) \subseteq \{\gamma_{i+1}\} \). Since \( \beta \notin S' \) by (19), we get \( \beta \notin \overline{\pi}_{j-1}(v_n) \).

Hence

(25) \( \beta \notin \overline{\pi}_{j-1}(T_j(v_n)) \).

Suppose \( \beta \notin \overline{\pi}_{j-1}(T_j(v_n)) \). Then \( \beta \notin \overline{\pi}_{j-1}(T_j) \) by (a2) and (25). Thus \( \beta \notin \overline{\pi}_{j-1}(T_j(v_n)) \) because \( T_j \) is obtained from \( T_j(v_n) \), by TAA under \( \varphi_{j-1} \). By (a2) and (18), we obtain \( \beta \notin \overline{\pi}_{j-1}(T_j - T_j(v_n)) \), \( \delta_{i} \notin \overline{\pi}_{j-1}(T_j - V(T_j(v_n))) \), and \( \delta_{i} \notin \overline{\pi}_{j-1}(T_j - T_j(v_n)) \). From the definition of \( q_0 \), we see that \( \overline{\pi}_0(v) = \overline{\pi}_0(v) \) for all \( v \in V(T_j \cup T_j'(w) - w) \) and \( q_0(f) = q_1(f) \) for all \( f \in E(T_j \cup T_j'(w)) \). Thus (f2) for \( \mu_{c(i)} \) follows instantly from (22) and (24).

Suppose \( \beta \in \overline{\pi}_{j-1}(T_j - V(T_j(v_n))) \). Recall that \( z \) is the first vertex in \( T_j - V(T_j(v_n)) \) in the order \( \prec \) with \( \beta \in \overline{\pi}_1(z) \). By (a2) and (25), we get \( \beta \notin \overline{\pi}_{j-1}(T_j(z) - z) \). Since \( T_j \) is obtained from \( T_j(v_n) \), by TAA under \( \varphi_{j-1} \), by (a2) we have \( \beta \notin \overline{\pi}_{j-1}(T_j(z) - T_j(v_n)) \) and hence \( \beta \notin \overline{\pi}_{j-1}(T_j(z) - T_j(v_n)) \). By (a2) and (18), we obtain \( \delta_{i} \notin \overline{\pi}_{j-1}(T_j - V(T_j(v_n))) \) and
\(\delta_n \notin g_1(T_j - T_j(v_n))\). From the definition of \(g_0\), we see that \(\delta_n \in \overline{\Omega}_0(z), \overline{\Omega}_0(v) = \overline{\Omega}_1(v)\) for all \(v \in V(T_j(z) \cup T_j'(w) - \{w, z\})\) and \(g_0(f) = g_1(f)\) for all \(f \in E(T_j(z) \cup T_j'(w))\). Thus \((f_3)\) for \(\mu_{i+1}\) follows instantly from \((22)\) and \((24)\).

Suppose we have established \((d1) - (d5)\) for \(\mu_{i+1}\). Let us proceed to the induction step for \(\mu_{i+1}\).

By \((f5)\) for \(\mu_{i+1}\) and \((6)\), all vertices in \(T_i - v_i\) are contained in \((\delta_1, \gamma_1)\)-cycles with respect to \(\mu_{i+1}\). So \(P_w(\delta_1, \gamma_1, \mu_{i+1}) \cap T_i = \{v_i\}\) or \(\emptyset\). Recall that \(v_i = v_n\). By the induction hypotheses on \((f2)\) and \((f3)\), we have \(\overline{\mu}_{i+1}(v_n) = \overline{\gamma}_1(v_n)\). By \((2)\) and \((3)\), we obtain \(\delta_1 \neq \delta_n \neq \gamma_1\). It follows from \((17)\) that \((26)\) is not an end of \(P_w(\delta_i, \gamma_1, \mu_{i+1})\). Furthermore, \(P_u(\delta_1, \gamma_1, \mu_{i+1})\) has the second end (other than \(w\)) outside \(T'_j\) and contains no edge from \(T'_j - T_j(v_n)\) by \((f6)\).

By \((f4)\) for \(\mu_{i+1}\), we have \(\gamma_{i+1} \in \overline{\mu}_{i+1}(w)\). By \((2)\), we obtain \(\gamma_1 \in \overline{\mu}_{i+1}(w)\) by \((22)\), so \((f4)\) holds for \(\mu_{i+1}\). It is routine to check that \((f1) - (f3)\) and \((f6)\) follow instantly from their counterparts for \(\mu_{i+1}\) and \((26)\), because \(T_j \not\subseteq T_i\). By \((6)\), each \((\delta_1, \gamma_1)\)-cycle \(Q\) with respect to \(\varphi_1\) intersecting \(T_i - v_i\), with \(1 \leq s \leq i - 1\) are contained in \(G[T_{i+1}]\) and hence in \(G[T_i]\). In view of \((25)\), \(Q\) is colored the same under \(\mu_{i+1}\) and \(\mu_{i+1}\). Thus \((f5)\) holds for \(\mu_{i+1}\). Hence \((23)\) is established.

Finally, define \(g_3 = \mu_{i+1}\). From \((23)\) for \(\mu_{i+1}\) we conclude that \(g_3\) satisfies all the properties \((e1) - (e4)\). This proves Claim 4.3 and hence Lemma 4.4.

**Lemma 4.5.** (Assuming \((4.1)\) and \((4.3)\)) Theorem 3.10(v) holds for all ETTs with \(n\) rungs and satisfying MP; that is, for any \((T_n, D_n, \varphi_n)\)-stable coloring \(\sigma_n\) and any defective color \(\delta\) of \(T_n\) with respect to \(\sigma_n\), if \(v\) is a vertex but not the smallest one (in the order \(\prec\)) in \(I[\varphi_n, \delta(T_n)]\), then \(v \preceq v_i\) for any supporting or extension vertex \(v_i\) with \(i \geq m(v)\).

**Proof.** By hypothesis, \(T\) is an ETT constructed from a \(k\)-triple \((G, e, \varphi)\) by using the Tashkinov series \(T = \{(T_i, \varphi_i, -1, S_i, -1, F_i, -1, \Theta_i, -1) : 1 \leq i \leq n + 1\}\), and \(T\) satisfies MP under \(\varphi_n\). Depending on the extension type \(\Theta_n\), we consider two cases.

**Case 1.** \(\Theta_n = PE\). In this case, according to Step 4 in Algorithm 3.1, \(\pi_n - 1\) is a \((T_n, D_n - 1) \cup \{\delta_n\}\)-stable coloring, \(v_n\) is a \((T_n, \pi_n - 1, \{\gamma_n, \delta_n\})\)-exit and \(\varphi_n = \pi_n - 1/P_v(\gamma_n, \delta_n, \pi_n - 1)\). Since \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable coloring, from Lemma 3.2(iv) we deduce that \(\partial_{\sigma_n, \gamma_n}(T_n) = \{f_n\}\). So \(\delta = \gamma_n\).

By Theorem 3.10(iv), \(P_v(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}\). Define \(\sigma_n - 1 = \sigma_n/P_v(\gamma_n, \delta_n, \sigma_n)\). By Lemma 3.5, we obtain

\[(1) \quad \sigma_n - 1 = (T_n, D_n - 1, \varphi_n - 1)\)-stable and hence is also \((T_n - 1, D_n - 1, \varphi_n - 1)\)-stable.

If \(i < n\), then \(v \in T_n - 1\) because \(m(v) \leq i < n\). Since \(v\) is not the smallest vertex in \(I[\partial_{\sigma_n, \delta}(T_n)]\) and \(\delta \neq \gamma_n\), from the definition of \(\sigma_n - 1\) it can be seen that \(\delta\) is a defective color of \(T_n - 1\) with respect to \(\sigma_n - 1\). Applying (4.3) and Theorem 3.10(v) to \(T_n - 1\) and \(\sigma_n - 1\) (see \((1)\)), we obtain \(v \preceq v_i\). So we assume that \(i = n\). Since \(v_n\) the maximum defective vertex with respect to \(T_n, D_n - 1, \varphi_n - 1\) (see the definition above \((3.1)\)), by \((1)\) we also have \(v \preceq v_n\).

**Case 2.** \(\Theta_n = SE\). In this case, \(\varphi_n \in (T_n, D_n - 1, \varphi_n - 1)\)-stable (see Algorithm 3.1). Since \(\sigma_n\) is \((T_n, D_n, \varphi_n)\)-stable and \(\varphi_n - 1(T_n) \cup D_n - 1 \subseteq \varphi_n(T_n) \cup D_n\) by Lemma 3.2(i), we deduce that \(\sigma_n\) is \((T_n, D_n - 1, \varphi_n - 1)\)-stable and hence is also \((T_n - 1, D_n - 1, \varphi_n - 1)\)-stable. If \(i < n\), then \(m(v) < n\). Since \(v \in T_n - 1\), \(\delta\) is a defective color of \(T_n - 1\) with respect to \(\sigma_n\). Thus \(v \preceq v_i\) by \((4.3)\) and
Theorem 3.10(v). So we assume that \( i = n \). Since \( v_n \) the maximum defective vertex with respect to \((T_n, D_n, \varphi_{n-1})\), we also have \( v \leq v_n \).

The proof of Theorem 3.10(vi) is based on the following technical lemma.

**Lemma 4.6.** (Assuming (4.1) and (4.4)) Let \( T_1 = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \) (resp. \( T_2 = \{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\} \)) be a Tashkinov series constructed from a \( k \)-triple \((G, e, \varphi_0)\) (resp. \((G, e, \sigma_0)\)) by using Algorithm 3.1. Suppose \( T_{n+1} \) satisfies MP under \( \varphi_n \), and \( \sigma_i \) is a \((T_i, D_i, \varphi_i)\)-stable coloring in \( C^k(G - e) \) for \( 1 \leq i \leq n - 1 \). Furthermore, \( \sigma_{n-1} \) is a \((T_n, D_{n-1}, \varphi_{n-1})\)-stable coloring. If \( \Theta_n = RE \), then there exists a Tashkinov series \( T_3 = \{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \), such that \( \sigma_n = \sigma_{n-1} \) and \( T_i^* = T_i \) for \( 1 \leq i \leq n \).

**Proof.** According to Step 1 in Algorithm 3.1, there exists a subscript \( h \leq n - 1 \) with \( \Theta_h = PE \) and \( S_h = \{\delta_h, \gamma_h\} \), such that for all \( i \) with \( h + 1 \leq i \leq n - 1 \), if any, we have \( \Theta_i = RE \) and \( S_i = \{\delta_i, \gamma_i\} = S_h \), where \( \delta_i = \delta_h \) and \( \gamma_i = \gamma_h \), and such that some \((\gamma_h, \delta_h)\)-cycle \( C \) with respect to \( \varphi_{n-1} \) contains both an edge \( f_n \in \partial\varphi_{n-1, \gamma_h}(T_n) \) and a segment \( L \) connecting \( V(T_h) \) and \( v_n \) with \( V(L) \subseteq V(T_h) \), where \( v_n \) is the end of \( f_n \) in \( T_n \). According to Step 2 in this algorithm, \( \varphi_n = \varphi_{n-1} \), \( T_{n+1} \) is a closure of \( T_n + f_n \) under \( \varphi_n \), \( \delta_n = \delta_h \), \( \gamma_n = \gamma_h \), \( S_n = \{\delta_n, \gamma_n\} \), and \( F_n = \{f_n\} \). Thus

(1) \( \varphi_n = \varphi_{n-1} = \ldots = \varphi_n \) and \( S_h = S_{h+1} = \ldots = S_n \).

(2) \( \sigma_n = \sigma_{h+1} = \ldots = \sigma_{n-1} \).

(3) \( \sigma_n \) is \((T_h, D_h, \varphi_h)\)-stable.

Let \( f_n, L \) and \( C \) be as specified in the first paragraph of our proof. Let \( L^* \) be a segment of \( C \) containing \( L \) and an edge in \( G[T_h] \) (see (3.4)). By the definition of \( D_{n-1} \), we have \( \varphi_{n-1}(T_n) \cup D_{n-1} = \varphi_{n-1}(T_n) \cup (\cup_{i<n} S_i) \) (see (1) in the proof of Lemma 3.2). So \( \{\delta_h, \gamma_h\} \subseteq \varphi_{n-1}(T_n) \cup D_{n-1} \). Since \( \sigma_{n-1} \) is \((T_n, D_{n-1}, \varphi_{n-1})\)-stable, \( L^* \) is also a \((\delta_h, \gamma_h)\)-path with respect to \( \sigma_{n-1} \) and hence with respect to \( \sigma_n \). Let \( C^* \) be the \((\delta_h, \gamma_h)\)-chain with respect to \( \sigma_n \) containing \( L^* \). Then \( f_n \in C^* \).

We claim that \( C^* \) is a \((\delta_h, \gamma_h)\)-path with respect to \( \sigma_n \). By (3) and Theorem 3.10(iv), \( P_{y_h}(\delta_h, \gamma_h, \sigma_n) \) is a \( T_h \)-exit path. Recall that \( \delta_h \) is missing at \( v_h \) in \( \varphi_n \) and hence in \( \sigma_n \) by (3), and that \( C^* \) contains at least two vertices from \( T_h \), so \( C^* \) and \( P_{y_h}(\delta_h, \gamma_h, \sigma_n) \) are two disjoint \((\delta_h, \gamma_h)\)-paths (with respect to \( \sigma_n \)) intersecting \( T_h \); this contradiction to Theorem 3.10(ii) justifies the claim.

The above claim and Algorithm 3.1 guarantee the existence of a Tashkinov series \( T_3 = \{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \), such that \( \sigma_n = \sigma_{n-1} \) and \( T_i^* = T_i \) for \( 1 \leq i \leq n \).

The lemma below states that ETTs along with the maximum property are also maintained under taking stable colorings.

**Lemma 4.7.** (Assuming (4.1) and (4.4)) Theorem 3.10(vi) holds for all ETTs with \( n \) rungs and satisfying MP; that is, every \((T_n, D_n, \varphi_n)\)-stable coloring \( \sigma_n \) is a \( \varphi_n \) mod \( T_n \) coloring. (So every ETT \( T^* \) corresponding to \((\sigma_n, T_n)\) satisfies MP under \( \sigma_n \) by Lemma 3.9.)
**Proof.** By hypothesis, $T$ is an ETT constructed from a $k$-triple $(G, e, \phi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$, and $T$ satisfies MP under $\varphi_n$. We aim to show (recall Definition 3.7), by induction on $r(T)$, the existence of an extended Tashkinov tree $T^*$ with corresponding Tashkinov series $T^* = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$, satisfying $\sigma_0 \in \mathcal{C}(G - e)$ and the following conditions for all $i$ with $1 \leq i \leq n$:

1. $T^*_n = T_n$ and
2. $\sigma_i$ is a $(T_i, D_i, \varphi_i)$-stable coloring in $\mathcal{C}^k(G - e)$, where $D_i = \cup_{h \leq i} S_h - \varphi_i(T_i)$.

For this purpose, we shall define a $(T_n, D_n, \varphi_n)$-stable coloring $\sigma_{n-1}$ based on $\sigma_n$, and apply induction hypothesis to $\sigma_{n-1}$.

Since $T_n$ is an ETT constructed from the $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T_n = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$, with $r(T_n) = n$, and since $T_n$ satisfies MP under $\varphi_n$, by (4.4) and Theorem 3.10(iii), we obtain

3. there exists a Tashkinov series $T^*_n = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$, satisfying $\sigma_0 \in \mathcal{C}^k(G - e)$ and (1) and (2) for all $i$ with $1 \leq i \leq n - 1$.

Our objective is to find $\sigma_{n-1}$, such that

4. $T^*_n$ can be set to $T_n$, and an ETT $T^*_{n+1}$ with respect to $e$ and $\sigma_n$ can be obtained from $T^*_n = T_n$ by using the same connecting edge, connecting color, and extension type $\Theta_n$ as $T_{n+1}$ in $T$.

Combining (3) and (4), we see that $\sigma_n$ is indeed a $\varphi_n \mod T_n$ coloring. To establish (4), we consider three cases, according to the extension type $\Theta_n$.

**Case 1.** $\Theta_n = RE$. In this case, define $\sigma_{n-1} = \sigma_n$. By hypothesis, $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring. So $\sigma_{n-1}$ is also $(T_n, D_n, \varphi_n)$-stable coloring. Since $\varphi_n = \varphi_{n-1}$ by Algorithm 3.1 and $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i), we deduce that $\sigma_{n-1}$ is $(T_n, D_n, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. By Lemma 3.2(iii), we have $\sigma_n(f) = \varphi_n(f)$ for any edge $f$ on $T_n$. It follows that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for any edge $f$ on $T_n$. Thus we can set $T^*_n = T_n$. Therefore, by Lemma 4.6, an ETT $T^*_{n+1}$ with respect to $e$ and $\sigma_n$ can be obtained from $T_n$ by using the same connecting edge, connecting color, and extension type RE as $T_{n+1}$ in $T$.

**Case 2.** $\Theta_n = SE$. In this case, according to Step 3 of Algorithm 3.1, $\varphi_n = \pi_{n-1}$, $T_{n+1}$ is a closure of $T_n + f_n$ under $\varphi_n$, $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$, where $\pi_{n-1}$ is a $(T_n, D_n, \varphi_{n-1})$-stable coloring so that $v_{\varphi_{n-1}} = v_n$, which is the maximum defective vertex with respect to $(T_n, D_n, \varphi_{n-1})$ (see the paragraph above (3.1)). By the definition of $\varphi_n$, we have

5. $\varphi_n$ is $(T_n, D_n, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_n, \varphi_{n-1})$-stable.

Define $\sigma_{n-1} = \sigma_n$. Since $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring, so is $\sigma_{n-1}$. By (5) and Lemma 3.2(i), we obtain

6. $\sigma_{n-1}$ is $(T_n, D_n, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_n, \varphi_{n-1})$-stable.

By Lemma 3.2(iii), we have $\varphi_{n-1}(T_n) \subseteq \varphi_{n-1}(T_n) \cup D_{n-1}$. It follows from (6) that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for any edge $f$ on $T_n$. Thus we can set $T^*_n = T_n$. Moreover, by (6) and Lemma 2.4, $v_n$ is also the maximum defective vertex with respect to $(T_n, D_n, \varphi_{n-1})$ (see the definition above (3.1)).

Since $\sigma_n$ is $(T_n, D_n, \varphi_n)$-stable, $\delta_n$ is also a defective color of $T_n$ under $\sigma_n$. Let $u_n$ be the vertex of $f_n$ outside $T_n$. By Algorithm 3.1 (see Steps 1 and 3), each $(T_n, D_n, \varphi_n)$-stable coloring
\[ \varphi_n' \text{ satisfies } \varphi_n'(T_n) \cap \varphi_n(u_n) = \emptyset. \]

Note that each \((T_n, D_n, \sigma_n)\)-stable coloring is a \((T_n, D_n, \varphi_n)\)-stable coloring by Lemma 2.4. So an ETT \(T_{n+1}^*\) with respect to \(e\) and \(\sigma_n\) can be obtained from \(T_n\) by using the same connecting edge, connecting color, and extension type \(SE\) as \(T_{n+1}\) in \(T\).

Recall that \(RE\) has priority over both \(SE\) and \(PE\) in the construction of a Tashkinov series using Algorithm 3.1, so we also need to check that no ETT \(T_{n+1}^*\) with respect to \(e\) and \(\sigma_n\) can be obtained from \(T_n\) by using \(RE\). Assume the contrary: \(T_{n+1}^*\) is such an ETT. Since \(T\) satisfies MP, so does the ETT \(T_{n+1}\). Let \(T\) be the Tashkinov series obtained from \(\{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}\) by adding a tuple \((T_{n+1}^*, \sigma_n, S_n^*, F_n^*, \Theta_n^*)\) corresponding to \(T_{n+1}^*\), where \(\Theta_n^* = RE\), and let \(T_i = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}\). Since \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable coloring by (6), it follows from Lemma 2.4 that \(\varphi_{n-1}\) is a \((T_n, D_{n-1}, \varphi_{n-1})\)-stable coloring.

Similarly, \(\varphi_i\) is a \((T_i, D_i, \sigma_i)\)-stable coloring for \(1 \leq i \leq n-1\), because \(\sigma_i\) is a \((T_i, D_i, \varphi_i)\)-stable coloring by (2) and (3). Applying Lemma 4.6 to \(T\) and \(T_i\), we see that an ETT with respect to \(e\) and the coloring \(\varphi_{n-1}\) in \(G\) can be obtained from \(T_n\) by using \(RE\), contradicting the hypothesis of the present case.

**Case 3.** \(\Theta_n = PE\). In this case, define \(\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)\). Since \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable coloring, by (4.4) and Theorem 3.10(iv), we obtain \(P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}\). Using Lemma 3.5, we have

\[(7) \sigma_{n-1} \text{ is } (T_n, D_{n-1}, \varphi_{n-1})\text{-stable and hence is also } (T_{n-1}, D_{n-1}, \varphi_{n-1})\text{-stable.} \]

By Lemma 3.2(iii), we have \(\varphi_{n-1}(T_n) \subseteq \varphi_{n-1}(T_n) \cup D_{n-1}\). It follows from (7) that \(\sigma_{n-1}(f) = \varphi_{n-1}(f)\) for any edge \(f\) on \(T_n\). Thus we can set \(T_n = T_n^*\). Moreover, by (7) and Lemma 2.4, \(v_n\) is also the maximum defective vertex with respect to \((T_n, D_{n-1}, \sigma_{n-1})\).

Since \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable, \(\delta_n\) is also a defective color of \(T_n\) under \(\sigma_n\). Let \(u_n\) be the vertex of \(f_n\) outside \(T_n\). By Algorithm 3.1 (see Steps 1 and 4), there exists a \((T_n, D_n, \varphi_n)\)-stable coloring \(\varphi_n\) such that \(\varphi_n'(T_n) \cap \varphi_n'(u_n) \neq \emptyset\). Note that \(\varphi_n'\) is a \((T_n, D_n, \sigma_n)\)-stable coloring by Lemma 2.4. So an ETT \(T_{n+1}^*\) with respect to \(e\) and \(\sigma_n\) can be obtained from \(T_n\) by using the same connecting edge, connecting color, and extension type \(PE\) as \(T_{n+1}\) in \(T\).

Using the same argument as in Case 2, we conclude that no ETT \(T_{n+1}^*\) with respect to \(e\) and \(\sigma_n\) can be obtained from \(T_n\) by using \(RE\).

\section{5 Good Hierarchies}

As is well known, Kempe changes play an important role in edge-coloring theory. To ensure that ETTs are preserved under such operations, in this section we develop an effective control mechanism, the so-called good hierarchies, which will serve as a powerful tool in the proof of Theorem 3.10(i). Throughout this section, we assume that

\subsection{5.1} Theorem 3.10(i) and (ii) holds for all ETTs with at most \(n - 1\) rungs and satisfying MP, and Theorem 3.10(iii)-(iv) hold for all ETTs with at most \(n\) rungs and satisfying MP.

Let \(J_n\) be a closure of \(T_n(v_n)\) under a \((T_n, D_n, \varphi_n)\)-stable coloring \(\sigma_n\). We use \(T_n \cup J_n\) to denote the tree sequence obtained from \(T_n\) by adding all vertices in \(V(J_n) - V(T_n)\) to \(T_n\) one by one, following the linear order \(<\) in \(J_n\), and using edges in \(J_n\).

\textbf{Lemma 5.1.} (Assuming (5.1)) Let \(T\) be an ETT constructed from a \(k\)-triple \((G, e, \varphi)\) by using the Tashkinov series \(T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}\). Suppose \(\Theta_n = PE\) and
$T$ enjoys MP under $\varphi_n$. If $J_n$ is a closure of $T_n(v_n)$ under a $(T_n, D_n, \varphi_n)$-stable coloring $\sigma_n$, then $V(T_n \cup J_n)$ is elementary with respect to $\sigma_n$.

Proof. Clearly, $T_n$ is an ETT with corresponding Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$, and satisfies MP under $\varphi_{n-1}$. Since $r(T_n) = n - 1$, by (5.1) and Theorem 3.10(i), $V(T_n)$ is elementary with respect to $\varphi_{n-1}$. Let $\pi_n$ and $\pi'_n$ be as specified in Step 4 of Algorithm 3.1. Since $\pi_n$ is an ETT corresponding to each edge $u$ of $T_n$, $\pi'_n$ is elementary with respect to $\varphi_{n-1}$. As $\varphi_n = \pi_n - 1 / P_{\pi_n}(v_n, \gamma_n, \pi'_{n-1})$ and $\delta_n \not\in \pi'_n(T_n)$, we further obtain

1. $V(T_n)$ is elementary with respect to $\varphi_n$ and hence elementary with respect to $\sigma_n$.

As $\sigma_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring, it follows from (5.1) and Theorem 3.10(iii) that $\sigma_n$ is $(T_j(v_n) - v_n, D_j - 1, \varphi_j - 1)$-stable, where $j = m(v_n)$. So $\sigma_n$ is a $(T_j, D_j - 1, \varphi_j - 1)$-stable coloring and hence is a $\varphi_j$-mod $T_j$ coloring by Theorem 3.10(vi). Furthermore, $\sigma_n(f) = \varphi_j - 1(f)$ for each edge $f$ on $T_j(v_n)$ (see (1) in the proof of Lemma 4.4). By (5.1) and Theorem 3.10(vi), $J_n$ is an ETT corresponding to $\sigma_n$, because it is obtained from $T_j$ by using the same connecting edge, connecting color, and extension type PE as $T_j$. Clearly, $J_n$ also satisfies the maximum property under $\sigma_n$. Since $J_n$ has $j - 1$ rungs, using (5.1), we obtain

2. $V(J_n)$ is elementary with respect to $\sigma_n$.

Suppose on the contrary that $V(T_n \cup J_n)$ is not elementary with respect to $\sigma_n$. Then $T_n \cup J_n$ contains two distinct vertices $u$ and $v$ such that $\sigma_n(u) \cap \sigma_n(v) = \emptyset$. By (1) and (2), we may assume that $u \in V(T_n) - V(J_n)$ and $v \in V(J_n) - V(T_n)$. Let $\alpha \in \sigma_n(u) \cap \sigma_n(v)$. Then $\alpha \not\in \delta_n$ by (2), because $\delta_n \in \sigma_n(v_n) = \sigma_n(v_n)$. Moreover, since $\gamma_n \in \sigma_n(v_n)$ and $V(T_n)$ is elementary with respect to $\varphi_n$, from Step 4 of Algorithm 3.1 and the definition of stable colorings, we deduce that $\gamma_n \not\in \sigma_n(T_n)$ and hence $\gamma_n \not\in \sigma_n(T_n)$. So $\alpha \not\in \gamma_n$. Consequently,

3. $\beta \not\in S_n$.

Since $T_n(v_n)$ contains the uncolored edge $e$, it contains a vertex $w \not\in v_n$. Note that $w$ is contained in both $T_n$ and $J_n$. Let $\beta \in \sigma_n(w)$. Since $\delta_n \in \sigma_n(v_n)$ and $\gamma_n \not\in \sigma_n(T_n)$, we obtain

4. $\beta \not\in S_n$ (see (2)).

As $V(J_n)$ is closed and elementary with respect to $\sigma_n$ (see (2)), the other end of $P_w(\alpha, \beta, \sigma_n)$ is $w$. From (3), (4), and Step 4 of Algorithm 3.1, we see that $\partial(T_n)$ contains no edge colored by $\alpha$ or $\beta$ under $\varphi_n$ and hence under $\sigma_n$, because $\sigma_n$ is $(T_n, D_n, \varphi_n)$-stable. Combining this with (1), we conclude that the other end of $P_w(\alpha, \beta, \sigma_n)$ is also $w$. Thus $P_w(\alpha, \beta, \sigma_n)$ terminates at both $u$ and $v$, a contradiction.

Let $T$ be an ETT as specified in Theorem 3.10; that is, $T$ is constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$. To prove that $V(T)$ is elementary with respect to $\varphi_n$, we shall turn to considering a restricted ETT $T'$ with ladder $T_1 \subset T_2 \subset \ldots \subset T_n \subset T'$ and $V(T') = V(T_{n+1})$, and then show that $V(T')$ is elementary with respect to $\varphi_n$. For convenience, we may simply view $T'$ as $T$.

In the remainder of this paper, we reserve the symbol $R_n$ for a closure of $T_n(v_n)$ under $\varphi_n$. Let $T_n \cup R_n$ be the tree sequence as defined above Lemma 5.1. We assume hereafter that

5.2 $T_{n+1}$ is a closure of $T_n \cup R_n$ under $\varphi_n$, which is a special closure of $T_n$ under $\varphi_n$ (see Step 4 in Algorithm 3.1).

By Lemma 5.1, $V(T_n \cup R_n)$ is elementary with respect to $\varphi_n$, so we may further assume that
(5.3) $T \neq T_n \lor R_n$ if $\Theta_n = PE$, which together with (5.2) implies that $T_n \lor R_n$ is not closed with respect to $\varphi_n$.

(5.4) If $\Theta_n = PE$, then each color in $\varphi_n(T_n) \cap \varphi_n(R_n)$ is closed in $T_n \lor R_n$ with respect to $\varphi_n$.

To justify this, note that each color in $\varphi_n(R_n)$ is closed in $R_n$ under $\varphi_n$ because $R_n$ is a closure. By Lemma 3.2(iv), each color in $\varphi_n(T_n) - \{\delta_n\}$ is closed in $T_n$ under $\varphi_n$. Hence each color in $\varphi_n(T_n) \cap \varphi_n(R_n) - \{\delta_n\}$ is closed in $T_n \lor R_n$ with respect to $\varphi_n$. Lemma 3.2(iv) also asserts that edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. So $\delta_n$ is closed in $T_n \lor R_n$ as well, because it is closed in $R_n$. Hence (5.4) follows.

To prove Theorem 3.10(i), we shall appeal to a hierarchy of $T$ of the form

(5.5) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$, such that $T_n \lor R_n \subset T_{n,1}$ if $\Theta_n = PE$ and $T_{n,i} = T(u_i)$ for $1 \leq i \leq q$, where $u_1 < u_2 < \ldots < u_q$ are some vertices in $T - V(T_n)$, called dividers of $T$. (So $T$ has $q$ dividers in total.)

As introduced in Algorithm 3.1, $D_n = \cup_{h \leq n} S_h - \varphi_n(T_n)$, where $S_h = \{\delta_h\}$ if $\Theta_h = SE$ and $S_h = \{\delta_h, \gamma_h\}$ otherwise. By Lemma 3.4, we have

(5.6) $|D_n| \leq n$.

Write $D_n = \{\eta_1, \eta_2, \ldots, \eta_n\}$. In Definition 5.2 given below and the remainder of this paper,

- $T_n^* = T_n \lor R_n$ if $\Theta_n = PE$ and $T_n^* = T_n$ otherwise, and $T_{n,j}^* = T_{n,j}$ if $j \geq 1$;
- $D_{n,j} = \cup_{h \leq n} S_h - \varphi_n(T_{n,j}^*)$ for $0 \leq j \leq q$;
- $v_{\eta_h}$ is defined to be the first vertex $u$ of $T$ in the order $\prec$ for which $\eta_h \in \varphi_n(u)$, if any, and defined to be the last vertex of $T$ in the order $\prec$ otherwise;
- $A_{b,j}^\gamma = \varphi_n(T_{n,j}^*) - \varphi_n(T_{n,j+1}(v_{\eta_h}) - T_{n,j}^*)$ for $0 \leq j \leq q$; and
- $\Gamma_j = \cup_{\eta_h \in D_{n,j}} \Gamma_{h,j}$ for $0 \leq j \leq q$.

Let $H$ be a subgraph of $G$ and let $C$ be a subset of $[k]$. We say that $H$ is $C$-closed with respect to $\varphi_n$ if $\partial_{\varphi_n,\alpha}(H) = \emptyset$ for any $\alpha \in C$, and say that $H$ is $C^\ominus$-closed with respect to $\varphi_n$ if it is $(\varphi_n(H) - C)$-closed with respect to $\varphi_n$.

**Definition 5.2.** Hierarchy (5.5) of $T$ is called good with respect to $\varphi_n$ if for any $j$ with $0 \leq j \leq q$ and any $\eta_h \in D_{n,j}$, there exists a 2-color subset $\Gamma_{h,j}^\gamma = \{\gamma_h^1, \gamma_h^2\} \subseteq [k]$, such that

(i) $\Gamma_{h,j}^\gamma \subseteq A_{h,j}^\gamma$ (so $\Gamma_{h,j}^\gamma \subseteq \varphi_n(T_{n,j})$ if $j = 0$ and $\Gamma_{h,j}^\gamma \subseteq \varphi_n(T_{n,j})$ if $j \geq 1$);

(ii) $\Gamma_{h,j}^\gamma \cap \Gamma_{h,j}^\gamma = \emptyset$ whenever $\eta_h$ and $\eta_{h'}$ are two distinct colors in $D_{n,j}$;

(iii) for any $j$ with $1 \leq j \leq q$, there exists precisely one color $\eta_{g_j} \in D_{n,j}$, such that $\Gamma_{g,j}^\gamma \subseteq \varphi_n(T_{n,j} - V(T_{n,j-1}))$ (so $\Gamma_{g,j}^\gamma \cap \Gamma_{g,j}^\gamma = \emptyset$) and $\Gamma_{h}^\gamma = \Gamma_{h}^\gamma - 1$ for all $\eta_h \in D_{n,j} - \{\eta_{g_j}\}$;

(iv) if $\Theta_n = PE$, then $T_n \lor R_n$ is not $(G^0)$-closed with respect to $\varphi_n$ and, subject to this, $|\varphi_n(T_n) \cap \varphi_n(R_n) - G^0|$ is maximized (this maximum value is at least 4, as we shall see); and

(v) $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma_{h,j}^\gamma - 1)^\ominus$-closed with respect to $\varphi_n$ for $1 \leq j \leq q$.

The sets $\Gamma_{h,j}^\gamma$ are referred to as $\Gamma$-sets of the hierarchy (or of $T$) under $\varphi_n$.

Some remarks may help to understand the concept of good hierarchies.

(5.7) From Condition (i) we see that neither the color $\gamma_h^1$ nor $\gamma_h^2$ can be used by edges on $T_{n,j+1}$ until after $\eta_h$ becomes missing at the vertex $v_{\eta_h}$ in $T_{n,j+1}$.
(5.8) Condition (iv) implies that \( T_{n,1} \neq T_n \lor R_n \) if \( \Theta_n = \text{PE} \).

(5.9) By definitions, \( D_{n,j} \subseteq D_{n,j-1} \), so \( \Gamma_j^{j-1} \) is well defined for any \( \eta_h \in D_{n,j} \) and \( \cup_{\eta_h \in D_{n,j}, \Gamma_j^{j-1}} \subseteq \Gamma_j^{j-1} \). In view of Condition (v), the first edge added to \( T_{n,j+1} - T_{n,j} \) is colored by a color \( \alpha \) in \( \Gamma_j^{j-1} \) for some \( g \) with \( \eta_g \in D_{n,j} \). From Condition (i) and (5.7) we see that \( \alpha \notin \Gamma_j^j \). So \( \Gamma_j^j \neq \Gamma_j^{j-1} \). According to Condition (iii), now \( \Gamma_j^j \) consists of two colors in \( \overline{\varphi_n}(T_{n,j} - V(T_{n,j-1}^*)) \).

Thus \( \Gamma_j^{j-1} \cap \Gamma_j^j = \emptyset \) and hence \( \alpha \notin \Gamma_j^j \).

(5.10) If a color \( \alpha \in \overline{\varphi_n}(T_{n,j} - V(T_{n,j-1}^*)) \) for some \( j \) with \( 1 \leq j \leq q \), then \( \alpha \notin \Gamma_j^{j-1} \) by Condition (i), and hence \( \alpha \) is closed in \( T_{n,j} \) with respect to \( \varphi_n \) by Condition (v). This simple observation will be used repeatedly in subsequent proofs.

(5.11) Note that not every ETT admits a good hierarchy. Suppose \( T \) does have such a hierarchy. To prove that \( V(T) \) is elementary with respect to \( \varphi_n \), as usual, we shall perform a sequence of Kempe changes. Since interchanging with colors in \( D_{n,j} \) often results in a coloring which is not stable, in our proof we shall use colors in \( \Gamma_j^j \) as stepping stones to switch with the color \( \eta_h \) in \( D_{n,j} \) while maintaining stable colorings in subsequent proofs. So we may think of \( \Gamma_j^j \) as a color set exclusively reserved for \( \eta_h \) and think of a good hierarchy as a control mechanism over Kempe changes.

We break the proof of Theorem 3.10(i) into the following two theorems. Although the first theorem appears to be weaker than Theorem 3.10(i), the second one implies that they are actually equivalent. We only present a proof of the second theorem in this section, and will give a proof of the first one in the next two sections.

**Theorem 5.3.** (Assuming (5.1)) Let \( T \) be an ETT constructed from a \( k \)-triple \((G,e,\varphi)\) by using the Tashkinov series \( T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \). Suppose \( T \) admits a good hierarchy and satisfies MP with respect to \( \varphi_n \). Then \( V(T) \) is elementary with respect to \( \varphi_n \).

**Theorem 5.4.** (Assuming (5.1) and Theorem 5.3) Let \( T \) be an ETT constructed from a \( k \)-triple \((G,e,\varphi)\) by using the Tashkinov series \( T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\} \). If \( T \) enjoys MP under \( \varphi_n \), then there exists a closed ETT \( T' \) constructed from \( T_n \) under \( \varphi_n \) by using the same connecting edge, connecting color, and extension type as \( T \), with \( r(T') = n \) and \( V(T') = V(T_{n+1}) \), such that \( T' \) admits a good hierarchy and satisfies MP with respect to \( \varphi_n \).

**Remark.** As we shall see, our proof of Theorem 5.4 is based on Theorem 5.3, while the proof of Theorem 5.3 is completely independent of Theorem 5.4.

**Proof of Theorem 5.4.** By (5.1) and Theorem 3.10(i), \( V(T_i) \) is elementary with respect to \( \varphi_{i-1} \) for \( 1 \leq i \leq n \). So each \( |T_i| \) is an odd number. Thus \( |T_i| - |T_{i-1}| \geq 2 \) for each \( 1 \leq i \leq n \). By Theorem 2.9, if \( |T_i| \leq 10 \), then \( G \) is an elementary multigraph, thereby proving Theorem 2.1 in this case. So we may assume that \( |T_i| \geq 11 \). Hence

1. \( |T_i| \geq 2i + 9 \) for \( 1 \leq i \leq n \).

We shall actually construct an ETT \( T' \) from \( T_n \) by using the same connecting edge, connecting color, and extension type as \( T \), which has a good hierarchy:

2. \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T' \), such that \( T_n \lor R_n \subset T_{n,1} \) if \( \Theta_n = \text{PE} \) and \( V(T') = V(T_{n+1}) \).

Since \( V(T_n) \) is elementary with respect to \( \varphi_{n-1} \), by (1) we have \( |\overline{\varphi_{n-1}}(T_n)| \geq 2n + 11 \) (as \( e \) is uncolored). From Algorithm 3.1 we see that \( |\overline{\varphi_{n-1}}(T_n)| = |\overline{\varphi_n}(T_n)| \). So
(3) \(|\varepsilon(T_n)| \geq 2n + 11\). Moreover, \(|D_{n,0}| \leq |D_n| \leq n\) by (5.6).

(4) If \(\Theta_n = PE\), then we can find a 2-color set \(\Gamma_{h}^{0} = \{\gamma_{h}, \gamma_{2h}\} \subseteq \varepsilon(T_n)\) for each \(\eta_h \in D_{n,0} = \bigcup_{h \leq n} S_h - \varepsilon(T_n \cup R_n)\), such that \(\Gamma_{g}^{0} \cap \Gamma_{h}^{0} = \emptyset\) whenever \(\eta_g\) and \(\eta_h\) are two distinct colors in \(D_{n,0}\), and such that \(T_n \cup R_n\) is not \((\Gamma^{0})^{-}\)-closed with respect to \(\varepsilon_n\), where \(\Gamma^{0} = \bigcup_{\eta_h \in D_{n,0}} \Gamma_{h}^{0}\).

To justify this, let \(\alpha\) be a color in \(\varepsilon_n(T_n \cup R_n)\) that is not closed in \(T_n \cup R_n\) under \(\varepsilon_n\); such a color exists by (5.3). In view of (3), \(\varepsilon_n(T_n) - \{\alpha\}\) contains at least \(2n + 10\) colors. So (4) follows if we pick all colors in \(\Gamma^{0}\) from \(\varepsilon_n(T_n) - \{\alpha\}\).

(5) If \(\Theta_n = PE\), then there exists a 2-color set \(\Gamma_{h}^{0} = \{\gamma_{h1}, \gamma_{h2}\} \subseteq \varepsilon_n(T_n)\) for each \(\eta_h \in D_{n,0}\) as described in (4), such that \(|\varepsilon_n(T_n) \cap \varepsilon_n(R_n) - \Gamma^{0}|\) is maximized, which is at least 4.

To justify this, let \(\alpha\) be as specified in the proof of (4). Then \(\alpha \notin \varepsilon_n(T_n) \cap \varepsilon_n(R_n)\) by (5.4).

If we pick all colors in \(\Gamma^{0}\) from \(\varepsilon_n(T_n) - \{\alpha\}\), with priority given to those in \(\varepsilon_n(T_n) - \varepsilon_n(R_n)\), then \(|\varepsilon_n(T_n) \cap \varepsilon_n(R_n) - \Gamma^{0}|\) \(\geq 4\) by (3), because the ends of the uncolored edge \(e\) are contained in both \(T_n\) and \(R_n\). So (5) is established.

Thus Definition 5.2(iv) is satisfied by these sets \(\Gamma_{h}^{0}\). Using (3), we can similarly get the following statement.

(6) If \(\Theta_n \neq PE\), then we can find a 2-color set \(\Gamma_{h}^{0} = \{\gamma_{h1}, \gamma_{h2}\} \subseteq \varepsilon_n(T_n)\) for each \(\eta_h \in D_{n,0} = D_n\), such that \(\Gamma_{g}^{0} \cap \Gamma_{h}^{0} = \emptyset\) whenever \(\eta_g\) and \(\eta_h\) are two distinct colors in \(D_{n,0}\).

Note that the ETT \(T'\) to be constructed is not necessarily \(T\), so \(T_{n,j}\) may not be a segment of \(T\) for \(1 \leq j \leq q\). Since \(T'\) is a tree sequence, we can obviously associate a linear order \(\prec\) with its vertices, so that \(\prec\) is identical with \(\prec\) when restricted to \(T_{n,0}\). Thus, in Algorithm 5.5 and 5.6, \(v_{m}\) is defined to be the first vertex of \(T'\) in the order \(\prec\) for which \(\eta_h \in \varepsilon_n(v_{m})\), if any, and is to be the last vertex of \(T'\) in the order \(\prec\) otherwise; and \(T_{n,j+1}(v_{m}) = T_{n,j+1}\) if \(v_{m}\) is not contained in \(T_{n,j+1}\).

Given \(\{\Gamma_{h}^{0}: \eta_h \in D_{n,0}\}\), let us construct \(T_{n,1}\) using the following procedure.

Algorithm 5.5

Step 0. Set \(T_{n,1} = T_n \cup R_n\) if \(\Theta_n = PE\) and \(T_{n,1} = T_n + f_n\) otherwise, where \(f_n\) is the connecting edge used in Steps 2 and 3 of Algorithm 3.1.

Step 1. While there exists \(f \in \partial(T_{n,1})\) with \(\varepsilon_n(f) \notin \varepsilon_n(T_{n,1})\), do: set \(T_{n,1} = T_{n,1} + f\) if \(\Gamma_{h}^{0} \cap \varepsilon_n(T_{n,1}(v_{m}) - T_{n,0}) = \emptyset\) for all \(\eta_h \in D_{n,0}\), where \(T_{n,0} = T_n \cup R_n\) if \(\Theta_n = PE\) and \(T_{n,0} = T_n\) otherwise.

Step 2. Return \(T_{n,1}\).

Note that if \(\Theta_n = PE\), then \(T_n \cup R_n\) is not \((\Gamma^{0})^{-}\)-closed with respect to \(\varepsilon_n\) by (4) and (5). So Step 1 is applicable to \(T_n \cup R_n\), and hence \(T_{n,1} \neq T_n \cup R_n\). If \(\Theta_n = RE\) or \(SE\), then \(T_{n,1} \neq T_n\) by the algorithm. For each \(\eta_h \in D_{n,0}\), it follows from (5), (6), and Step 1 that \(\Gamma_{h}^{0} \subseteq \varepsilon_n(T_n) - \varepsilon_n(T_{n,1}(v_{m}) - T_{n,0})\) if \(\Theta_n = PE\) and \(\Gamma_{h}^{0} \subseteq \varepsilon_n(T_n) - \varepsilon_n(T_{n,1}(v_{m}) - T_{n,0})\) otherwise. So \(\Gamma_{h}^{0} \subseteq \Lambda_{h}^{0}\). Moreover, \(T_{n,1}\) is \((\bigcup_{\eta_h \in D_{n,0}} \Gamma_{h}^{0})^{-}\)-closed with respect to \(\varepsilon_n\). To justify this, assume the contrary: there exists \(f \in \partial(T_{n,1})\) with \(\varepsilon_n(f) \notin \varepsilon_n(T_{n,1}) - (\bigcup_{\eta_h \in D_{n,0}} \Gamma_{h}^{0})\). Then either \(\varepsilon_n(f) \notin \varepsilon_n(T_{n,1}) - (\bigcup_{\eta_h \in D_{n,0}} \Gamma_{h}^{0})\) or \(\varepsilon_n(f) \in \Gamma_{h}^{0}\) for some \(\eta_h \in D_{n,0}\) but \(\eta_h \notin D_{n,1}\); in the latter case, \(\eta_h\) is a missing color at the vertex \(v_{m}\) in \(T_{n,1}\). Thus we can further grow \(T_{n,1}\) by using \(f\) and Step 1 in either case, a contradiction. Therefore, \(T_{n,1}\) and \(\{\Gamma_{h}^{0}: \eta_h \in D_{n,0}\}\) satisfy all the conditions stated in Definition 5.2.
Suppose we have constructed $T_{n,i}$ and $\{\Gamma^j_h : \eta_h \in D_{n,i-1}\}$ for all $i$ with $1 \leq i \leq j$, which are as described in Definition 5.2. If $T_{n,j}$ is closed with respect to $\varphi_n$ (equivalently $V(T_{n,j}) = V(T_{n+1})$), set $T' = T_{n,j}$. Otherwise, we proceed to the construction of $T_{n,j+1}$ and $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$ using the following procedure.

**Algorithm 5.6**

**Step 0.** Set $\Gamma^j_h = \Gamma^j_h - 1$ for each $\eta_h \in D_{n,j}$.

**Step 1.** Let $f$ be an edge in $\partial(T_{n,j})$ with $\varphi_n(f) \in \Gamma^j_h$ for some $\eta_h \in D_{n,j}$, let $T_{n,j+1} = T_{n,j} + f$, and let $\{\gamma_{h_1}, \gamma_{h_2}\}$ be a 2-subset of $\varphi_n(T_{n,j} - V(T_{n,j-1}))$. Replace $\Gamma^j_h$ by $\{\gamma_{h_1}, \gamma_{h_2}\}$.

**Step 2.** While there exists $f \in \partial(T_{n,j+1})$ with $\varphi_n(f) \in \varphi_n(T_{n,j+1})$, do: set $T_{n,j+1} = T_{n,j+1} + f$ if $\Gamma^j_h \cap \varphi_n(T_{n,j+1}(v_{\eta_h} - T_{n,j})) = \emptyset$ for all $\eta_h \in D_{n,j}$.

**Step 3.** Return $T_{n,j+1}$ and $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$.

Let us make some observations about this algorithm and its output.

As $T_{n,j}$ is not closed with respect to $\varphi_n$, $V(T_{n,j})$ is a proper subset of $V(T_{n+1})$. By Definition 5.2(v), $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^j_h - 1)$-closed with respect to $\varphi_n$. So there exists a color $\beta \in \cup_{\eta_h \in D_{n,j}} \Gamma^j_h - 1$, such that $\partial \varphi_n, \partial(T_{n,j}) \neq \emptyset$. Hence the edge $f$ specified in Step 1 is available.

For $1 \leq i \leq j$, we have $|\varphi_n(T_{n,i})| \geq |\varphi_n(T_{n,j})| \geq 2n + 1$ and $|D_{n,i}| \leq |D_{n,0}| \leq |D_n| \leq n$ by (3). So $\varphi_n(T_{n,i}) - (\cup_{\eta_h \in D_{n,j}} \Gamma^j_h - 1) \neq \emptyset$; let $\alpha$ be a color in this set. By Theorem 5.3 (see the remark right above the proof of this theorem), $V(T_{n,i})$ is elementary with respect to $\varphi_n$, which implies that $|D_{n,i}|$ is odd, because $\alpha$ is closed in $T_{n,j}$ under $\varphi_n$ by Definition 5.2(v). It follows that $|T_{n,j}| - |T_{n,j-1}| \geq 2$. So $\varphi_n(T_{n,j} - V(T_{n,j-1}))$ contains at least two distinct colors, and hence the 2-subset $\{\gamma_{h_1}, \gamma_{h_2}\}$ involved in Step 1 exists.

Note that each color in $\varphi_n(T_{n,j+1}) - (\cup_{\eta_h \in D_{n,j+1}} \Gamma^j_h)$ is closed in $T_{n,j+1}$ with respect to $\varphi_n$, for otherwise, $T_{n,j+1}$ can be augmented further using Step 2 (see the paragraph succeeding Algorithm 5.5 for details). Thus $T_{n,j+1}$ is $(\cup_{\eta_h \in D_{n,j+1}} \Gamma^j_h)$-closed with respect to $\varphi_n$ for $1 \leq j \leq q - 1$. From the algorithm we see that $\Gamma^j_h \subseteq \varphi_n(T_{n,j} - \varphi_n(T_{n,j+1}(v_{\eta_h} - T_{n,j})) = \varphi_n(T_{n,j+1})$ for all $\eta_h \in D_{n,j}$. So $T_{n,j+1}$ and $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$ satisfy all the conditions in Definition 5.2 and hence are as desired.

Repeating the process, we can eventually get a closed ETT $T'$, with $V(T') = V(T_{n+1})$, that admits a good hierarchy with respect to $\varphi_n$. Clearly, $T'$ also satisfies MP under $\varphi_n$. \hfill \blacksquare

A coloring $\sigma \in C^k(G - e)$ is called a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring if it is both $(T_n, D_n, \varphi_n)$-stable and $(R_n, \emptyset, \varphi_n)$-stable; that is, the following conditions are satisfied:

- $\sigma(f) = \varphi_n(f)$ for any edge $f$ incident to $T_n$ with $\varphi_n(f) \in \varphi_n(T_n) \cup D_n$;
- $\sigma(f) = \varphi_n(f)$ for any edge $f$ incident to $R_n$ with $\varphi_n(f) \in \varphi_n(R_n)$; and
- $\sigma(v) = \varphi_n(v)$ for any $v \in V(T_n \cup R_n)$.

(5.12) If $\sigma$ is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, then $\sigma(f) = \varphi_n(f)$ for any edge $f$ on $T_n \cup R_n$. To justify this, note that, for any edge $f$ on $T_n$, this equality holds by Lemma 3.2(iii). For any edge $f$ in $R_n - T_n$, we have $\varphi_n(f) \in \varphi_n(R_n)$ by the definition of $R_n$ and TAA. It follows from the above definition that $\sigma(f) = \varphi_n(f)$. 

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From Lemma 2.4 it is clear that being \((T_n \oplus R_n, D_n, \cdot)\)-stable is also an equivalence relation on \(\mathcal{C}(G - e)\). Moreover, every \((T_n \vee R_n, D_n, \varphi_n)\)-stable coloring is \((T_n \oplus R_n, D_n, \varphi_n)\)-stable, but the converse need not hold.

**Lemma 5.7.** Let \(T\) be an ETT constructed from a \(k\)-tuple \((G, e, \varphi)\) by using the Tashkinov series \(T = \{(T_i, \varphi_i, 1, S_i, 1, F_i, \Theta_i, 1) : 1 \leq i \leq n + 1\}\). Suppose \(\Theta_n = PE\) and \(T\) enjoys MP under \(\varphi_n\). Let \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T\) be a hierarchy of \(T\), and let \(\sigma_n\) be a \((T_n \oplus R_n, D_n, \varphi_n)\)-stable coloring. If \(T\) can be built from \(T_n \vee R_n\) by using TAA under \(\sigma_n\), then \(T\) is also an ETT satisfying MP with respect to \(\sigma_n\), and \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T\) remains to be a hierarchy of \(T\) under \(\sigma_n\).

**Proof.** Since \(\sigma_n\) is a \((T_n \oplus R_n, D_n, \varphi_n)\)-stable coloring, we have \(\sigma(f) = \varphi_n(f)\) for any edge \(f\) on \(T_n \vee R_n\) by (5.12). By definition, \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable coloring, so it is a \(\varphi_n\)-mod \(T_n\) coloring by (5.1) and Theorem 3.10(vi). Thus \(T_n\) is an ETT corresponding to \(\sigma_n\). As \(R_n\) is a closure of \(T_n(v_n)\) under \(\varphi_n\) and \(\sigma_n\) is \((R_n, \emptyset, \varphi_n)\)-stable, \(R_n\) is also a closure of \(T_n(v_n)\) under \(\sigma_n\). By hypothesis, \(T\) can be built from \(T_n \vee R_n\) by using TAA under \(\sigma_n\). So \(T\) is an ETT corresponding to the coloring \(\sigma_n\) and satisfies MP under \(\sigma_n\) by Theorem 3.10(vi). Obviously, \(T_n, 0 \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T\) remains to be a hierarchy of \(T\) under \(\sigma_n\).

We define one more term before proceeding. Let \(T\) be a tree sequence with respect to \(G\) and \(e\). A coloring \(\pi \in \mathcal{C}(G - e)\) is called \((T, \varphi_n)\)-invariant if \(\pi(f) = \varphi_n(f)\) for any \(f \in E(T - e)\) and \(\pi(v) = \overline{\varphi_n}(v)\) for any \(v \in V(T)\). Clearly, being \((T, \cdot)\)-invariant is also an equivalence relation on \(\mathcal{C}(G - e)\). Note that for any subset \(C\) of \([k]\), a \((T, C, \varphi_n)\)-stable coloring \(\pi\) is also \((T, \varphi_n)\)-invariant, provided that \(\pi(T) \subseteq \overline{\varphi_n}(T) \cup C\).

**Lemma 5.8.** (Assuming (5.1)) Let \(T\) be an ETT constructed from a \(k\)-tuple \((G, e, \varphi)\) by using the Tashkinov series \(T = \{(T_i, \varphi_i, 1, S_i, 1, F_i, \Theta_i, 1) : 1 \leq i \leq n + 1\}\). Suppose \(T\) enjoys MP under \(\varphi_n\). Let \(\sigma_n\) be obtained from \(\varphi_n\) by recoloring some \((\alpha, \beta)\)-chains fully contained in \(G - V(T)\). Then the following statements hold:

(i) \(\sigma_n\) is both \((T, D_n, \varphi_n)\)-stable and \((T, \varphi_n)\)-invariant. In particular, if \(\Theta_n = PE\), then \(\sigma_n\) is \((T_n \oplus R_n, D_n, \varphi_n)\)-stable.

(ii) \(T\) is an ETT satisfying MP with respect to \(\sigma_n\).

(iii) If \(T\) admits a good hierarchy \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T\) under \(\varphi_n\), then this hierarchy of \(T\) remains good under \(\sigma_n\), with the same \(T\)-sets (see Definition 5.2). Furthermore, if \(T\) is \((\cup_{h \in D_n, q+1} \Gamma_h^n)\)-closed with respect to \(\varphi_n\), then \(T\) is also \((\cup_{h \in D_n, q+1} \Gamma_h^n)\)-closed with respect to \(\sigma_n\).

**Proof.** Since the recolored \((\alpha, \beta)\)-chains are fully contained in \(G - V(T)\), we have

1. \(\sigma_n(f) = \varphi_n(f)\) for each edge \(f\) incident to \(V(T)\) and \(\overline{\varphi_n}(v) = \overline{\sigma_n}(v)\) for each \(v \in V(T)\).

Our proof relies heavily on this observation.

(i) By (1) and definitions, it is clear that \(\sigma_n\) is both a \((T, D_n, \varphi_n)\)-stable and a \((T, \varphi_n)\)-invariant coloring. In particular, if \(\Theta_n = PE\), then \(\sigma_n\) is \((T_n \oplus R_n, D_n, \varphi_n)\)-stable, which implies that \(\sigma_n\) is \((T_n \oplus R_n, D_n, \varphi_n)\)-stable.

(ii) In view of (1), we can construct \(T\) from \(T_n\) under \(\sigma_n\) in exactly the same way as under \(\varphi_n\). From (1) we also deduce that \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-stable coloring. Hence, by Theorem 3.10(vi), \(T\) remains to be an ETT and satisfies MP under \(\sigma_n\).
(iii) From (1), (5.5) and Lemma 5.7 (when \( \Theta_n = PE \)), we see that the given hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T \) is also a hierarchy of \( T \) under \( \sigma_n \). By hypothesis, this hierarchy is good with respect to \( \varphi_n \). Consider the \( \Gamma \)-sets specified in Definition 5.2 with respect to \( \varphi_n \). Using (1) it is routine to check that these \( \Gamma \)-sets satisfy all the conditions in Definition 5.2 with respect to \( \sigma_n \). So the given hierarchy of \( T \) remains good under \( \sigma_n \), with the same \( \Gamma \)-sets. Furthermore, if \( T \) is \( (\bigcup_{h \in D_{n,q+1}} \Gamma_{h}^q) \)-closed with respect to \( \varphi_n \), then \( T \) is also \( (\bigcup_{h \in D_{n,q+1}} \Gamma_{h}^q) \)-closed with respect to \( \sigma_n \).

\[\square\]

6 Basic Properties

As we have seen, Theorem 3.10(i) follows from Theorems 5.3 and 5.4. In the preceding section we have proved Theorem 5.4. The remainder of this paper is devoted to a proof of Theorem 5.3. In this section we make some technical preparations.

Let \( T \) be an ETT that admits a good hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) and satisfies MP with respect to the generating coloring \( \varphi_n \). To prove Theorem 5.3 (that is, \( V(T) \) is elementary with respect to \( \varphi_n \)), we apply induction on \( q \), and the induction base is Theorem 3.10(i) for \( T_n \). For convenience, we view \( T_{n,0} \) as an ETT with \(-1\) divider and \( n \) rungs in the following assumption. Throughout this section we assume that

(6.1) In addition to (5.1), Theorem 5.3 holds for every ETT that admits a good hierarchy and satisfies MP, with \( n \) rungs and at most \( q - 1 \) dividers, where \( q \geq 0 \).

Let us first prove two technical lemmas that will be used in the proof of Theorem 5.3.

Lemma 6.1. (Assuming (5.1)) Let \( T \) be an ETT constructed from a \( k \)-triple \((G,e,\varphi)\) by using the Tashkinov series \( T = \{\{T_i,\varphi_{i-1},S_{i-1},F_{i-1},\Theta_{i-1}\} : 1 \leq i \leq n + 1\}\). Suppose \( \Theta_n = PE \) and \( T \) enjoys MP under \( \varphi_n \). Let \( \sigma_n \) be a \((T_n \oplus R_n,D_n,\varphi_n)\)-stable coloring and let \( \alpha \) and \( \beta \) be two colors in \([k]\). Then the following statements hold:

(i) \( \alpha \) and \( \beta \) are \( R_n \)-interchangeable under \( \sigma_n \) if \( \alpha \in \sigma_n(R_n) \);

(ii) \( \alpha \) and \( \beta \) are \( T_n \)-interchangeable under \( \sigma_n \) if \( \alpha \in \sigma_n(T_n) \);

(iii) \( \alpha \) and \( \beta \) are \( T_n \cup R_n \)-interchangeable under \( \sigma_n \) if \( \alpha \in \sigma_n(T_n \cup R_n) \) is closed in \( T_n \cup R_n \) under \( \sigma_n \); and

(iv) \( \alpha \) and \( \beta \) are \( T_n \cup R_n \)-interchangeable under \( \sigma_n \) if \( \alpha \in \sigma_n(T_n) \) and \( \beta \in \sigma_n(R_n) \).

Proof. Since \( \sigma_n \) is a \((T_n \oplus R_n,D_n,\varphi_n)\)-stable coloring, it is \((T_n,D_n,\varphi_n)\)-stable by definition. Let \( j = m(v_n) \). It follows from (5.1) and Theorem 3.10(iii) that \( \sigma_n \) is a \((T_j(v_n) - v_n,D_{j-1},\varphi_{j-1})\)-stable coloring. So \( \sigma_n \) is \((T_{j-1},D_{j-1},\varphi_{j-1})\)-stable and hence, by (5.1) and Theorem 3.10(vi), it is a \( \varphi_{j-1} \mod T_{j-1} \) coloring. Furthermore, \( \sigma(f) = \varphi_n(f) \) for any edge \( f \) in \( T_n \cup R_n \) by (5.12) and \( \sigma_n(v) = \varphi_n(v) \) for all \( v \in V(T_n \cup R_n) \).

(i) Since \( R_n \) is a closure of \( T_n(v_n) \) under \( \varphi_n \) and \( \sigma_n \) is \((R_n,\emptyset,\varphi_n)\)-stable, \( R_n \) is also a closure of \( T_n(v_n) \) under \( \sigma_n \). Since \( R_n \) is obtained from \( T_{j-1} \) by using the same connecting edge, connecting color, and extension type as \( T_j \), by (5.1) and Theorem 3.10(vi), \( R_n \) is an ETT corresponding to \((\sigma_n,T_{j-1})\) and satisfies MP under \( \sigma_n \). Let \( \alpha \) and \( \beta \) be as specified in the lemma. As \( r(R_n) = j - 1 \), by (5.1) and Theorem 3.10(ii), there is at most one \((\alpha,\beta)\)-path with respect to \( \sigma_n \) intersecting \( R_n \). Hence \( \alpha \) and \( \beta \) are \( R_n \)-interchangeable under \( \sigma_n \).
Let us make some observations before proving statements (ii) and (iii). By (5.4), each color in $\mathcal{V}_n(T_n) \cap \mathcal{V}_n(R_n)$ is closed in $T_n \cup R_n$ with respect to $\varphi_n$. Since $\sigma_n$ is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, by definition we obtain

1. each color in $\mathcal{V}_n(T_n) \cap \mathcal{V}_n(R_n)$ is closed in $T_n \cup R_n$ under $\sigma_n$.
2. $\alpha$ and $\beta$ are $T_n$-interchangeable under $\sigma_n$ if $\alpha \in \mathcal{V}_n(T_n)$, $\alpha \neq \delta_n$, and $\beta \neq \delta_n$.

To justify this, note that $\alpha \neq \gamma_n$, because $\gamma_n \notin \mathcal{V}_n(T_n) = \sigma_n(T_n)$. So $\alpha \notin S_n$. Nevertheless, there are two possibilities for $\beta$.

Let us first consider the case when $\beta \neq \gamma_n$. Since $\sigma_n$ is $(T_n, D_n, \varphi_n)$-stable, $P_{\gamma_n}((\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$ by (5.1) and Theorem 3.10(iv). Define $\sigma'_n = \sigma_n/P_{\gamma_n}((\gamma_n, \delta_n, \sigma_n)$. By Lemma 3.5, $\sigma'_n$ is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. From (5.1) and Theorem 3.10(ii) we deduce that $\alpha$ and $\beta$ are $T_n$-interchangeable under $\sigma'_n$. So they are $T_n$-interchangeable under $\sigma_n$ because $\{\alpha, \beta\} \cap S_n = \emptyset$. It remains to consider the case when $\beta = \gamma_n$. In this case, $\gamma_n$ is the only edge in $\partial \varphi_n \gamma_n(T_n) = \partial \varphi_n \gamma_n(T_n)$ by Lemma 3.2(iv). Since $V(T_n)$ is elementary with respect to $\varphi_n$, it is also elementary with respect to $\sigma_n$. As $\partial \varphi_n \gamma_n(T_n) = \emptyset$, there is at most one $(\alpha, \gamma_n)$-path with respect to $\sigma_n$ intersecting $T_n$. So $\alpha$ and $\beta$ are $T_n$-interchangeable under $\sigma_n$. Thus (2) is established.

By (1), $\delta_n$ is closed in $T_n \cup R_n$ with respect to $\sigma_n$. So statement (ii) follows instantly from (2) and statement (iii).

(iii) Assume the contrary: there are at least two $(\alpha, \beta)$-paths $P_1$ and $P_2$ with respect to $\sigma_n$ intersecting $T_n \cup R_n$. We may assume that

3. $\alpha \in \mathcal{V}_n(T_n) \cap \mathcal{V}_n(R_n)$.

To justify this, let $A$ be the set of four ends of $P_1$ and $P_2$. Then at least two vertices from $A$ are outside $T_n \cup R_n$ because, by Lemma 5.1, $V(T_n \cup R_n)$ is elementary with respect to $\sigma_n$. Using (i), it is then routine to check that $P_1 \cup P_2$ contains two vertex-disjoint subpaths $Q_1$ and $Q_2$, which are $T_n$-exit paths with respect to $\sigma_n$. Let $u \in V(T_n) \cap V(R_n)$, let $\eta \in \mathcal{V}_n(u)$, and let $\sigma'_n = \sigma_n/(G - T_n \cup R_n, \alpha, \eta)$. By (1), $\eta$ is closed in $T_n \cup R_n$ with respect to $\sigma_n$; so is $\alpha$ by hypothesis. Hence $\sigma'_n$ is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, and $Q_1$ and $Q_2$ are two $T_n$-exit paths with respect to $\sigma'_n$. Since $P_u(\eta, \beta, \sigma'_n)$ contains at most one of $Q_1$ and $Q_2$, replacing $\sigma_n$ and $\alpha$ by $\sigma'_n$ and $\eta$, respectively, we obtain (3).

Let $v$ be a vertex in $V(T_n) \cap V(R_n)$ with $\alpha \in \mathcal{V}_n(v)$. Clearly, we may assume that $P_1 = P_v(\alpha, \beta, \sigma_n)$. By (i), we may further assume that $P_2$ is disjoint from $R_n$. $P_2$ intersects $T_n$. Therefore $\alpha$ and $\beta$ are not $T_n$-interchangeable under $\sigma_n$. Since $\gamma_n \notin \mathcal{V}_n(T_n) = \sigma_n(T_n)$, we have $\alpha \neq \gamma_n$. By (2), we may assume that $\alpha = \delta_n$ or $\beta = \delta_n$.

Suppose $\beta = \delta_n$. From Step 4 of Algorithm 3.1, it is clear that edges in $\partial \varphi_n \delta_n(T_n)$ are all incident to $V(T_n) \cap V(R_n)$ by Lemma 3.2(iv). Thus both $P_1$ and $P_2$ intersect $V(T_n) \cap V(R_n)$, contradicting statement (i).

Suppose $\alpha = \delta_n$. By (1), $\delta_n$ is closed in $T_n \cup R_n$ under $\sigma_n$. Since $V(T_n) \cap V(R_n)$ contains both ends of the uncolored edge $e$, there exists a color $\theta \in \mathcal{V}_n(T_n) \cap \mathcal{V}_n(R_n) - \{\delta_n\}$. Let $\sigma''_n = \sigma_n/(G - T_n \cup R_n, \delta_n, \theta)$. Then $\sigma''_n$ is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. From the existence of $P_1$ and $P_2$, we see that $\theta$ and $\beta$ are not $T_n \cup R_n$-interchangeable under $\sigma''_n$, contradicting our observation above the case $\alpha \neq \delta_n \neq \beta$.

(iv) Assume the contrary: there are at least two $(\alpha, \beta)$-paths $P_1$ and $P_2$ with respect to $\sigma_n$ intersecting $T_n \cup R_n$. Let $u$ be a vertex in $T_n$ with $\alpha \in \mathcal{V}_n(u)$ and let $v$ be a vertex in $R_n$ with $\beta \in \mathcal{V}_n(v)$. By (ii) (resp. (i)), $P_u(\alpha, \beta, \sigma_n)$ (resp. $P_v(\alpha, \beta, \sigma_n)$) is the only $(\alpha, \beta)$-path with respect to $\sigma_n$ intersecting $T_n$ (resp. $R_n$). Hence none of these four paths has an end in.
(4) $u \in V(T_n) - V(R_n)$, and $v \in V(R_n) - V(T_n)$. $P_1 = P_u(\alpha, \beta, \sigma_n)$, and $P_2 = P_v(\alpha, \beta, \sigma_n)$

By (4) and statement (ii), $P_1(\alpha, \beta, \sigma_n)$ is disjoint from $T_n$. Let $\sigma'_n = \sigma_n/P_v(\alpha, \beta, \sigma_n)$. By Lemma 5.8, $\sigma'_n$ is a $(T_n, D_n, \varphi_n)$-stable coloring. By Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to $\sigma_n$. Since $\alpha \in \sigma_n(u)$ and $\beta \in \sigma_n(v)$, from TAA we see that no edge in $R_n(v) - T_n(v)$ is colored by $\alpha$ or $\beta$ under both $\varphi_n$ and $\sigma_n$. Thus edges in $R_n(v) - T_n(v)$ are colored exactly the same under $\sigma'_n$ as under $\sigma_n$ and $\sigma_n(x) = \sigma'_n(x)$ for any $x \in V(R_n(v) - v)) \cup V(T_n)$. Let $R'_n$ be a closure of $T_n(v)$ under $\sigma'_n$. Then $v \in V(R'_n)$. In view of Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to $\sigma'_n$. However, $\alpha \in \sigma'_n(u) \cap \sigma'_n(v)$, a contradiction.

As introduced in Section 5, $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise. Throughout a coloring $\sigma_n \in \mathcal{C}^k(G - e)$ is called a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring if it is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring when $\Theta_n = PE$ and is a $(T_n, D_n, \varphi_n)$-stable coloring when $\Theta_n \neq PE$. By Lemma 3.2(iii) and (5.12), every $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring is $(T^*_{n,0}, \varphi_n)$-invariant. It follows from Lemma 2.4 that being $(T^*_{n,0}, D_n, \cdot)$-strongly stable is an equivalence relation on $\mathcal{C}^k(G - e)$.

**Lemma 6.2.** (Assuming (6.1)) Let $T$ be an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$, and let $\sigma_n$ be a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. Suppose $T'$ is an ETT obtained from $T^*_{n,0}$ corresponding to $(\sigma_n, T_n)$ (see Definition 3.7 and Theorem 3.10(vi)) that has a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p} = T'$, where $1 \leq p \leq q$ (see (6.1)). Suppose further that $T'$ is $(\cup m \in D_{n,p} \Gamma_{m-1}^-)$-closed with respect to $\sigma_n$. Let $\alpha \in \sigma_n(T')$ and $\beta \in [k] - \{\alpha\}$. If $\alpha$ is closed in $T'$ under $\sigma_n$, then $\alpha$ and $\beta$ are $T'$-interchangeable under $\sigma_n$.

A very useful corollary of this lemma is given below.

**Corollary 6.3.** (Assuming (6.1)) Let $T$ be an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$. Suppose $T$ has a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$. Let $p$ be a subscript with $1 \leq p \leq q$, and let $\alpha \in \sigma_n(T_{n,p})$ and $\beta \in [k] - \{\alpha\}$. If $\alpha$ is closed in $T_{n,p}$ under $\varphi_n$, then $\alpha$ and $\beta$ are $T_{n,p}$-interchangeable under $\varphi_n$.

**Proof of Lemma 6.2.** Assume the contrary: there are two $(\alpha, \beta)$-paths $Q_1$ and $Q_2$ with respect to $\sigma_n$ intersecting $T' = T_{n,p}$; subject to this, $p$ is minimum. Let us make some simple observations about $T'$ before proceeding. Since $\sigma_n$ is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, by Theorem 3.10(vi) we have

1. $T'$ satisfies MP under $\sigma_n$, and hence $V(T')$ is elementary with respect to $\sigma_n$ by (6.1) and Theorem 5.3.

By hypothesis, $\alpha$ is closed in $T'$ with respect to $\sigma_n$, which together with (1) implies that

2. $|T'|$ is odd.

In our proof we shall repeatedly use the following hypothesis:

3. $T'$ is $(\cup m \in D_{n,p} \Gamma_{m-1}^-)$-closed with respect to $\sigma_n$.

Depending on whether $\beta$ is contained in $\sigma_n(T')$, we consider two cases.

**Case 1.** $\beta \in \sigma_n(T')$. 41
In this case, \(|\partial_{\sigma_n,\beta}(T')|\) is even by (1) and (2). From the existence of \(Q_1\) and \(Q_2\), we see that \(G\) contains two vertex-disjoint \((T', \sigma_n, \{\alpha, \beta\})\)-exit paths \(P_1\) and \(P_2\). For \(i = 1, 2\), let \(a_i\) and \(b_i\) be the ends of \(P_i\) with \(b_i \in V(T')\). Renaming subscripts if necessary, we may assume that \(b_1 \prec b_2\).

We distinguish between two subcases according to the location of \(b_2\).

**Subcase 1.1.** \(b_2 \in V(T'_0) - V(T_{n,0}')\) if \(p = 1\) and \(b_2 \in V(T') - V(T_{n,p-1}')\) if \(p \geq 2\).

Since the edge on \(P_1\) incident to \(b_1\) is a boundary edge of \(T'\) and is colored by \(\beta\), we have \(\beta \in \Gamma_h^{-1}\) for some \(h\) with \(\eta_h \in D_n,p\) by (3), which together with Definition 5.2(i) implies that \(v_\beta \in V(T_{n,p-1})\), where \(v_\beta\) is the vertex in \(T'\) (see (1)) for which \(\beta \in \sigma_n(v_\beta)\). Let \(\gamma \in \sigma_n(b_2)\).

By the assumption of the present subcase and Definition 5.2(i), we have \(\gamma \notin \Gamma_p^{-1}\). Hence \(\gamma\) is closed with respect to \(\sigma_n\) in \(T'\) by (3). So

(4) both \(\alpha\) and \(\gamma\) are closed in \(T'\) under \(\sigma_n\).

Let \(\mu_1 = \sigma_n/(G - T', \alpha, \gamma)\). Clearly, \(\mu_1\) is a \((T_{n,0}', D_n, \varphi_n)\)-strongly stable coloring. By Lemma 5.8,

(5) the given hierarchy of \(T'\) remains good under \(\mu_1\), with the same \(\Gamma\)-sets as those under \(\sigma_n\) (see Definition 5.2). Furthermore, \(T' = (\bigcup_{\eta_h \in D_n,p} \Gamma_h^{p-1})^{-}\)-closed under \(\mu_1\).

Note that \(P_1\) and \(P_2\) are two \((T', \mu_1, \{\gamma, \beta\})\)-exit paths. Let \(\mu_2 = \mu_1/P_2(\gamma, \beta, \mu_1)\). Since \(P_2(\gamma, \beta, \mu_1) \cap T' = \{b_2\}\), all edges incident to \(V(T'\{b_2\} - b_2)\) are colored the same under \(\mu_2\) as under \(\mu_1\). Using (5) and Lemma 5.8, it is easy to see that

(6) \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_2)\) is a good hierarchy of \(T'(b_2)\) under \(\mu_2\), with the same \(\Gamma\)-sets as \(T'\) under \(\sigma_n\).

Clearly, \(\mu_2\) is a \((T_{n,0}', D_n, \varphi_n)\)-strongly stable coloring. By Theorem 3.10(vi), \(T'(b_2)\) satisfies MP under \(\mu_2\). Thus from (6.1) we conclude that \(V(T'(b_2))\) is elementary with respect to \(\mu_2\).

However, \(\beta \in \varphi_p(T_{n,p-1}) \cap \varphi_p(b_2)\), a contradiction.

**Subcase 1.2.** \(b_2 \in V(T_{n,0}')\) if \(p = 1\) and \(b_2 \in V(T_{n,p-1}')\) if \(p \geq 2\).

We propose to show that

(7) there exists a color \(\theta \in \sigma_n(T_n)\) that is closed in both \(T_{n,0}'\) and \(T_{n,1}\) under \(\sigma_n\) if \(p = 1\), and a color \(\theta \in \sigma_n(T_{n,p-1})\) that is closed in both \(T_{n,p-1}'\) and \(T_{n,p}\) under \(\sigma_n\) if \(p \geq 2\).

Our proof is based on the following simple observation (see (3) in the proof of Theorem 5.4).

(8) \(|\sigma_n(T_n)| \geq 2n + 11\) (as \(e\) is uncolored) and \(|D_n| \leq |D_n| \leq n\) for \(0 \leq i \leq q\).

Let us first assume that \(p = 1\). When \(\Theta_n \neq PE\), let \(\theta\) be a color in \(\sigma_n(T_n) - (\bigcup_{\eta_h \in D_n,1} \Gamma_h^0)\); such a color exists by (8). From Algorithm 3.1 we see that \(T_n\) is closed under \(\varphi_n\) and hence under \(\sigma_n\). By (3) and Definition 5.2(v), \(T_{n,1}\) is \((\bigcup_{\eta_h \in D_n,1} \Gamma_h^{p-1})^{-}\)-closed under \(\sigma_n\). So \(\theta\) is as desired. When \(\Theta_n = PE\), we have \(|\sigma_n(T_n) \cap \varphi_n(R_n - \Gamma^0)| \geq 4\) by Definition 5.2(iv). Let \(\theta \in \varphi_p(T_n) \cap \varphi_n(R_n - \Gamma^0 - \delta^0)\). Then \(\theta\) is closed in \(T_n \cap R_n\) under \(\varphi_n\) by (5.4). By the hypothesis of the present lemma, \(\sigma_n\) is a \((T_{n,0}', D_n, \varphi_n)\)-strongly stable coloring. So \(\theta \notin \Gamma^0\) and is closed in \(T_n \cap R_n\) under \(\sigma_n\). By Definition 5.2(v), \(T_{n,1}\) is \((\bigcup_{\eta_h \in D_n,1} \Gamma_h^0)^{-}\)-closed with respect to \(\sigma_n\). So \(\theta\) is also as desired.

Next we assume that \(p \geq 2\). By (8), we have \(|\sigma_n(T_{n,p-2})| \geq |\sigma_n(T_n)| \geq 2n + 11\) and \(|D_{n,p-2}| \leq |D_n| \leq n\). So there exists a color \(\theta\) in \(\sigma_n(T_{n,p-2}) - (\bigcup_{\eta_h \in D_n,p-1} \Gamma_h^{p-2})\). Since \(\sigma_n(T_{n,p-2}) \subseteq \sigma_n(T_{n,p-1})\), we have \(\theta \in \sigma_n(T_{n,p-1}) - (\bigcup_{\eta_h \in D_n,p-1} \Gamma_h^{p-2})\). By Definition 5.2(v), \(\theta\) is closed in \(T_{n,p-1}\) under \(\sigma_n\). From the definition of \(\theta\) and Definition 5.2(iii), we also see that \(\theta \notin \Gamma^{p-1}\). So \(\theta \in \sigma_n(T_p) - \Gamma^{p-1} \subseteq \sigma_n(T_{n,p}) - (\bigcup_{\eta_h \in D_n,p} \Gamma_h^{p-1})\). Again by Definition 5.2(v), \(\theta\) is closed in \(T_{n,p}\) under \(\sigma_n\). Hence (7) is established.
Let \( \mu_3 = \sigma_n / (G - T', \alpha, \theta) \). Since both \( \alpha \) and \( \theta \) are closed in \( T' \) with respect to \( \sigma_n \), by Lemma 5.8, \( \mu_3 \) is a \( (T^*_{n,0}, D_n, \varphi_n) \)-strongly stable coloring. Furthermore, \( T_{n,p} \) admits a good hierarchy and satisfies MP with respect to \( \mu_3 \). Thus \( T_{n,p-1} \) also admits a good hierarchy and satisfies MP with respect to \( \mu_3 \) if \( p \geq 2 \). By (7), \( \theta \) is closed in \( T^*_{n,0} \) if \( p = 1 \) and closed in \( T_{n,p-1} \) if \( p \geq 2 \) under \( \mu_3 \). Note that both \( P_1 \) and \( P_2 \) are \( (T^*_{n,p-1}, \mu_3, \{\theta, \beta\}) \)-exit paths. So \( \theta \) and \( \beta \) are not \( T^*_{n,0} \)-interchangeable under \( \mu_3 \) if \( p = 1 \) and not \( T_{n,p-1} \)-interchangeable under \( \mu_3 \) if \( p \geq 2 \), which contradicts Lemma 6.1(iii) or the minimality assumption on \( p \).

**Case 2.** \( \beta \notin \sigma_n(T') \).

In this case, \( |\partial \sigma_n, \beta(T')| \) is odd and at least three by (1) and (2). From the existence of \( Q_1 \) and \( Q_2 \), we see that \( G \) contains at least three \( (T, \sigma_n, \{\alpha, \beta\}) \)-exit paths \( P_1, P_2, P_3 \). For \( i = 1, 2, 3 \), let \( a_i \) and \( b_i \) be the ends of \( P_i \) with \( b_i \in V(T) \), and \( f_i \) be the edge of \( P_i \) incident to \( b_i \). Renaming subscripts if necessary, we may assume that \( b_1 < b_2 < b_3 \).

**Subcase 2.1.** \( b_3 \in V(T') - V(T^*_{n,0}) \) if \( p = 1 \) and \( b_3 \in V(T') - V(T_{n,p-1}) \) if \( p \geq 2 \).

For convenience, we call the tuple \( (\sigma_n, T', \alpha, \beta, P_1, P_2, P_3) \) a counterexample and use \( K \) to denote the set of all such counterexamples. With a slight abuse of notation, we still use \( (\sigma_n, T', \alpha, \beta, P_1, P_2, P_3) \) to denote a counterexample in \( K \) with the minimum \( |P_1| + |P_2| + |P_3| \).

Let \( \gamma \in \partial (b_3) \). By the hypothesis of the present subcase and Definition 5.2(i), we have \( \gamma \notin \Gamma^p \). So \( \gamma \) is closed in \( T' \) under \( \sigma_n \) by (3). Note that \( \gamma \) might be some \( \eta \in D_n \).

Let \( \mu_4 = \sigma_n / (G - T', \alpha, \gamma) \). By Lemma 5.8, \( \mu_4 \) is a \( (T^*_{n,0}, D_n, \varphi_n) \)-strongly stable coloring. Furthermore, \( T' \) admits a good hierarchy and satisfies MP under \( \mu_4 \). Note that \( P_1, P_2, P_3 \) are three \( (T', \mu_4, \{\gamma, \beta\}) \)-paths.

Consider \( \mu_5 = \mu_4 / P_{b_3}(\gamma, \beta, \mu_4) \). Clearly, \( \beta \in \partial_3 (b_3) \) and \( \beta \notin \Gamma^p \). Since \( P_{b_3}(\gamma, \beta, \mu_4) \cap T' = \{b_3\} \), it is easy to see that \( \mu_5 \) is a \( (T^*_{n,0}, D_n, \varphi_n) \)-strongly stable coloring. By (5.1) and Theorem 3.10(vi), \( T'(b_3) \) is an ETT satisfying MP under \( \mu_5 \). By Lemma 5.7 and Lemma 5.8, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_3) - b_3 \) is a good hierarchy of \( T'(b_3) - b_3 \) under \( \mu_5 \), with the same \( \Gamma \)-sets as \( T' \) under \( \sigma_n \) (see Definition 5.2).

(9) \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_3) \) is a good hierarchy of \( T'(b_3) \) under \( \mu_5 \), with the same \( \Gamma \)-sets as \( T' \) under \( \sigma_n \) (see Definition 5.2).

Let \( H \) be obtained from \( T'(b_3) \) by adding \( f_1 \) and \( f_2 \). Since \( \beta \notin \Gamma^p \), it can be seen from (9) that

(10) \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset H \) is a good hierarchy of \( H \) under \( \mu_5 \), with the same \( \Gamma \)-sets as \( T' \) under \( \sigma_n \).

By (5.1) and Theorem 3.10(vi), \( H \) satisfies MP under \( \mu_5 \). Set \( T'' = H \). Let us grow \( T'' \) by using the following algorithm:

(11) While there exists \( f \in \partial(T'') \) with \( \mu_5(f) \in \partial_3(T'') \), do: set \( T'' = T'' + f \) if \( \Gamma^p \cap \mu_5(T''(v_{\eta})) - T_{n,p-1} = \emptyset \) for all \( \eta \in D_{n,p-1} \).

Note that this algorithm is exactly the same as Step 2 in Algorithm 5.6. From (11) we see that

(12) \( T'' \) is \( (\cup_{\eta \in D_{n,p}} \Gamma^p_{\eta} - 1) \)-closed with respect to \( \mu_5 \), where \( D_{n,p} = \cup_{\eta \leq n} S_{\eta} - \partial_3(T'') \) (so \( D_{n,p} \subseteq D_{n,p-1} \)).

In view of (10) and (11), we conclude that

(13) \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'' \) is a good hierarchy of \( H \) under \( \mu_5 \), with the same \( \Gamma \)-sets as \( T' \) under \( \sigma_n \).
Clearly, $T''$ satisfies MP under $\mu_5$. By (13), (6.1), and Theorem 5.3, $V(T'')$ is elementary with respect to $\mu_5$. Observe that none of $a_1, a_2, a_3$ is contained in $T''$, for otherwise, let $a_i \in V(T_2)$ for some $i$ with $1 \leq i \leq 3$. Since $\{\beta, \gamma\} \cap \overline{\mu_5}(a_i) \neq \emptyset$ and $\beta \in \overline{\mu_5}(b_3)$, we obtain $\gamma \in \overline{\sigma_2}(a_i)$. Hence from TAA we see that $P_1, P_2, P_3$ are all entirely contained in $G[T'' \setminus \{a_i\}]$, which in turn implies $\gamma \in \overline{\sigma_2}(a_j)$ for $j = 1, 2, 3$. So $V(T'')$ is not elementary with respect to $\mu_5$, a contradiction.

Each $P_i$ contains a subpath $L_i$, which is a $T''$-exit path with respect to $\mu_5$. Since $f_1$ is not contained in $L_1$, we obtain $|L_1| + |L_2| + |L_3| < |P_1| + |P_2| + |P_3|$. Thus, in view of (12), the existence of the counterexample $(\mu_5, T'', \gamma, \beta, L_1, L_2, L_3)$ violates the minimality assumption on $(\sigma_n, T', \alpha, \beta, P_1, P_2, P_3)$.

**Subcase 2.2.** $b_3 \in V(T'_{n,0})$ if $p = 1$ and $b_3 \in V(T_{n,p-1})$ if $p \geq 2$.

The proof in this subcase is essentially the same as that in Subcase 1.2. Let $\theta$ be a color as described in (7). Consider $\mu_3 = \sigma_n/(G - T', \alpha, \theta)$. Then we can verify that $\theta$ and $\beta$ are not $T'_{n,0}$-interchangeable under $\mu_3$ if $p = 1$ and not $T_{n,p-1}$-interchangeable under $\mu_3$ if $p \geq 2$, which contradicts Lemma 6.1(iii) or the minimality assumption on $p$; for the omitted details, see the proof in Subcase 1.2.

Let us make some further preparations before proving Theorem 5.3. Let $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ be a good hierarchy of $T$ (see (5.5) and Definition 5.2). Recall that $T'_{n,0} = T_n \lor R_n$ if $\Theta_n = PE$ and $T'_{n,0} = T_n$ otherwise, $T'_{n,0} \subset T_{n,1}$ by (5.5), and $T'_{n,q} = T_{n,q}$ if $q \geq 1$. Let $T$ be constructed from $T'_{n,q}$ using TAA by recursively adding edges $e_1, e_2, \ldots, e_p$ and vertices $y_1, y_2, \ldots, y_p$, where $y_i$ is the end of $e_i$ outside $T(y_{i-1})$ for $i \geq 1$, with $T(y_0) = T'_{n,q}$. Write $T = T'_{n,q} \cup \{e_1, y_1, e_2, \ldots, e_p, y_p\}$. The path number of $T$, denoted by $p(T)$, is defined to be the smallest subscript $i \in \{1, 2, \ldots, p\}$ such that the sequence $(y_i, e_{i+1}, \ldots, e_p, y_p)$ corresponds to a path in $G$.

A coloring $\sigma_n \in C^k(G - e)$ is called a $(T'_{n,q}, D_n, \varphi_n)$-strongly stable coloring if it is both a $(T'_{n,q}, D_n, \varphi_n)$-strongly stable and a $(T'_{n,q}, \varphi_n)$-invariant coloring. Since every $(T'_{n,0}, D_n, \varphi_n)$-strongly stable coloring is $(T'_{n,0}, \varphi_n)$-invariant by Lemma 3.2(iii) and (5.12), this concept is a natural extension of $(T'_{n,0}, D_n, \varphi_n)$-strongly stable colorings. Let $v$ be a vertex of $G$. By $T \prec v$ we mean that $u \prec v$ for any $u \in V(T)$. Given a color $\alpha \in [k]$, we use $v_\alpha$ to denote the first vertex $u$ of $T$ in the order $\prec$ for which $\alpha \in \varphi_n(u)$, if any, and defined to be the last vertex of $T$ in the order $\prec$ otherwise.

Recall that our proof of Theorem 5.3 proceeds by induction on $q$ (see (6.1)). The induction step will be carried out by contradiction. Throughout the remainder of this section and Subsection 7.1, $(T, \varphi_n)$ stands for a minimum counterexample to Theorem 5.3; that is,

(6.2) $T$ is an ETT that admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1}$

in $T$ and satisfies MP with respect to the generating coloring $\varphi_n$;

(6.3) subject to (6.2), $V(T)$ is not elementary with respect to $\varphi_n$;

(6.4) subject to (6.2) and (6.3), $p(T)$ is minimum; and

(6.5) subject to (6.2)-(6.4), $|T| - |T_{n,q}|$ is minimum.

Our objective is to find another counterexample $(T', \sigma_n)$ to Theorem 5.3, which violates the minimality assumption (6.4) or (6.5) on $(T, \varphi_n)$.

The following fact will be used frequently in subsequent proof.

(6.6) $V(T(y_{p-1}))$ is elementary with respect to $\varphi_n$. 

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Let us exhibit some basic properties satisfied by the minimum counterexample \((T, \varphi_n)\) as specified above.

**Lemma 6.4.** For \(0 \leq i \leq p\), the inequality
\[
|\varphi_n(T(y_i))| - |\varphi_n(T_{n,0}^* - V(T_n))| - |\varphi_n(T(y_i) - T_{n,q}^*)| \geq 2n + 11
\]
holds, where \(T(\theta_n) = T_{n,q}^*\). Furthermore, if
\[
|\varphi_n(T(y_i))| - |\varphi_n(T_{n,0}^* - V(T_n))| - |\varphi_n(T(y_i) - T_{n,q}^*)| - |\Gamma_q \cup D_{n,q}| \leq 4,
\]
then there exist 7 distinct colors \(\eta_h \in D_{n,q} \cap \varphi_n(T(y_i))\) such that \((\Gamma_q \cup \{\eta_h\}) \cap \varphi_n(T(y_i) - T_{n,q}^*) = \emptyset\), where \(\Gamma_q\) and \(\Gamma_h\) are introduced in Definition 5.2.

**Proof.** Since the number of vertices in \(T(y_i) - V(T_{n,q}^*)\) is \(i\), and the number of edges in \(T(y_i) - T_{n,q}^*\) is also \(i\), we obtain \(|\varphi_n(T(y_i) - V(T_{n,q}^*))| \geq |\varphi_n(T(y_i) - T_{n,q}^*)|\). Hence
\[
|\varphi_n(T_{n,0}^*)| - |\varphi_n(T_{n,0}^* - V(T_n))| - |\varphi_n(T(y_i) - T_{n,q}^*)| \geq 2n + 11,
\]
where the last inequality can be found in the proof of Theorem 5.4 (see (3) therein). So the first inequality is established.

Suppose the second inequality also holds. Then these two inequalities guarantee the existence of at least \(2n + 7\) colors in the intersection of \(\varphi_n(T(y_i)) - \varphi_n(T_{n,0}^* - V(T_n))\) and \(\Gamma_q \cup D_{n,q}\). Let \(C\) denote this intersection. Then \(|C| \geq 2n + 7\). By (5.6), we have \(|D_{n,q}| \leq |D_n| \leq n\) and \(|\Gamma_q| \leq 2|D_{n,q}| \leq 2n\). So \(|\Gamma_q \cup D_{n,q}| \leq 3n\). Since \(|\Gamma_q| \leq |\Gamma_q \cup D_{n,q}|\), it follows that \(2n + 7 \leq 3n\), which implies \(n \geq 7\). Note that \(C = \bigcup_{\eta_h \in D_{n,q}} (\Gamma_q \cup \{\eta_h\}) \cap C\) and \(|(\Gamma_q \cup \{\eta_h\}) \cap C| \leq 3\) for any \(\eta_h\) in \(D_{n,q}\). Since \(|C| \geq 2n + 7\) and \(n \geq 7\), by the Pigeonhole Principle, there exist at least 7 distinct colors \(\eta_h\) in \(D_{n,q}\), such that \(|(\Gamma_q \cup \{\eta_h\}) \cap C| = 3\), or equivalently, \((\Gamma_q \cup \{\eta_h\}) \cap C \subseteq C\). For each of these \(\eta_h\), clearly \(\eta_h \in D_{n,q} \cap \varphi_n(T(y_i))\) and \((\Gamma_q \cup \{\eta_h\}) \cap \varphi_n(T(y_i) - T_{n,q}^*) = \emptyset\).

**Lemma 6.5.** Suppose \(q \geq 1\) and \(\alpha \in \varphi_n(T_{n,q})\). If there exists a subscript \(i\) with \(0 \leq i \leq q\), such that \(\alpha\) is closed in \(T_{n,i}^*\) with respect to \(\varphi_n\), then \(\alpha \notin \varphi_n(T_{n,q}^* - T_{n,r})\), where \(r\) is the largest such \(i\). If there is no such subscript \(i\), then \(\alpha \in \bigcup_{\eta_h \in D_{n,q}} (\Gamma_q \cup \{\eta_h\}) \subseteq \Gamma_{j-1}^r\) for \(1 \leq j \leq q\), \(\Theta_n = PE\), \(v_\alpha \in V(T_n) - V(R_n)\), and \(\alpha \notin \varphi_n(T_{n,q} - T_n)\).

**Proof.** Let us first assume the existence of a subscript \(i\) with \(0 \leq i \leq q\), such that \(\alpha\) is closed in \(T_{n,i}^*\) with respect to \(\varphi_n\). By definition, \(r\) is the largest such \(i\). Suppose the contrary: \(\alpha \in \varphi_n(T_{n,q} - T_{n,r})\). Then \(r < q\) and there exists a subscript \(r + 1 \leq s \leq q\) such that \(\alpha \in \varphi_n(T_{n,s} - T_{n,s-1})\). From the definition of \(r\), we see that \(\alpha\) is not closed in \(T_{n,s}\) with respect to \(\varphi_n\). It follows from Definition 5.2(v) that \(\alpha \in \Gamma_{h-1}^s\) for some \(\eta_h \in D_{n,s}\). By the definitions of
$D_{n,s}$ and $D_{n,s-1}$, we have $D_{n,s} \subseteq D_{n,s-1}$. So $\eta_h \in D_{n,s-1}$. Since $\alpha$ in $\Gamma_{h}^{s-1}$ is used by at least one edge in $T_{n,s} - T_{n,s-1}^{*}$, from Definition 5.2(i) (with $j = s - 1$) we deduce that $\eta_h$ is a color missing at some vertex in $T_{n,s}$ (see (5.7) and (5.9)). Thus $\eta_h \notin D_{n,s}$ by definition, a contradiction.

Next we assume that there exists no subscript $i$ with $0 \leq i \leq q$, such that $\alpha$ is closed in $T_{n,i}$ with respect to $\varphi_n$. Since $\alpha \in \varphi_n(T_{n,q})$, it follows from (5.10) that $\alpha \in \varphi_n(T_{n,0})$. By Definition 5.2(v), we obtain

(1) $\alpha \in \cup_{h \in D_{n,s}} \Gamma_{h}^{j-1} \subseteq \Gamma_{j-1}^{j-1}$ for $1 \leq j \leq q$.

Hence $\alpha \in \Gamma_{j}^{j}$ for all $0 \leq j \leq q - 1$. From the definition of $\Gamma_{0}$, we see that $v_{a} \in V(T_{n})$. By the assumption on $\alpha$, Algorithm 3.1 and (5.4), we further obtain $\Theta_{n} = PE$ and $v_{a} \in V(T_{n}) - V(R_{n})$.

Since $R_{n}$ is a closure of $T_{n}(v_{n})$ under $\varphi_n$, using (6.6) and TAA we get

(2) $\alpha \notin \varphi_n(T_{n,q} - V(T_{n}))$ and $\alpha \notin \varphi_n(R_{n} - T_{n})$.

(3) $\alpha \notin \varphi_n(T_{n,q} - T_{n,0})$.

Assume the contrary: $\alpha \in \varphi_n(T_{n,q} - T_{n,0})$. Then there exists a subscript $1 \leq s \leq q$ such that $\alpha \in \varphi_n(T_{n,s} - T_{n,s-1}^{*})$. By (1), we have $\alpha \in \Gamma_{h}^{s-1}$ for some $\eta_h \in D_{n,s}$. As $D_{n,s} \subseteq D_{n,s-1}$, we obtain $\eta_h \in D_{n,s-1}$. Since $\alpha$ is used by at least one edge in $T_{n,s} - T_{n,s-1}^{*}$, from Definition 5.2(i) (with $j = s - 1$) we deduce that $\eta_h$ is a color missing at some vertex in $T_{n,s}$ (see (5.7) and (5.9)). Thus $\eta_h \notin D_{n,s}$ by definition, a contradiction.

Combining (2) and (3), we conclude that $\alpha \notin \varphi_n(T_{n,q} - T_{n})$.  

Our proof of Theorem 5.3 relies heavily on the following two technical lemmas.

**Lemma 6.6.** Let $\alpha$ and $\beta$ be two colors in $\varphi_n(T_{n,q}(y_{p-1}))$. Suppose $v_{a} \prec v_{b}$ and $\alpha \notin \varphi_n(T_{n,q} - T_{n,q}^{*})$ if $\{\alpha, \beta\} - \varphi_n(T_{n,q}) \neq \emptyset$. Then $P_{v_{a}}(\alpha, \beta, \varphi_n) = P_{v_{b}}(\alpha, \beta, \varphi_n)$ if one of the following cases occurs:

(i) $q \geq 1$, and $\alpha \in \varphi_n(T_{n,q})$ or $\{\alpha, \beta\} \cap D_{n,q} = \emptyset$;

(ii) $q = 0$, and $\alpha \in \varphi_n(T_{n,0})$ or $\{\alpha, \beta\} \cap D_{n} = \emptyset$; and

(iii) $\alpha \in \varphi_n(T_{n,q}^{*})$ and is closed in $T_{n,q}^{*}$ with respect to $\varphi_n$.

Furthermore, in Case (iii), $P_{v_{a}}(\alpha, \beta, \varphi_n) = P_{v_{b}}(\alpha, \beta, \varphi_n)$ is the only $(\alpha, \beta)$-path with respect to $\varphi_n$ intersecting $T_{n,q}^{*}$.

**Proof.** Let $a = v_{a}$ and $b = v_{b}$. We distinguish among three cases according to the locations of $a$ and $b$.

**Case 1.** $\{a,b\} \subseteq V(T_{n,q}^{*})$.

By (6.6), $V(T_{n,q}^{*})$ is elementary with respect to $\varphi_n$. So $a$ (resp. $b$) is the only vertex in $T_{n,q}^{*}$ missing $\alpha$ (resp. $\beta$). If both $\alpha$ and $\beta$ are closed in $T_{n,q}^{*}$ with respect to $\varphi_n$, then no boundary edge of $T_{n,q}^{*}$ is colored by $\alpha$ or $\beta$. Hence $P_{a}(\alpha, \beta, \varphi_n) = P_{b}(\alpha, \beta, \varphi_n)$ is the only path intersecting $T_{n,q}^{*}$. We may assume that $\alpha$ or $\beta$ is not closed in $T_{n,q}^{*}$ with respect to $\varphi_n$. It follows that if $q = 0$, then $\Theta_{n} = PE$, for otherwise, Algorithm 3.1 would imply that both $\alpha$ and $\beta$ are closed in $T_{n} = T_{n,0}^{*}$, a contradiction. Therefore

(1) $T_{n,0}^{*} = T_{n} \vee R_{n}$ if $q = 0$.

Let us first assume that precisely one of $\alpha$ and $\beta$ is closed in $T_{n,q}^{*}$ with respect to $\varphi_n$. In this subcase, by Corollary 6.3 if $q \geq 1$ and by (1) and Lemma 6.1(iii) if $q = 0$, colors $\alpha$ and $\beta$ are $T_{n,q}^{*}$-interchangeable under $\varphi_n$, so $P_{a}(\alpha, \beta, \varphi_n) = P_{b}(\alpha, \beta, \varphi_n)$ is the only path intersecting $T_{n,q}^{*}$.
Next we assume that neither \( \alpha \) nor \( \beta \) is closed in \( T_{n,q}^* \) with respect to \( \varphi_n \). In this subcase, we only need to show that \( P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n) \). Symmetry allows us to assume that \( a < b \).

Let \( r \) be the subscript with \( \beta \in \overline{\varphi}_n(T_{n,r}^* - V(T_{n,r-1}^*)) \), where \( 0 \leq r \leq q \) and \( T_{n,-1}^* = \emptyset \). Then \( a, b \in V(T_{n,r}^*) \). By (6.2), \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) is a good hierarchy of \( T \). If \( r \geq 1 \), then \( \beta \) is closed in \( T_{n,r} \) with respect to \( \varphi_n \) by Definition 5.2 (see (5.10)). From the above discussion about \( T_{n,q}^* \) (with \( r \) in place of \( q \)), we similarly deduce that \( P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n) \).

So we may assume that \( r = 0 \). If \( \theta_n \neq PE \), then both \( \alpha \) and \( \beta \) are closed in \( T_n \) with respect to \( \varphi_n \) (see Algorithm 3.1), so \( P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n) \) by (6.6). If \( \Theta_n = PE \), then it follows from Lemma 6.1(i), (ii) and (iv) that \( P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n) \).

**Case 2.** \( \{a, b\} \cap V(T_{n,q}^*) = \emptyset \).

By the hypotheses of the present case and the present lemma, we have \( \{\alpha, \beta\} \cap D_{n,q} = \emptyset \) if \( q \geq 1 \) and \( \{\alpha, \beta\} \cap D_n = \emptyset \) if \( q = 0 \). So

\[
(2) \quad \alpha, \beta \notin D_{n,q} \cup \overline{\varphi}_n(T_{n,q}^*) \text{ if } q \geq 1 \text{ and } \alpha, \beta \notin D_n \cup \overline{\varphi}_n(T_{n,0}^*) \text{ if } q = 0.
\]

Since \( \alpha \notin \varphi_n(T(b) - T_{n,q}^*) \), from (2) and Lemma 3.2(ii) we see that

\[
(3) \quad \alpha, \beta \notin \varphi_n(T(b)).
\]

Suppose on the contrary that \( P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n) \). Consider \( \sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n) \). Using (2) and (3), it is routine to check that \( \sigma_n \) is a \( (T_{n,q}^*, D_n, \varphi_n) \)-strongly stable coloring, and \( T(b) \) is an ETT satisfying MP with respect to \( \sigma_n \). Moreover, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b) \) is a good hierarchy of \( T(b) \) under \( \sigma_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \) (see Definition 5.2). As \( \alpha \in \sigma_n(a) \cap \sigma_n(b) \), the pair \( (T(b), \sigma_n) \) is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on \( (T, \varphi_n) \).

**Case 3.** \( a \in V(T_{n,q}^*) \) and \( b \notin V(T_{n,q}^*) \).

By the hypotheses of the present case and the present lemma, (6.6) and TAA, we obtain

\[
(4) \quad \alpha \notin \varphi_n(T(b) - T_{n,q}^*) \text{ and } \beta \notin \overline{\varphi}_n(T(b) - b) \text{. So } \beta \text{ is not used by any edge in } T(b) - T_{n,q}^*, \text{ except possibly } e_1 \text{ when } q = 0 \text{ and } T_{n,0}^* = T_n \text{ (now } e_1 = f_n \text{ in Algorithm 3.1 and } \beta \in D_n). \]

Let us first assume that \( \alpha \) is closed in \( T_{n,q}^* \) with respect to \( \varphi_n \). By Corollary 6.3 if \( q \geq 1 \) and by Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)) if \( q = 0 \), colors \( \alpha \) and \( \beta \) are \( T_{n,q}^* \)-interchangeable under \( \varphi_n \). So \( P_a(\alpha, \beta, \varphi_n) \) is the only \( (\alpha, \beta) \)-path intersecting \( T_{n,q}^* \). Suppose on the contrary that \( P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n) \). Then \( P_b(\alpha, \beta, \varphi_n) \) is vertex-disjoint from \( T_{n,q}^* \) and hence contains no edge incident to \( T_{n,q}^* \).

Consider \( \sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n) \). Using (4), it is routine to check that \( \sigma_n \) is a \( (T_{n,q}^*, D_n, \varphi_n) \)-strongly stable coloring, and \( T(b) \) is an ETT satisfying MP with respect to \( \sigma_n \). Moreover, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b) \) is a good hierarchy of \( T(b) \) under \( \sigma_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \). As \( \alpha \in \sigma_n(a) \cap \sigma_n(b) \), the pair \( (T(b), \sigma_n) \) is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on \( (T, \varphi_n) \).

So we assume hereafter that

\[
(5) \quad \alpha \text{ is not closed in } T_{n,q}^* \text{ with respect to } \varphi_n. \]

Our objective is to show that \( P_a(\alpha, \beta, \varphi_n) = P_b(\alpha, \beta, \varphi_n) \). Assume the contrary: \( P_a(\alpha, \beta, \varphi_n) \neq P_b(\alpha, \beta, \varphi_n) \). We distinguish between two subcases according to the value of \( q \).

**Subcase 3.1.** \( q = 0 \).

By the hypothesis of the present lemma, \( \alpha \in \overline{\varphi}_n(T_n) \) or \( \{\alpha, \beta\} \cap D_n = \emptyset \). So \( \alpha \notin D_n \). From (5) and Algorithm 3.1 we deduce that \( T_{n,0} \neq T_n \). Hence

\[
(6) \quad \Theta_n = PE, \text{ which together with (5) and (5.4) yields } a \notin V(T_n) \cap V(R_n).
\]

Consider \( \sigma_n = \varphi_n/P_b(\alpha, \beta, \varphi_n) \). We claim that
(7) $\sigma_n$ is a $(T_{n,0}^*, D_n, \varphi_n)$-strongly stable coloring.

To justify this, note that if $a \in V(T_n) - V(R_n)$, then $\alpha, \beta \notin \varphi_n(R_n)$ by (6.6) and the hypothesis of the present case. In view of Lemma 6.1(ii), $P_b(\alpha, \beta, \varphi_n)$ is disjoint from $T_n$ and hence contains no edge incident to $T_n$. From TAA we see that no edge in $R_n - T(v_n)$ is colored by $\alpha$ or $\beta$ under $\varphi_n$. So (7) holds. Suppose $a \in V(R_n) - V(T_n)$. By the hypothesis of the present lemma, $\{\alpha, \beta\} \cap D_n = \emptyset$. By (6.6), we also have $\alpha, \beta \notin \varphi_n(T_n)$. Thus $\alpha, \beta \notin \varphi_n(R_n) \cup D_n$. From Lemma 3.2(ii) it follows that no edge in $T_n$ is colored by $\alpha$ or $\beta$. Using Lemma 6.1(i), $P_b(\alpha, \beta, \varphi_n)$ is disjoint from $R_n$ and hence contains no edge incident to $R_n$. Thus (7) also holds.

From (4), (7) and (6.6) we see that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma_n}(u) = \overline{\varphi_n}(u)$ for each $u \in V(T(b) - b)$. So $T(b)$ is an ETT satisfying MP with respect to $\sigma_n$, and $T_n = T_{n,0}^* \subset T(b)$ is a good hierarchy of $T(b)$ under $\sigma_n$, with the same $\Gamma$-sets as $T$ under $\varphi_n$. As $\alpha \in \varphi_n(a) \cap \varphi_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on $(T, \varphi_n)$.

**Subcase 3.2.** $q \geq 1$.

Let us first assume that $\alpha$ is closed in $T_{n,i}^*$ with respect to $\varphi_n$ for some $i$ with $0 \leq i \leq q$. Let $r$ be the largest subscript $i$ with this property. Then $r \leq q - 1$ by (5). By Lemma 6.5, we have $\alpha \notin \varphi_n(T_{n,q} - T_{n,r}^*)$, which together with (4) yields

$$\text{(8) } \alpha \notin \varphi_n(T(b) - T_{n,r}^*).$$

By Corollary 6.3 if $r \geq 1$ and by Theorem 3.10(ii) or Lemma 6.1(iii) if $r = 0$, colors $\alpha$ and $\beta$ are $T_{n,r}^*$-interchangeable under $\varphi_n$. So $P_{b}(\alpha, \beta, \varphi_n)$ is the only $(\alpha, \beta)$-path with respect to $\varphi_n$ intersecting $T_{n,r}^*$. Hence $P_{b}(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T_{n,r}^*$ and therefore contains no edge incident to $T_{n,r}^*$. Consider $\sigma_n = \varphi_n/P_{b}(\alpha, \beta, \varphi_n)$. By Lemma 5.8, $\sigma_n$ is a $(T_{n,r}^*, D_n, \varphi_n)$-strongly stable coloring, and $T_{n,r}^*$ is an ETT having a good hierarchy and satisfying MP with respect to $\sigma_n$. By (4) and TAA, $\beta$ is not used by any edge in $T(b) - T_{n,r}^*$, except possibly $e_1$ when $r = 0$ and $T_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$). It follows from (8), the above discussion and (6.6) that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma_n}(u) = \overline{\varphi_n}(u)$ for each $u \in V(T(b) - b)$. So $T(b)$ is an ETT satisfying MP with respect to $\sigma_n$. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of $T(b)$ under $\sigma_n$, with the same $\Gamma$-sets as $T$ under $\varphi_n$. As $\alpha \in \varphi_n(a) \cap \varphi_n(b)$, the existence of $(T(b), \sigma_n)$ contradicts the minimality assumption (6.5) on $(T, \varphi_n)$.

Next we assume that $\alpha$ is not closed in $T_{n,i}^*$ with respect to $\varphi_n$ for any $i$ with $0 \leq i \leq q$. In view of Lemma 6.5, we obtain

$$\text{(9) } \alpha \in \bigcup_{h \in D_{n,j}} \Gamma^{-1}_h \subseteq \Gamma^{-1}_j \text{ for } 1 \leq j \leq q, \Theta_n = PE, a \in V(T_n) - V(R_n), \text{ and }$$
$$\alpha \notin \varphi_n(T_{n,q} - T_n).$$

It follows from (4), (9) and TAA that

$$\text{(10) } \alpha \notin \varphi_n(T(b) - T_n) \text{ and } \beta \notin \varphi_n(T(b) - T_{n,0}).$$

Since $R_n$ is a closure of $T_n(v_n)$ under $\varphi_n$, using (9), (6.6) and TAA we obtain

$$\text{(11) } \alpha, \beta \notin \varphi_n(R_n) \text{ and } \beta \notin \varphi_n(R_n - T_n).$$

By Lemma 6.1(ii), colors $\alpha$ and $\beta$ are $T_n$-interchangeable under $\varphi_n$. So $P_{a}(\alpha, \beta, \varphi_n)$ is the only $(\alpha, \beta)$-path with respect to $\varphi_n$ intersecting $T_n$. Hence $P_{b}(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T_n$ and therefore contains no edge incident to $T_n$. Consider $\sigma_n = \varphi_n/P_{b}(\alpha, \beta, \varphi_n)$.

---

1. By the definition of $r$, color $\alpha$ is not closed in $T_{n,t}$ with respect to $\varphi_n$ for any $t$ with $r + 1 \leq t \leq q$. It may become closed in $T_{n,t}$ with respect to $\sigma_n$ for some of these $t$. Yet, even in this situation Definition 5.2(v) remains valid with respect to $\sigma_n$. 48
Lemma 6.7. Let \( \alpha \) and \( \beta \) be two colors in \( \overline{\varphi}_n(T(y_{p-1})) \), let \( Q \) be an \((\alpha, \beta)\)-chain with respect to \( \varphi_n \), and let \( \sigma_n = \varphi_n/Q \). Suppose one of the following cases occurs:

1) \( q \geq 1 \), \( \alpha \in \overline{\varphi}_n(T_{n,q}) \), and \( Q \) is an \((\alpha, \beta)\)-path disjoint from \( P_{v_\alpha}(\alpha, \beta, \varphi_n) \);

2) \( q = 0 \), \( \alpha \in \overline{\varphi}_n(T_n) \), or \( \alpha \in \overline{\varphi}_n(T_{n,0}^*) \) with \( \alpha, \beta \notin D_n \), and \( Q \) is an \((\alpha, \beta)\)-path disjoint from \( P_{v_\alpha}(\alpha, \beta, \varphi_n) \);

3) \( T_{n,q} \prec v_\alpha \prec v_\beta \), \( \alpha, \beta \notin D_{n,q} \), \( \alpha \notin \varphi_n(T(v_\beta) - T(v_\alpha)) \), and \( Q \) is an arbitrary \((\alpha, \beta)\)-chain.

Then the following statements hold:

(i) \( \sigma_n \) is a \((T_{n,q}^*, D_n, \varphi_n^\ast)\)-strongly stable coloring;

(ii) \( T_{n,q}^* \) is an ETT satisfying MP with respect to \( \sigma_n \); and

(iii) if \( q \geq 1 \), then \( T_{n,q} \) admits a good hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \) under \( \sigma_n \), with the same \( \Gamma \)-sets (see Definition 5.2) as \( T \) under \( \varphi_n \), and \( T_{n,q} \) is \((\cup_{y \in D_n} \Gamma_y^{-1})^{-1}\)-closed with respect to \( \sigma_n \).

Furthermore, in Case 3, \( T \) is also an ETT satisfying MP with respect to \( \sigma_n \), and \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a hierarchy of \( T \) under \( \sigma_n \), with the same \( \Gamma \)-sets (see Definition 5.2) as \( T \) under \( \varphi_n \).

Proof. Write \( a = v_\alpha \) and \( b = v_\beta \). Let us consider the three cases described in the lemma separately.

Case 1. \( q \geq 1 \), \( \alpha \in \overline{\varphi}_n(T_{n,q}) \), and \( Q \) is an \((\alpha, \beta)\)-path disjoint from \( P_{v_\alpha}(\alpha, \beta, \varphi_n) \).

We consider two subcases according to the location of \( b \).

Subcase 1.1. \( b \in V(T_{n,q}) \).

Let us first assume that there exists a subscript \( i \) with \( 0 \leq i \leq q \), such that \( \alpha \) or \( \beta \) is closed in \( T_{n,i}^* \) with respect to \( \varphi_n \). Let \( r \) be the largest such \( i \). By Lemma 6.5, we have

1) \( \{a, b\} \subseteq V(T_{n,r}^*) \) and \( \alpha, \beta \notin \varphi_n(T_{n,q} - T_{n,r}^*) \);

2) \( \alpha \) and \( \beta \) are \( T_{n,r}^* \)-interchangeable under \( \varphi_n^* \). So \( P_{v_\alpha}(\alpha, \beta, \varphi_n^*) = P_{v_\beta}(\alpha, \beta, \varphi_n^*) \).

To justify this, note that if \( r \geq 1 \), then (2) holds by Corollary 6.3. So we assume that \( r = 0 \). Then \( \alpha \) or \( \beta \) is closed in \( T_{n,0}^* \) with respect to \( \varphi_n \). Hence, by Lemma 6.1(iii) if \( \Theta_n = PE \) and by (5.1) and Theorem 3.10(ii) otherwise, \( \alpha \) and \( \beta \) are \( T_{n,0}^* \)-interchangeable under \( \varphi_n \). This proves (2).

It follows from (2) that \( Q \) is vertex-disjoint from \( T_{n,r}^* \) and hence contains no edge incident to \( T_{n,r}^* \). By Lemma 5.8, \( \sigma_n = \varphi_n/Q \) is a \((T_{n,r}^*, D_n, \varphi_n^*)\)-strongly stable coloring, and \( T_{n,r}^* \) is an ETT satisfying MP with respect to \( \sigma_n \). By (1), we obtain \( \sigma_n(f) = \varphi_n(f) \) for each edge \( f \) of \( T_{n,q} \) and \( \overline{\sigma}_n(u) = \overline{\varphi}_n(u) \) for each vertex \( u \) of \( T_{n,q} \). Therefore \( \sigma_n \) is a \((T_{n,q}, D_n, \varphi_n)\)-strongly stable coloring. It is then routine to check that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \) is a good hierarchy of
\( T_{n,q} \) under \( \sigma_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n^2 \), and \( T_{n,q} \) is \((\cup_{\eta_h \in D_{n,q}} \Gamma_{h}^{\varphi_n^2})^{-}\)-closed with respect to \( \sigma_n \).

Next we assume that there exists no subscript \( i \) with \( 0 \leq i \leq q \), such that \( \alpha \) or \( \beta \) is closed in \( T_{n,i}^* \) with respect to \( \varphi_n \). By Lemma 6.5, we have

\[
(3) \quad \alpha, \beta \in (\cup_{\eta_h \in D_{n,i}} \Gamma_{h}^{\varphi_n^2}) \subseteq \Gamma_{\varphi_n^2}^{\varphi_n^2} \text{ for } 1 \leq j \leq q, \quad \Theta_n = PE, \quad v_\alpha, v_\beta \in V(T_n) - V(R_n), \quad \text{and } \alpha, \beta \notin \varphi_n(T_{n,q} - T_n).
\]

Since \( R_n \) is a closure of \( T_n(v_n) \) under \( \varphi_n \), using (6.6) and TAA we obtain

\[
(4) \quad \alpha, \beta \notin \varphi_n(R_n).
\]

By Lemma 6.1(iii), colors \( \alpha \) and \( \beta \) are \( T_n \)-interchangeable under \( \varphi_n \). So \( P_n(\alpha, \beta, \varphi_n) \) is the only \((\alpha, \beta)\)-path with respect to \( \varphi_n \) intersecting \( T_n \). Hence \( Q \) is vertex-disjoint from \( T_n \) and therefore contains no edge incident to \( T_n \). By Lemma 5.8, \( \sigma_n = \varphi_n/Q \) is a \((T_n, D_n, \varphi_n)\)-stable coloring, and \( T_n \) is an ETT satisfying MP with respect to \( \sigma_n \). By (3), (4) and (6.6), we further deduce that \( \sigma_n \) is a \((T_n^*, D_n, \varphi_n)\)-stable coloring, \( \sigma_n(f) = \varphi_n(f) \) for each edge \( f \) of \( T_{n,q} \), and \( \sigma_n(u) = \varphi_n(u) \) for each vertex \( u \) of \( T_{n,q} \). It is then routine to check that the desired statements hold.

**Subcase 1.2.** \( b \notin V(T_{n,q}) \).

Let us first assume that there exists a subscript \( i \) with \( 0 \leq i \leq q \), such that \( \alpha \) is closed in \( T_{n,i}^* \) with respect to \( \varphi_n \). Let \( r \) be the largest such \( i \). By Lemma 6.5 and TAA, we have

\[
(5) \quad a \subseteq V(T_{n,r}^*) \text{ and } \alpha \notin \varphi_n(T_{n,q} - T_{n,r}^*). \quad \text{Furthermore, no edge in } T_{n,q} - T_{n,r}^* \text{ is colored by } \beta, \text{ except possibly } e_1 \text{ when } r = 0 \text{ and } T_{n,0}^* = T_n \text{ (now } e_1 = f_n \text{ in Algorithm 3.1 and } \beta \in D_n). \quad \text{Using the same argument of (2), we obtain}
\]

\[
(6) \quad \alpha \text{ and } \beta \text{ are } T_{n,r}^* \text{-interchangeable under } \varphi_n.
\]

It follows from (6) that \( Q \) is vertex-disjoint from \( T_{n,r}^* \) and hence contains no edge incident to \( T_{n,r}^* \). By Lemma 5.8, \( \sigma_n = \varphi_n/Q \) is a \((T_{n,r}^*, D_n, \varphi_n)\)-strongly stable coloring, and \( T_{n,r}^* \) is an ETT satisfying MP with respect to \( \sigma_n \). Using (5), we obtain \( \sigma_n(f) = \varphi_n(f) \) for each edge \( f \) of \( T_{n,q} \) and \( \sigma_n(u) = \varphi_n(u) \) for each vertex \( u \) of \( T_{n,q} \). Therefore \( \sigma_n \) is a \((T_{n,q}, D_n, \varphi_n)\)-strongly stable coloring, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \) is a good hierarchy of \( T_{n,q} \) under \( \sigma_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \), and \( T_{n,q} \) is \((\cup_{\eta_h \in D_{n,q}} \Gamma_{h}^{\varphi_n^2})^{-}\)-closed with respect to \( \sigma_n \).

Next we assume that there exists no subscript \( i \) with \( 0 \leq i \leq q \), such that \( \alpha \) is closed in \( T_{n,i}^* \) with respect to \( \varphi_n \). By Lemma 6.5, we have

\[
(7) \quad \alpha \in (\cup_{\eta_h \in D_{n,j}} \Gamma_{h}^{\varphi_n^2}) \subseteq \Gamma_{\varphi_n^2}^{\varphi_n^2} \text{ for } 1 \leq j \leq q, \quad \Theta_n = PE, \quad v_\alpha \in V(T_n) - V(R_n), \quad \text{and } \alpha \notin \varphi_n(T_{n,q} - T_n).
\]

It follows that (4) also holds. By Lemma 6.1(ii), colors \( \alpha \) and \( \beta \) are \( T_n \)-interchangeable under \( \varphi_n \). So \( P_n(\alpha, \beta, \varphi_n) \) is the only \((\alpha, \beta)\)-path with respect to \( \varphi_n \) intersecting \( T_n \). Hence \( Q \) is vertex-disjoint from \( T_n \) and therefore contains no edge incident to \( T_n \). By Lemma 5.8, \( \sigma_n = \varphi_n/Q \) is a \((T_n, D_n, \varphi_n)\)-stable coloring, and \( T_n \) is an ETT satisfying MP with respect to \( \sigma_n \). Since \( b \notin V(T_n) \), no edge in \( T_{n,q} - T_{n,0}^* \) is colored by \( \beta \) by TAA, because \( T_{n,0}^* = T_n \lor R_n \) by (7). Using (4) and (7), it is routine to check that the desired statements hold.

**Case 2.** \( q = 0 \), \( \alpha \in \varphi_n(T_n) \), or \( \alpha \in \varphi_n(T_{n,0}^*) \) with \( \alpha, \beta \notin D_n \), and \( Q \) is an \((\alpha, \beta)\)-path disjoint from \( P_{v_n}(\alpha, \beta, \varphi_n) \).

\(^2\)Color \( \alpha \) or \( \beta \) may become closed in \( T_{n,t} \) with respect to \( \sigma_n \) for some \( t \) with \( r + 1 \leq t \leq q \). Yet, even in this situation Definition 5.2(v) remains valid with respect to \( \sigma_n \).
Let us first assume that \( \alpha \) or \( \beta \) is closed in \( T_{n,0}^* \) with respect to \( \varphi_n \). By Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)), colors \( \alpha \) and \( \beta \) are \( T_{n,0}^* \)-interchangeable under \( \varphi_n \). So \( P_\alpha(\alpha, \beta, \varphi_n) \) is the only \( (\alpha, \beta) \)-path intersecting \( T_{n,0}^* \), and hence \( Q \) is vertex-disjoint from \( T_{n,0}^* \). It is then routine to check that \( \sigma_n = \varphi_n/Q \) is a \((T_{n,0}^*, D_n, \varphi_n)\)-strongly stable coloring, and \( T_{n,0}^* \) is an ETT satisfying MP with respect to \( \sigma_n \) by Theorem 3.10(vi). So we assume hereafter that

(8) neither \( \alpha \) nor \( \beta \) is closed in \( T_{n,0}^* \) with respect to \( \varphi_n \).

By the hypothesis of the present case, \( \alpha \in \overline{T}_n(T_n) \) or \( \{\alpha, \beta\} \cap D_n = \emptyset \). So \( \alpha \notin D_n \). From (8) and Algorithm 3.1 we deduce that \( T_{n,0}^* \neq T_n \). Hence

(9) \( \Theta_n = PE \), which together with (5.4) yields \( a, b \notin V(T_n) \cap V(R_n) \).

Let us show that

(10) \( \sigma_n = \varphi_n/Q \) is a \((T_{n,0}^*, D_n, \varphi_n)\)-strongly stable coloring.

To justify this, note that if one of \( a \) and \( b \) is contained in \( V(T_n) - V(R_n) \) and the other is contained in \( V(R_n) - V(T_n) \), then \( \alpha \) and \( \beta \) are \( T_{n,0}^* \)-interchangeable under \( \varphi_n \) by Lemma 6.1(iv).

So \( Q \) is vertex-disjoint from \( T_{n,0}^* \) and hence (10) holds. In view of (9), we may assume that

(11) if \( a, b \in V(T_{n,0}^*) \), then either \( a, b \in V(T_n) - V(R_n) \) or \( a, b \in V(R_n) - V(T_n) \).

Let us first assume that \( a \in V(T_n) - V(R_n) \). Then \( \alpha \notin \overline{T}_n(T_n) \) by (6.6) and \( b \in V(T_n) - V(R_n) \) if \( b \in V(T_{n,0}^*) \) by (11). So \( \alpha \) and \( \beta \) are \( T_n \)-interchangeable under \( \varphi_n \) by Lemma 6.1(ii) and \( \beta \notin \overline{T}_n(R_n) \) by (6.6). It follows that \( Q \) is vertex-disjoint from \( T_n \). From TAA we see that no edge in \( R_n - T(v_n) \) is colored by \( \alpha \) or \( \beta \) under \( \varphi_n \). Hence (10) holds.

Next we assume that \( a \in V(R_n) - V(T_n) \). Then \( \alpha \notin \overline{T}_n(T_n) \) by (6.6) and \( b \in V(R_n) - V(T_n) \) if \( b \in V(T_{n,0}^*) \) by (11). So \( \alpha \) and \( \beta \) are \( R_n \)-interchangeable under \( \varphi_n \) by Lemma 6.1(i) and \( \beta \notin \overline{T}_n(T_n) \) by (6.6). It follows that \( Q \) is vertex-disjoint from \( R_n \). By the hypothesis of the present case, \( \{\alpha, \beta\} \cap D_n = \emptyset \). From Algorithm 3.1 and TAA we see that no edge in \( T_n \) is colored by \( \alpha \) or \( \beta \) under \( \varphi_n \). So \( \alpha, \beta \notin \overline{T}_n(T_n) \cup D_n \) and hence (10) holds.

From (10) we deduce that \( T_{n,0}^* \) is an ETT satisfying MP with respect to \( \sigma_n \).

**Case 3.** \( T_{n,q}^* \prec v_\alpha \prec v_\beta \), \( \alpha, \beta \notin D_{n,q}, \alpha \notin \varphi_n(T(v_\beta) - T(v_\alpha)) \), and \( Q \) is an arbitrary \((\alpha, \beta)\)-chain.

By (6.6), \( V(T(y_{n-1})) \) is elementary with respect to \( \varphi_n \). So \( \alpha, \beta \notin \overline{T}_n(T_{n,q}) \). By hypothesis, \( \alpha, \beta \notin D_{n,q} \). Hence

(12) \( \alpha, \beta \notin \overline{T}_n(T_{n,q}) \cup D_{n,q} \).

Since \( \overline{T}_n(T_n) \cup D_n \subseteq \overline{T}_n(T_{n,q}) \cup D_{n,q} \), we have \( \alpha, \beta \notin \overline{T}_n(T_n) \cup D_n \). From TAA and the hypothesis of the present case, we further deduce that

(13) \( \alpha, \beta \notin \varphi_n(T(b)) \).

In view of Lemma 6.6, we obtain

(14) \( P_\alpha(\alpha, \beta, \varphi) = P_\beta(\alpha, \beta, \varphi) \). (Possibly \( Q \) is this path.)

Since \( T_{n,q}^* \prec a < b \), using (12)-(14), it is straightforward to verify that \( \sigma_n = \varphi_n/Q \) is a \((T_{n,q}^*, D_{n,q}, \varphi_n)\)-strongly stable coloring.

From (12) and (13) we also see that \( T(b) \) can be obtained from \( T_{n,q}^* \) by using TAA, no matter whether \( Q = P_\alpha(\alpha, \beta, \varphi) \). Thus \( T \) is an ETT corresponding to \( (\sigma_n, T_n) \). It is clear that \( T \) also satisfies MP under \( \sigma_n \), and \( T_{n} = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a hierarchy of \( T \) under \( \sigma_n \), with the same \( T \)-sets as \( T \) under \( \varphi_n \).
7 Elementariness and Interchangeability

In Section 5 we have developed a control mechanism, good hierarchies, over Kempe changes. In Section 6 we have derived some properties satisfied by these hierarchies. Now we are ready to present a proof of Theorem 5.3 by using a novel recoloring technique based on good hierarchies.

7.1 Proof of Theorem 5.3

By hypothesis, $T$ is an ETT constructed from a $k$-triple $(G, e, \varphi)$ by using the Tashkinov series $T = \{(T_i, \varphi_{i-1}, S_i, F_i, \Theta_{i-1}) : 1 \leq i \leq n + 1\}$. Furthermore, $T$ admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} = T$ and satisfies MP with respect to $\varphi_n$. Our objective is to show that $V(T)$ is elementary with respect to $\varphi_n$.

As introduced in the preceding section, $T = T_{n,q} \cup \{e_1, y_1, e_2, \ldots, e_p, y_p\}$, where $y_i$ is the end of $e_i$ outside $T(y_{i-1})$ for $i \geq 1$, with $T(y_0) = T_{n,q}$. Suppose on the contrary that $V(T)$ is not elementary with respect to $\varphi_n$. Then

(7.1) $\varphi_n(T(y_{p-1})) \cap \varphi_n(y_p) \neq \emptyset$ by (6.6).

For ease of reference, recall that (see (3) in the proof of Theorem 5.4)

(7.2) $|\varphi_n(T_n)| \geq 2n + 11$ (as $e$ is uncolored) and $|D_{n,j}| \leq |D_n| \leq n$ for $0 \leq j \leq q$.

In our proof we shall frequently make use of a coloring $\sigma_n \in \mathcal{C}^k(G-e)$ with properties (i)-(iii) as described in Lemma 6.7; that is,

(7.3) $\sigma_n$ is a $(T_{n,q}^*, D_n, \varphi_n)$-strongly stable coloring, and $T_{n,q}^*$ is an ETT satisfying MP with respect to $\sigma_n$. Furthermore, if $q \geq 1$, then $T_{n,q}$ admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ under $\sigma_n$, with the same $\Gamma$-sets (see Definition 5.2) as $T$ under $\varphi_n$, and $T_{n,q}$ is $(\cup_{y_j \in D_{n,q}} \Gamma_n^{-1})$-closed with respect to $\sigma_n$.

Claim 7.1. $p \geq 2$.

Assume the contrary: $p = 1$; that is, $T = T_{n,q} \cup \{e_1, y_1\}$. Then

(1) there exists a color $\alpha$ in $\varphi_n(T_{n,q}) \cap \varphi_n(y_1)$ by (7.1).

We consider two cases according to the value of $q$.

Case 1. $q = 0$. In this case, from (1) and Algorithm 3.1 we see that $\Theta_n \neq SE$. Let us first assume that $\Theta_n = RE$. Let $\delta_n, \gamma_n$ be as specified in Step 2 of Algorithm 3.1. Since $\alpha, \delta_n \in \varphi_n(T_n)$, both of them are closed in $T_n$ with respect to $\varphi_n$. Hence $P_{y_1}(\alpha, \delta_n, \varphi_n)$ is vertex-disjoint from $T_n$. Let $\sigma_n = \varphi_n/P_{y_1}(\alpha, \delta_n, \varphi_n)$. Then $\delta_n \in \varphi_n(T_n) \cap \varphi_n(y_1)$. By Lemma 5.8, $\sigma_n$ is a $(T_{n,q}^*, D_n, \varphi_n)$-stable coloring. It follows from Theorem 3.10(vi) that $\sigma_n$ is the same $\Gamma$-sets (see Definition 5.2) as $T$ under $\varphi_n$, and $T_{n,q}$ is $(\cup_{y_j \in D_{n,q}} \Gamma_n^{-1})$-closed with respect to $\sigma_n$.

We consider two cases according to the value of $q$.

Case 1. $q = 0$. In this case, from (1) and Algorithm 3.1 we see that $\Theta_n \neq SE$. Let us first assume that $\Theta_n = RE$. Let $\delta_n, \gamma_n$ be as specified in Step 2 of Algorithm 3.1. Since $\alpha, \delta_n \in \varphi_n(T_n)$, both of them are closed in $T_n$ with respect to $\varphi_n$. Hence $P_{y_1}(\alpha, \delta_n, \varphi_n)$ is vertex-disjoint from $T_n$. Let $\sigma_n = \varphi_n/P_{y_1}(\alpha, \delta_n, \varphi_n)$. Then $\delta_n \in \varphi_n(T_n) \cap \varphi_n(y_1)$. By Lemma 5.8, $\sigma_n$ is a $(T_{n,q}^*, D_n, \varphi_n)$-stable coloring. It follows from Theorem 3.10(vi) that $\sigma_n$ is a $\varphi_n$ mod $T_n$ coloring. From Definition 3.7 and Step 1 of Algorithm 3.1, we see that $f_n = e_1$ is still an RE connecting edge under $\sigma_n$ and is contained in a $(\delta_n, \gamma_n)$-cycle under $\sigma_n$, which is impossible because $\delta_n \in \varphi_n(y_1)$.

So we may assume that $\Theta_n = PE$. Let $\beta = \varphi_n(e_1)$. From TAA we see that $\beta \in \varphi_n(T_{n,0})$. Let $\theta \in \varphi_n(T_n) \cap \varphi_n(R_n)$. Then $\theta$ is closed in $T_{n,0}^*$ under $\varphi_n$ by (5.4). In view of Lemma 6.1(iii), $P_{y_1}(\alpha, \theta, \varphi_n)$ is the only $\theta$-path intersecting $T_{n,0}^*$. Thus $P_{y_1}(\alpha, \theta, \varphi_n) \cap T_{n,0}^* = \emptyset$. Let $\sigma_n = \varphi_n/P_{y_1}(\alpha, \theta, \varphi_n)$. By Lemma 6.7 (the second case), $\sigma_n$ is a $(T_{n,0}^*, D_n, \varphi_n)$-strongly stable coloring, so $\theta$ is also closed in $T_{n,0}^*$ with respect to $\sigma_n$. In view of Lemma 6.1(iii), $\beta$ and $\theta$ are $T_{n,0}^*$-interchangeable under $\sigma_n$. As $P_{y_1}(\theta, \beta, \sigma_n) \cap T_{n,0}^* \neq \emptyset$, there are at least two $(\theta, \beta)$-paths with respect to $\sigma_n$ intersecting $T_{n,0}^*$; a contradiction.
Case 2. $q \geq 1$. In this case, by Definition 5.2(v), we have

(2) $T_{n,q}$ is $(\bigcup_{y_0 \in D_{n,q}} \Gamma_{y_0}^{-1})$-closed with respect to $\varphi_n$

So $e_1$ is colored by some color $\gamma_1$ in $\bigcup_{y_0 \in D_{n,q}} \Gamma_{y_0}^{-1}$. By Definition 5.2(i) and (5.9), we have $\gamma_1 \notin \Gamma^y$. Let $\theta \in \varphi_n(T_{n,q}) - \varphi_n(T_{n,q-1})$. Then $\theta \notin \Gamma^{-1}$ (so $\theta \neq \gamma_1$) by Definition 5.2(i) and (5.10). Furthermore, $\theta$ is closed in $T_{n,q}$ under $\varphi_n$ by (2). In view of Corollary 6.3, $\alpha$ and $\theta$ are $T_{n,q}$-interchangeable under $\varphi_n$. So $P_{\gamma_1}(\alpha, \theta, \varphi_n) = P_{\gamma_1}(\alpha, \theta, \varphi_n)$ is the unique $(\alpha, \theta)$-path intersecting $T_{n,q}$. Hence $P_{\gamma_1}(\alpha, \theta, \varphi_n) \cap T_{n,q} = \emptyset$. Let $\sigma_n = \varphi_n/P_{\gamma_1}(\alpha, \theta, \varphi_n)$. Then $\sigma_n$ satisfies all the properties described in (7.3) by Lemma 6.7. Since $e_1$ is still colored by $\gamma_1 \in \Gamma^y$-under $\sigma_n$ and $\gamma_1 \notin \Gamma^y$, we can obtain $T$ from $T_{n,q}$ by TAA under $\sigma_n$, so $T$ is an ETT satisfying MP under $\sigma_n$. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\sigma_n$, with the same $T$-sets as those under $\varphi_n$. Hence $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $P_{\gamma_1}(\alpha, \gamma_1, \sigma_n) \cap T_{n,q} \neq \emptyset$, there are at least two $(\theta, \gamma_1)$-paths with respect to $\sigma_n$ intersecting $T_{n,q}$, contradicting Lemma 6.6(iii) (with $\sigma_n$ in place of $\varphi_n$), because $\theta, \gamma_1 \in \sigma_n(T_{n,q})$ and $\theta$ is also closed in $T_{n,q}$ under $\sigma_n$ by (2). Hence Claim 7.1 is justified.

Recall that the path number $p(T)$ of $T$ is the smallest subscript $i \in \{1, 2, \ldots, p\}$, such that the sequence $(y_i, e_{i+1}, \ldots, e_p, y_p)$ corresponds to a path in $G$, where $p \geq 2$ by Claim 7.1. Depending on the value of $p(T)$, we distinguish among three situations, labeled as Situation 7.1, Situation 7.2, and Situation 7.3.

**Situation 7.1.** $p(T) = 1$. Now $T - V(T_{n,q}^*)$ is a path obtained by using TAA under $\varphi_n$.

**Claim 7.2.** We may assume $\varphi_n(y_i) \cap \varphi_n(y_p) \neq \emptyset$ for some $i$ with $1 \leq i \leq p - 1$.

To justify this, let $\alpha \in \varphi_n(T_{n,q}(y_{p-1})) \cap \varphi_n(y_p)$ (see (7.1)). If $\alpha \in \varphi_n(y_i) \cap \varphi_n(y_p)$ for some $i$ with $1 \leq i \leq p - 1$, we are done. So we assume that

1. $\alpha \in \varphi_n(T_{n,q}(y_p))$ and $\alpha \notin \varphi_n(y_i)$ for all $1 \leq i \leq p - 1$.
2. If $\Theta_n = PE$ and $q = 0$, then we may further assume that $\alpha \in \varphi_n(T_n)$.

By (1), we have $\alpha \in \varphi_n(T_{n,0}^*)$. Suppose $\alpha \in \varphi_n(R_n - V(T_n))$. Then $\alpha \notin \Gamma^0$ by Definition 5.2(i). In view of (7.2), we have $|\varphi_n(T_n)| \geq 11 + 2n$ and $|\Gamma^0| \leq 2|D_{n,0}| \leq 2n$. So there exists $\beta \in \varphi_n(T_n) - \Gamma^0$. By Lemma 6.1(iv), $\alpha$ and $\beta$ are $T_{n,0}^*$-interchangeable under $\varphi_n$. Thus $P_{\gamma_1}(\alpha, \beta, \varphi_n) = P_{\gamma_1}(\alpha, \beta, \varphi_n)$ and $P_{\gamma_1}(\alpha, \beta, \varphi_n)$ is disjoint from $T_{n,0}^*$. Let $\sigma_n = \varphi_n/P_{\gamma_1}(\alpha, \beta, \varphi_n)$. By Lemma 6.7, $\sigma_n$ is a $(T_{n,0}^*, D_{n,0}, \varphi_n)$-strongly stable coloring, and $T_{n,0}^*$ is an ETT satisfying MP with respect to $\sigma_n$. Note that $T$ can also be obtained from $T_{n,0}^*$ by TAA under $\sigma_n$, because $\alpha, \beta \in \varphi_n(T_{n,0}^*)$. Hence $T$ satisfies MP under $\sigma_n$ as well. Since $\alpha, \beta \notin \Gamma^0$ and $\alpha, \beta \notin \varphi_n(T_{n,q}(y_{p-1}) - V(T_{n,0}^*))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} = T$ remains to be good under $\sigma_n$, with the same $T$-sets as those under $\varphi_n$. Therefore $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $\beta \in \varphi_n(T_n) \cap \varphi_n(y_p)$, replacing $\varphi_n$ by $\sigma_n$ and $\alpha$ by $\beta$ if necessary, we see that (2) holds.

Depending on whether $\alpha$ is used by edges in $T - T_{n,q}^*$, we consider two cases.

**Case 1.** $\alpha \notin \varphi_n(T - T_{n,q}^*)$. In this case, let $\beta \in \varphi_n(y_{p-1})$. Then $\beta$ is not used by any edge in $T - T_{n,q}^*$, except possibly $e_1$ when $q = 0$ and $T_{n,0}^* = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \varphi_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \varphi_n(T_n)$ if $q = 0$. It follows from Lemma 6.6 that $P_{\gamma_1}(\alpha, \beta, \varphi_n) = P_{\gamma_1}(\alpha, \beta, \varphi_n)$. So $P_{\gamma_1}(\alpha, \beta, \varphi_n)$ is disjoint
from $P_v(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_y(\alpha, \beta, \varphi_n)$. By Lemma 6.7, $\sigma_n$ satisfies all the properties described in (7.3). In particular, if $e_1 = f_0$ and $\varphi_n(e_1) = \beta = D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_y(\alpha, \beta, \varphi_n)$. So $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\sigma_n(u) = \varphi_n(u)$ for each $u \in V(T(y_p-1))$. Thus $T$ can be obtained from $T_{n,q}$ by TAA and satisfies MP under $\sigma_n$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\sigma_n$, with the same $-n$-sets as those under $\varphi_n$. Therefore, $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $\beta \in \sigma_n(y_{p-1}) \cap \sigma_n(y_p)$, replacing $\varphi_n$ by $\sigma_n$ if necessary, we see that Claim 7.2 is true.

**Case 2.** $\alpha \in \varphi_n(T - T_{n,q})$. In this case, let $e_j$ be the edge with the smallest subscript in $T - T_{n,q}$ such that $\varphi(e_j) = \alpha$. We distinguish between two subcases according to the value of $j$.

**Subcase 2.1.** $j \geq 2$. In this subcase, let $\beta \in \varphi_n(y_{j-1})$. Then $\beta$ is not used by any edge in $T_j - T_{n,q}$, except possibly $e_1$ when $q = 0$ and $T_{n,0} = T_n$ (now $e_1 = f_0$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \varphi_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \varphi_n(T_{n,q})$ if $q = 0$. It follows from Lemma 6.6 that $P_v(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$. So $P_y(\alpha, \beta, \varphi_n)$ is disjoint from $P_v(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_y(\alpha, \beta, \varphi_n)$. By Lemma 6.7, $\sigma_n$ satisfies all the properties described in (7.3). In particular, if $e_1 = f_0$ and $\varphi_n(e_1) = \beta \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_y(\alpha, \beta, \varphi_n)$. So $T$ can be obtained from $T_{n,q}$ by TAA and $\sigma_n$ and hence satisfies MP under $\sigma_n$.

Note that $\beta \notin \Gamma^q$ by Definition 5.2(i) and that $\varphi_n(u) = \varphi_n(u)$ for each $u \in V(T(y_p-1))$. If $\alpha \notin \Gamma^q$, then clearly $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy of $T$ under $\sigma_n$, with the same $-n$-sets as those under $\varphi_n$. If $\alpha \in \Gamma^q$, say $\alpha \in \Gamma^q_\eta$ for some $\eta \in D_n$, then Definition 5.2(i) implies that $\eta \in \varphi_n(w)$ for some $w \leq y_{j-1}$. Since only edges outside $T(w)$ may change colors between $\alpha$ and $\beta$ as we transform $\varphi_n$ into $\sigma_n$, it follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\sigma_n$, with the same $-n$-sets as those under $\varphi_n$. Hence $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\beta \in \sigma_n(y_{p-1}) \cap \sigma_n(y_p)$, replacing $\varphi_n$ by $\sigma_n$ if necessary, we see that Claim 7.2 holds.

**Subcase 2.2.** $j = 1$. In this subcase, $\alpha = \varphi(e_1)$. Note that $\alpha \notin \Gamma^q$ by Definition 5.2(i) and (5.9). We propose to show that

(3) there exists a color $\gamma$ in $\varphi_n(T_{n,q}) - \Gamma^q$ if $q \geq 1$ and in $\varphi_n(T_{n,q}) - \Gamma^0$ if $q = 0$, such that $\gamma$ is closed in $T_{n,q}$ with respect to $\varphi_n$.

Let us first assume that $q \geq 1$. By (7.2), we obtain $|\varphi_n(T_{n,q})| \geq |\varphi_n(T_{n,q})| \geq 2n + 11$ and $|\Gamma^q| \leq 2|D_{n,q-1}| \leq 2n$. So $|\varphi_n(T_{n,q}) - \Gamma^q| \geq 11$. By Definition 5.2(iii), we have $|\Gamma^q - \Gamma^q| = 2$. So $|\varphi_n(T_{n,q}) - (\Gamma^q - \Gamma^q)| \geq 9$. Let $\gamma$ be a color in $\varphi_n(T_{n,q}) - (\Gamma^q - \Gamma^q)$. By Definition 5.2(v), $\gamma$ is closed in $T_{n,q}$ with respect to $\varphi_n$.

Next we assume that $q = 0$. Again, by (7.2), we have $|\varphi_n(T_{n,q})| \geq 2n + 11$ and $|\Gamma^q| \leq 2|D_{n,0}| \leq 2n$. Let $\gamma$ be a color in $\varphi_n(T_{n,q}) - \Gamma^0$ if $\Theta_n \neq PE$ and a color in $\varphi_n(T_{n,q}) \cap \varphi_n(R_n) - \Gamma^0$ if $\Theta_n = PE$ (see Definition 5.2(iv)). By Algorithm 3.1 and (5.4), $\gamma$ is closed in $T_{n,q}$ with respect to $\varphi_n$. So (3) holds.

By (3) and Lemma 6.6, $P_v(\alpha, \gamma, \varphi_n) = P_v(\alpha, \gamma, \varphi_n)$ is the only $(\alpha, \gamma)$-path intersecting $T_{n,q}$. So $P_y(\alpha, \gamma, \varphi_n)$ is disjoint from $T_{n,q}$ and hence it does not contain $e_1$. Let $\sigma_n = \varphi_n / P_y(\alpha, \gamma, \varphi_n)$. Then $\sigma_n$ satisfies all the properties described in (7.3) by Lemma 6.7. Moreover, $\sigma_n(u) = \varphi_n(u)$ for all $u \in V(T(y_p-1))$. Since $\alpha, \gamma \in \varphi_n(T_{n,q})$, we have $\alpha, \gamma \in \sigma_n(T_{n,q})$. Hence we can obtain $T$ from $T_{n,q}$, by using TAA under $\sigma_n$, so $T$ satisfies MP under $\sigma_n$. Since $\alpha, \gamma \notin \Gamma^q$, the hierarchy
$T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under $\sigma_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore, $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $e_1$ is outside $P_{y_n}(\alpha, \gamma, \varphi_n)$, we have $\sigma_n(e_1) = \alpha$. As $\gamma \in \varphi_n(y_{p}) \cap \varphi_n(v)$ for some $v \in V(T_{n,q})$ and $\alpha \neq \gamma$, the present subcase reduces to Case 1 if $\gamma \notin \sigma_n(T - T_{n,q})$ or to Subcase 2.1 if $\gamma \in \sigma_n(T - T_{n,q})$. This proves Claim 7.2.

Claim 7.3. We may assume that $\varphi_n(y_{p-1}) \cap \varphi_n(y_{p}) \neq \emptyset$.

To justify this, let $K$ be the set of all minimum counterexamples $(T, \varphi_n)$ to Theorem 5.3 (see (6.2)-(6.5)), and let $i$ be the largest subscript with $1 \leq i \leq p - 1$, such that there exists a member $(T, \mu_n)$ of $K$ with $\varphi_n(y_{i}) \cap \varphi_n(y_{p}) \neq \emptyset$; this $i$ exists by Claim 7.2. We aim to show that $i = p - 1$. Thus Claim 7.3 follows by replacing $\varphi_n$ with $\mu_n$, if necessary.

With a slight abuse of notation, we assume that $\varphi_n(y) \cap \varphi_n(y_{p}) \neq \emptyset$ and assume, on the contrary, that $i < p - 2$. Let $\alpha \in \varphi_n(y_{i}) \cap \varphi_n(y_{p})$. Using (6.6) and TAA, we obtain

(1) $\alpha \notin \varphi_n(T(y_{i-1}))$, where $T(y_{i}) = T_{n,q}$. So $\alpha$ is not used by any edge in $T(y_{i+1}) - T_{n,q}$, except possibly $e_1$ when $q = 0$ and $T_{n,0} = T_{n,q} = \emptyset$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \alpha \in D_n$).

Recall that Definition 5.2 involves $\Gamma^q_h = \{\gamma^q_{h_1}, \gamma^q_{h_2}\}$ for each $\eta_h \in D_{n,q}$. Nevertheless, in our proof we only consider a fixed $\eta_h \in D_{n,q}$.

Case 1. $\alpha \notin D_{n,q}$. In this case, let $\theta \in \varphi_n(y_{i+1})$. From TAA and (6.6) it follows that

(3) $\theta \notin \varphi_n(T(y_{i}))$, so $\theta$ is not used by any edge in $T(y_{i+1}) - T_{n,q}$, except possibly $e_1$ when $q = 0$ and $T_{n,0} = T_{n,q} = \emptyset$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \theta \in D_n$).

If $\theta \notin D_{n,q}$, then $\{\alpha, \theta\} \cap D_{n,q} = \emptyset$. By the definitions of $D_n$ and $D_{n,q}$, we have $\varphi_n(T_n) \cup D_n \subseteq \varphi_n(T_{n,q}) \cup D_{n,q}$, which together with (1) and (3) implies $\{\alpha, \theta\} \cap D_n = \emptyset$. Hence $P_{y_n}(\alpha, \theta, \varphi_n) = P_{y_{i+1}}(\alpha, \theta, \varphi_n)$ by Lemma 6.6. Let $\sigma_n = \varphi_n/P_{y_n}(\alpha, \theta, \varphi_n)$. Since both $y_{i}$ and $y_{i+1}$ are contained in $T - V(T_{n,q})$ and (1) holds, by Lemma 6.7 (the third case), $\sigma_n$ satisfies all the properties described in (7.3). Furthermore, $T$ is also an ETT satisfying MP with respect to $\sigma_n$, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of $T$ under $\sigma_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Hence $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \varphi_n(y_{p}) \cap \varphi_n(y_{p+1})$, we reach a contradiction to the maximality assumption on $i$.

So we may assume that $\theta \in D_{n,q}$. Let $\theta = \eta_h \in D_{n,q}$. In view of (2) and Lemma 6.6, we obtain $P_{y_n}(\alpha, \gamma, \varphi_n) = P_{y_n}(\alpha, \gamma, \varphi_n)$, which is disjoint from $P_{y_n}(\alpha, \gamma, \varphi_n)$. Let $\sigma_n = \varphi_n/P_{y_n}(\alpha, \gamma, \varphi_n)$. By Lemma 6.7, $\sigma_n$ satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \alpha \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_{y_n}(\alpha, \gamma, \varphi_n)$. By (6.6), (1) and (2), we have $\sigma_n(u) = \varphi_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge $f$ in $T(y_{i+1})$. So $T$ can be obtained from $T_{n,q}$ by TAA under $\sigma_n$, and hence satisfies MP under $\sigma_n$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of $T$ under $\sigma_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Hence $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), with $\gamma_1 \in \varphi_n(y_{p}) \cap \varphi_n(T_{n,q})$.
Using (2) and Lemma 6.6, we obtain $P_{v_n}(\eta_h, \gamma_1, \sigma_n) = P_{y_n}(\eta_h, \gamma_1, \sigma_n)$, which is disjoint from $P_{y_n}(\eta_h, \gamma_1, \sigma_n)$. Let $\sigma'_n = \sigma_n / P_{y_n}(\eta_h, \gamma_1, \sigma_n)$. By Lemma 6.7, $\sigma'_n$ satisfies all the properties described in (7.3) (with $\sigma'_n$ in place of $\sigma_n$). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \eta_h \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that $e_1$ is outside $P_{y_n}(\eta_h, \gamma_1, \sigma_n)$. By (6.6), (2) and (3), we have $\sigma'_n(u) = \sigma_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma'_n(f) = \sigma_n(f)$ for each edge $f$ in $T(y_{p+1})$. So $T$ can be obtained from $T^*_{n,q}$ by TAA under $\sigma'_n$, and hence satisfies MP under $\sigma'_n$. Furthermore, since $\eta_h \in \sigma'_n(y_{p+1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \cdots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under $\sigma'_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore $(T, \sigma'_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\eta_h \in \sigma'_n(y_p) \cap \sigma'_n(y_{p+1})$, we reach a contradiction to the maximality assumption on $i$.

Case 2. $\alpha \in D_{n,q}$. In this case, let $\alpha = \eta_h \in D_{n,q}$. Then $\Gamma_h = \{\gamma_1, \gamma_2\}$ (see the paragraph above (2)). Renaming subscript if necessary, we may assume that $\varphi_n(e_{i+1}) \not= \gamma_1$. By (1) and (2), we have

(4) $\gamma_1 \not\in \varphi_n(T(y_{p+1}) - T^*_{n,q})$ and $\eta_h$ is not used by any edge in $T(y_{p+1}) - T^*_{n,q}$, except possibly $e_1$ when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (4) and Lemma 6.6, we obtain $P_{v_n}(\eta_h, \gamma_1, \varphi_n) = P_{y_n}(\eta_h, \gamma_1, \varphi_n)$, which is disjoint from the path $P_{y_n}(\eta_h, \gamma_1, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_n}(\eta_h, \gamma_1, \varphi_n)$. By Lemma 6.7, $\sigma_n$ satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_h \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_{y_n}(\eta_h, \gamma_1, \varphi_n)$. By (6.6) and (4), we have $\sigma_n(u) = \varphi_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge $f$ in $T(y_{p+1})$. So $T$ can be obtained from $T^*_{n,q}$ by TAA under $\sigma_n$, and hence satisfies MP under $\sigma_n$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \cdots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of $T$ under $\sigma_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), with $\gamma_1 \in \sigma_n(y_p) \cap \sigma_n(T_{n,q})$. Let $\theta \in \sigma_n(y_{p+1})$. From TAA we see that

(5) $\theta$ is not used by any edge in $T(y_{p+1}) - T^*_{n,q}$ under $\sigma_n$, except possibly $e_1$ when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\sigma_n(e_1) = \theta \in D_n$).

By (6.6), we have $\theta \not= \gamma_1$. Using (4) and Lemma 6.6, we get $P_{v_n}(\theta, \gamma_1, \sigma_n) = P_{y_n}(\theta, \gamma_1, \sigma_n)$.

Let $\sigma'_n = \sigma_n / P_{y_n}(\theta, \gamma_1, \sigma_n)$. By Lemma 6.7, $\sigma'_n$ satisfies all the properties described in (7.3) (with $\sigma'_n$ in place of $\sigma_n$). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \theta \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that $e_1$ is outside $P_{y_n}(\theta, \gamma_1, \sigma_n)$. From (6.6) and (4) we deduce that $\sigma'_n(u) = \sigma_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma'_n(f) = \sigma_n(f)$ for each edge $f$ in $T(y_{p+1})$. So $T$ can also be obtained from $T^*_{n,q}$ by TAA under $\sigma'_n$, and hence satisfies MP under $\sigma'_n$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \cdots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of those under $\sigma'_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore $(T, \sigma'_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \sigma'_n(y_p) \cap \sigma'_n(y_{p+1})$, we reach a contradiction to the maximality assumption on $i$. Hence Claim 7.3 is established.

By Claim 7.3, we have $\varphi_n(y_{p-1}) \cap \varphi_n(y_p) \not= \emptyset$. Let $\alpha \in \varphi_n(y_{p-1}) \cap \varphi_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let $\sigma_n$ be obtained from $\varphi_n$ by recoloring $e_p$ with $\alpha$ and let $T' = T(y_{p-1})$. Then $\beta \in \sigma_n(y_{p-2}) \cap \sigma_n(y_{p+1})$ and $T_n = T_{n,0} \subset T_{n,1} \subset \cdots \subset T_{n,q} \subset T'$ is a good hierarchy of $T'$ under $\sigma_n$. By Claim 7.1, $p \geq 2$. So $(T', \sigma_n)$ is a counterexample to Theorem 5.3 (see (6.2)-(6.4)), which violates the minimality assumption (6.5) on $(T, \varphi_n)$. This completes our discussion about Situation 7.1.
Situation 7.2. \( p(T) = p \geq 1 \). Now \( e_p \) is not incident to \( y_{p-1} \).

By (7.1), there exists a color \( \alpha \in \varphi_n(T(y_{p-1})) \cap \varphi_n(y_p) \). We divide this situation into 3 cases and further into 6 subcases (see figure below), depending on whether \( v_{\alpha} = y_{p-1} \) and \( \alpha \in D_{n,q} \). Our proof of Subcase 1.1 is self-contained. Yet, in our discussion Subcase 1.2 may be redirected to Subcase 1.1 and Subcase 2.1, and Subcase 2.1 may be redirected to Subcase 1.1, etc. Figure 1 illustrates such redirections (note that no cycling occurs).

![Figure 1. Redirections](image-url)

Throughout this situation we reserve the symbol \( \theta \) for \( \varphi_n(e_p) \). Clearly, \( \theta \neq \alpha \).

**Case 1.** \( \alpha \in \varphi_n(y_p) \cap \varphi_n(y_{p-1}) \) and \( \alpha \in D_{n,q} \).

Let \( \alpha = \eta_m \in D_{n,q} \). For simplicity, we abbreviate the two colors \( \gamma_{m1}^q \) and \( \gamma_{m2}^q \) in \( \Gamma_m^q \) (see Definition 5.2) to \( \gamma_1 \) and \( \gamma_2 \), respectively. Since \( \eta_m \in \varphi_n(y_p) \cap \varphi_n(y_{p-1}) \), from TAA and Definition 5.2(i) we see that

1. \( \gamma_1, \gamma_2 \notin \varphi_n(T(y_{p-1}) - T_{n,q}^*), \) and \( \eta_m \) is not used by any edge in \( T - T_{n,q}^* \), except possibly \( e_1 \) when \( q = 0 \) and \( T_{n,q}^* = T_n \) (now \( e_1 = f_n \) in Algorithm 3.1 and \( \varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n \)).

   By (1) and Lemma 6.6 (with respect to \( (T, \varphi_n) \)), we have

2. \( P_{v_j}(\eta_m, \gamma_j, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_j, \varphi_n) \) for \( j = 1, 2 \).

Let us consider two subcases according to whether \( \theta \notin \varphi_n(y_{p-1}) \).

**Subcase 1.1.** \( \theta \notin \varphi_n(y_{p-1}) \).

In our discussion about this subcase, we shall appeal to the following two tree sequences:

- \( T^- = (T_{n,q}^*, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p) \) and
- \( T^+ = (T_{n,q}^*, e_1, y_1, e_2, \ldots, y_{p-2}, e_p, y_{p-1}, e_p, e_{p-1}, y_{p-1}) \).

Note that \( T^- \) is obtained from \( T \) by deleting \( y_{p-1} \) and \( T^+ \) arises from \( T \) by interchanging the order of \( (e_{p-1}, y_{p-1}) \) and \( (e_p, y_p) \). Furthermore, both \( T^- \) and \( T^+ \) can be obtained from \( T_{n,q}^* \) by using TAA under \( \varphi_n \), so both of them are ETTs and satisfy MP with respect to \( \varphi_n \).

Let us first assume that \( \theta \notin \Gamma^q \). Now it is easy to see that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^- \) is a good hierarchy of \( T^- \) under \( \varphi_n \), with the same \( \Gamma \)-sets (see Definition 5.2) as \( T \). (Notice that if \( \theta \in \Gamma^q \), say \( \theta \in \Gamma^q_{h} \), and \( \eta_h \in \varphi_n(y_{p-1}) \), then \( T^- \) no longer satisfies Definition 5.2(i).) Observe that \( \gamma_1 \notin \varphi_n(y_p) \), for otherwise, \( \gamma_1 \) is missing at two vertices in \( T^- \). Thus \( (T^- , \varphi_n) \) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on \( (T, \varphi_n) \). Let us turn to considering \( T^+ \). Since \( \theta \notin \varphi_n(y_{p-1}) \) and \( \theta \notin \Gamma^q \), it is clear that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^+ \) is a good hierarchy of \( T^+ \) under \( \varphi_n \), with the
same Γ-sets as T. Moreover, by (1), we have $\gamma_1 \not\in \varphi_n(T^*(y_p) - T^*_{n,q})$. It follows from Lemma 6.6 (with respect to $(T^*, \varphi_n)$) that $P_{v_1}(\eta_m, \gamma_1; \varphi_n) = P_{y_p}(\eta_m, \gamma_1, \varphi_n)$, contradicting (2).

Next we assume that $\theta \in \Gamma^q$. Then $\theta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$. If $\eta_h \not\in \varphi_n(y_p-1)$, then $\eta_h \in \varphi_n(T(y_p-2))$ by Definition 5.2(i). So we can still ensure that both $T^-$ and $T^*$ have good hierarchies under $\varphi_n$. Thus, using the same argument as employed in the preceding paragraph, we can reach a contradiction. Hence we may assume that $\eta_h \in \varphi_n(y_p-1)$.

Clearly, $\theta \neq \gamma_1$ or $\gamma_2$. Renaming subscripts if necessary, we may assume that

$$(3) \theta \neq \gamma_2.$$ 

Since $P_{v_2}(\eta_m, \gamma_2, \varphi_n) = P_{y_p}(\eta_m, \gamma_2, \varphi_n)$ by (2), this path is disjoint from $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. Let $\mu_1 = \varphi_n/P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By Lemma 6.7, $\mu_1$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By (1) and (3), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\varphi_n(u) = \varphi_n(u)$ for each $u \in V(T(y_p-1))$. So we can obtain $T$ from $T^*_n$ by using TAA under $\mu_1$; thereby $T$ satisfies MP under $\mu_1$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_1$, with the same Γ-sets as those under $\varphi_n$. Therefore, $(T,n,1)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\gamma_2$ is missing at two vertices.

By Lemma 6.4, we have $|\varphi_1(T(y_p-2))| - |\varphi_1(T_{n,0}) - V(T_n)| + |\varphi_1(T(y_p-2) - T^*_{n,q})| \geq 2n + 11$, where $T(y_0) = T^*_n$.

By (4) and Lemma 6.6, $P_{v_2}(\beta, \gamma_2, \mu_1) = P_{v_2}(\beta, \gamma_2, \mu_1)$, so it is disjoint from $P_{v_2}(\beta, \gamma_2, \mu_1)$. Let $\mu_2 = \mu_1/P_{y_p}(\beta, \gamma_2, \mu_1)$. By Lemma 6.7, $\mu_2$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). By (1), (3) and (4), we have $\beta, \gamma_2 \notin \mu_1(T(y_p) - T^*_{n,q})$. So $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$ and $\varphi_2(u) = \varphi_1(u)$ for each $u \in V(T(y_p-1))$. Hence we can obtain $T$ from $T^*_n$ by using TAA under $\mu_2$; thereby $T$ satisfies MP under $\mu_2$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_2$, with the same Γ-sets as those under $\mu_1$. Therefore, $(T,n,2)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta$ is missing at two vertices. Since $\theta \in \Gamma^q_h$ and $\eta_h \in \varphi_n(y_p-1) = \varphi_1(y_p-1) = \varphi_2(y_p-1)$, we obtain

$$(5) \theta \notin \mu_2(T(y_p-1) - T^*_{n,q}).$$ 

By (4), we also have

$$(6) \beta \notin \mu_2(T - T^*_{n,q}).$$ 

It follows from (5), (6) and Lemma 6.6 that $P_{y_p}(\beta, \theta, \mu_2) = P_{y_p}(\beta, \theta, \mu_2)$, so it is disjoint from $P_{y_p}(\beta, \theta, \mu_2)$. Finally, set $\mu_3 = \mu_2/P_{y_p}(\beta, \theta, \mu_2)$. By Lemma 6.7, $\mu_3$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). From (5) and (6) we see that $T$ can be obtained from $T^*_n$ by using TAA under $\mu_3$. Hence $T$ satisfies MP under $\mu_3$. Note that $\mu_3(f) = \mu_2(f)$ for each $f \in E(T(y_p-1))$, $\mu_3(e_p) = \beta$, and $\varphi_2(u) = \varphi_1(u)$ for each $u \in V(T(y_p-1))$. Moreover, $\beta \notin \Gamma^q$ by (4). It is a routine matter to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_3$, with the same Γ-sets as those under $\mu_2$. Since $\mu_3(e_p) = \beta \notin \Gamma^q$ and $v_{y,p} \prec y_p-1$, we see that $T^-$ has a good hierarchy and satisfies MP with respect to $\mu_3$. As $\theta$ is missing at two vertices in $T^-$, we conclude that $(T^-, \mu_3)$ is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which contradicts the minimality assumption (6.4) or (6.5) on $(T, \varphi_n)$.

**Subcase 1.2.** $\theta \in \varphi_n(y_p-1)$. 58
In this subcase, from (6.6) and TAA we see that
(7) \( \theta \notin \varphi_n(T(y_p-2)) \), so \( \theta \notin \Gamma^\gamma \) and hence \( \theta \notin \gamma_1, \gamma_2 \). Furthermore, \( \theta \) is not used by any edge in \( T(y_p-1) \) except possibly \( e_1 \) when \( q = 0 \) and \( T^*_n,0 = T_n \) (now \( e_1 = f_n \) in Algorithm 3.1 and \( \varphi_n(e_1) = \theta \in D_n \)).

Since \( P_{\gamma_1}(\eta_m, \gamma_1, \varphi_n) = P_{y_p}(\eta_m, \gamma_1, \varphi_n) \) by (2), this path is disjoint from \( P_{y_p}(\eta_m, \gamma_1, \varphi_n) \). Let \( \mu_1 = \varphi_n/P_{y_p}(\eta_m, \gamma_1, \varphi_n) \). By Lemma 6.7, \( \mu_1 \) satisfies all the properties described in (7.3) (with \( \mu_1 \) in place of \( \sigma_n \)). In particular, if \( e_1 = f_n \) and \( \varphi_n(e_1) = \eta_m \in D_n \), then \( \mu_1(e_1) = \varphi_n(e_1) \), which implies that \( e_1 \) is outside \( P_{y_p}(\eta_m, \gamma_1, \varphi_n) \). By (1) and (7), we have \( \mu_1(f) = \varphi_n(f) \) for each \( f \in E(T) \) and \( p_1(u) = \varphi_n(u) \) for each \( u \in V(T(y_p-1)) \). So we can obtain \( T \) from \( T^*_n \) by using TAA under \( \mu_1 \), and hence \( T \) satisfies MP under \( \mu_1 \). Furthermore, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a good hierarchy of \( T \) under \( \mu_1 \), with the same \( \Gamma \)-sets as those under \( \varphi_n \). Therefore, \( (T, \mu_1) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \gamma_1 \) is missing at two vertices.

From (1) and the definition of \( \mu_1 \), we see that
(8) \( \gamma_1 \notin \mu_1(T - T^*_n) \).

From (8) and Lemma 6.6, we deduce that \( P_{\gamma_1}(\theta, \gamma_1, \mu_1) = P_{y_p}(\theta, \gamma_1, \mu_1) \), which is disjoint from \( P_{y_p}(\theta, \gamma_1, \mu_1) \). Let \( \mu_2 = \mu_1/P_{y_p}(\theta, \gamma_1, \mu_1) \). By Lemma 6.7, \( \mu_2 \) satisfies all the properties described in (7.3) (with \( \mu_2 \) in place of \( \sigma_n \)). In particular, if \( e_1 = f_n \) and \( \mu_2(e_1) = \theta \in D_n \), then \( \mu_2(e_1) = \mu_1(e_1) \), which implies that \( e_1 \) is outside \( P_{y_p}(\theta, \gamma_1, \mu_1) \). In view of (7) and (8), \( T \) can be obtained from \( T_{n,q} \) by using TAA under \( \mu_2 \). Hence \( T \) satisfies MP under \( \mu_2 \). Note that \( \mu_2(f) = \mu_1(f) \) for each \( f \in E(T(y_p-1)) \) and \( \mu_2(e_p) = \gamma_1 \notin \varphi_n(e_p) \) and \( \mu_2(e_p) = \gamma_1 \notin \varphi_n(e_p) \), the present subcase reduces to Subcase 1.1 if \( \theta \in D_{n,q} \) and reduces to Subcase 2.1 (to be discussed below) if \( \theta \notin D_{n,q} \).

Case 2. \( \alpha \in \varphi_n(y_p) \cap \varphi_n(y_p-1) \) and \( \alpha \notin D_{n,q} \).

By the definitions of \( D_n \) and \( D_{n,q} \), we have \( \varphi_n(T_n) \cup D_n \subseteq \varphi_n(T^*_n) \cup D_{n,q} \). Using (6.6) and this set inclusion, we obtain
(9) \( \alpha \notin \varphi_n(y_p-2) \) and \( \alpha \notin D_{n,q} \). So \( \alpha \notin \varphi_n(T - T^*_n) \) by TAA (see, for instance, (1)).

Recall that \( T(y_p) = T^*_n \) and \( \theta = \varphi_n(e_p) \). We consider two subcases according to whether \( \theta \in \varphi_n(y_p-1) \).

Subcase 2.1. \( \theta \notin \varphi_n(y_p-1) \).

In our discussion about this subcase, we shall also appeal to the following two tree sequences:

- \( T^- = (T^*_n, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p) \) and
- \( T^+ = (T^*_n, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1}) \).

As stated before, \( T^- \) is obtained from \( T \) by deleting \( y_{p-1} \) and \( T^+ \) arises from \( T \) by interchanging the order of \( (e_{p-1}, y_{p-1}) \) and \( (e_p, y_p) \). Furthermore, both \( T^- \) and \( T^+ \) can be obtained from \( T^*_n \) by using TAA under \( \varphi_n \), so both of them are ETTs and satisfy MP with respect to \( \varphi_n \). Observe that
(10) \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T^* \) is a good hierarchy of \( T^* \) under \( \varphi_n \), except when \( \theta \in \Gamma^\gamma_h \) for some \( \eta_h \in D_{n,q} \) and \( \eta_h \in \varphi_n(y_p-1) \).
Let us first assume that the exceptional case in (10) does not occur; that is, there exists no \( \eta_h \in D_{n,q} \) such that \( \eta_h \in \varphi_n(y_{p-1}) \) and \( \theta \in T^q \). It is easy to see that now \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset \varphi_n \) is a good hierarchy of \( T^q \) under \( \varphi_n \).

By Lemma 6.4, we have \( |\varphi_n(T(y_{p-2})) - |\varphi_n(T_{n,0} - V(T_n))| - |\varphi_1(T(y_{p-2}) - T^q_{n,q})| \geq 2n + 11 \) holds, where \( T(y_0) = T^q_n \). Since \( |T^q_n| \leq 2|D_{n,q}| \leq 2n \) by Lemma 3.4, using (6.6) we obtain

\( \text{(11) there exists a color } \beta \in \varphi_n(T(y_{p-2})) - \varphi_n(T_{n,0} - V(T_n)) - \varphi_n(T - T^q_{n,q}) - T^q.} \)

Note that \( \beta \notin \varphi_n(y_p) \), for otherwise, \( (T^q, \varphi_n) \) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on \( (T, \sigma_n) \). Since \( \alpha, \beta \notin \varphi_n(T - T^q_{n,q}) \) by (9) and (11), applying Lemma 6.6 to \( (T, \varphi_n) \) and \( \alpha, \beta, \varphi_n \), respectively, we obtain \( P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{p-1}}(\alpha, \beta, \varphi_n) \) and \( P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{p}}(\alpha, \beta, \varphi_n) \), a contradiction.

So that we can assume that the exceptional case in (10) occurs; that is, there exists \( \eta_h \in D_{n,q} \) such that \( \eta_h \in \varphi_n(y_{p-1}) \) and \( \theta \in T^q_n \). For simplicity, we abbreviate the two colors \( \gamma_{h_1}^q \) and \( \gamma_{h_2}^q \) in \( T^q_n \) (see Definition 5.2) to \( \gamma_1 \) and \( \gamma_2 \), respectively. Renaming subscripts if necessary, we may assume that \( \theta = \gamma_1 \). By Definition 5.2(i) and TAA, we have

\( \text{(12) } \gamma_2 \notin \varphi_n(T - T^q_{n,q}) \) and \( \eta_h \) is not used by any edge in \( T - T^q_{n,q} \), except possibly \( e_1 \) when \( q = 0 \) and \( T^q_{n,q} = T_n \) (now \( e_1 = f_n \) in Algorithm 3.1 and \( \varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n \)).

By (9), (12) and Lemma 6.6, we obtain \( P_{v_{\varphi_n}}(\alpha, \gamma_2, \varphi_n) = P_{y_{p-1}}(\alpha, \gamma_2, \varphi_n) \), which is disjoint from \( P_{y_{p}}(\alpha, \gamma_2, \varphi_n) \). Let \( \mu_1 = \varphi_n/P_{y_{p}}(\alpha, \gamma_2, \varphi_n) \). By Lemma 6.7, \( \mu_1 \) satisfies all the properties described in (7.3) (with \( \mu_2 \) in place of \( \sigma_n \)). Since \( \alpha, \gamma_2 \notin \varphi_n(T(y_{p-1}) - \varphi_n(T - T^q_{n,q})) \) by (9) and (12), we have \( \mu_1(f) = \varphi_n(f) \) for each \( f \in E(T) \) and \( \varphi_n(u) = \varphi_n(u) \) for each \( u \in V(T(y_{p-1})) \). So we can obtain \( T \) from \( T^q_{n,q} \) by using TAA under \( \mu_1 \), and hence \( T \) satisfies MP under \( \mu_1 \). Furthermore, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a good hierarchy of \( T \) under \( \mu_1 \), with the same \( \Gamma \)-sets as those under \( \varphi_n \). Therefore, \( (T, \mu_1) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \gamma_2 \) is missing at two vertices.

If \( \eta_h \in \varphi_n(y_p) \), then \( \eta_h \in \varphi_1(y_p) \cap \varphi_1(y_{p-1}) \) and \( \mu_1(e_p) = \gamma_1 \notin \varphi_n(y_{p-1}) \). Thus the present subcase reduces to Subcase 1.1. So we may assume that \( \eta_h \notin \varphi_1(y_p) \). By (12) and the definition of \( \mu_1 \), we have

\( \text{(13) } \gamma_2 \notin \mu_1(T - T^q_{n,q}) \) and \( \eta_h \) is not used by any edge in \( T - T^q_{n,q} \) under \( \mu_1 \), except possibly \( e_1 \) when \( q = 0 \) and \( T^q_{n,q} = T_n \) (now \( e_1 = f_n \) in Algorithm 3.1 and \( \mu_1(e_1) = \eta_h \in D_{n,q} \)).

By (13) and Lemma 6.6, we obtain \( P_{v_{\varphi_n}}(\eta_h, \gamma_2, \mu_1) = P_{y_{p-1}}(\eta_h, \gamma_2, \mu_1) \), which is disjoint from \( P_{y_{p}}(\eta_h, \gamma_2, \mu_1) \). Let \( \mu_2 = \mu_1/P_{y_{p}}(\eta_h, \gamma_2, \mu_1) \). By Lemma 6.7, \( \mu_2 \) satisfies all the properties described in (7.3) (with \( \mu_2 \) in place of \( \sigma_n \)). In particular, if \( e_1 = f_n \) and \( \mu_1(e_1) = \eta_h \in D_{n,q} \), then \( \mu_2(e_1) = \mu_1(e_1) \), which implies that \( e_1 \) is outside \( P_{y_{p}}(\eta_h, \gamma_2, \mu_1) \). By (13), we have \( \mu_2(f) = \mu_1(f) \) for each \( f \in E(T) \) and \( \varphi_n(u) = \varphi_n(u) \) for each \( u \in V(T(y_{p-1})) \). So we can obtain \( T \) from \( T^q_{n,q} \) by using TAA under \( \mu_2 \), and hence \( T \) satisfies MP under \( \mu_2 \). Furthermore, \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a good hierarchy of \( T \) under \( \mu_2 \), with the same \( \Gamma \)-sets as those under \( \mu_1 \). Therefore, \( (T, \mu_2) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \eta_h \in \varphi_2(y_p) \cap \varphi_2(y_{p-1}) \) and \( \mu_2(e_p) = \gamma_1 \notin \varphi_2(y_{p-1}) \). Thus the present subcase reduces to Subcase 1.1.

**Subcase 2.2.** \( \theta \in \varphi_n(y_{p-1}) \).

Let us first assume that \( \theta \in D_{n,q} \); that is, \( \theta = \eta_m \) for some \( \eta_m \in D_{n,q} \). For simplicity, we use \( e_1 \) and \( e_2 \) to denote the two colors \( \gamma_{n_1}^q \) and \( \gamma_{n_2}^q \) in \( T^q_n \) (see Definition 5.2), respectively. By Definition 5.2(i) and TAA, we have

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By Lemma 6.4, there exist 7 distinct colors \( \Phi \) to Theorem 5.3 (see (6.2)-(6.5)), in which \( T_{n,0}, T_{n,1}, \ldots, T_{n,q} \subset T_{n,q+1} = T \) remains to be a good hierarchy of \( T \) under \( \mu_1 \), with the same \( \Gamma \)-sets as those under \( \varphi_n \). Therefore, \( (T, \mu_1) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \epsilon_1 \) is missing at two vertices.

By (15) and Lemma 6.6, we obtain \( P_{v_0}(\eta, \epsilon_1, \mu_1) = P_{v_0}(\eta, \epsilon_1, \mu_1) \), which is disjoint from \( P_{v_0}(\eta, \epsilon_1, \mu_1) \). Let \( \mu_2 = \mu_1/P_{v_0}(\eta, \epsilon_1, \mu_1) \). By Lemma 6.7, \( \mu_2 \) satisfies all the properties described in (7.3) (with \( \mu_2 \) in place of \( \varphi_n \)). Thus the present subcase reduces to Subcase 1.1.

Next we assume that \( \epsilon \in D_{n,q} \). We propose to show that

(16) there exists a color \( \beta \in \varphi(T_{y_2}) - \varphi(T_{T_{y_2}} - V(T_{y_2})) - \varphi(T_{T_{y_2}} - D_{n,q}) \), such that either \( \beta \not\in \Gamma^\varphi \) or \( \beta \in \Gamma^\varphi \) for some \( \eta \in D_{n,q} \cap \varphi(T_{y_2}) \).

To justify this, note that if \( |\varphi(T_{y_2})| = |\varphi(T_{y_2}) - V(T)| - |\varphi(T_{y_2}) - T_{y_2}| - |\Gamma^\varphi \cup D_{n,q}| \geq 5 \), then \( |\varphi(T_{y_2}) - V(T)| - |\varphi(T_{y_2}) - T_{y_2}| - |\Gamma^\varphi \cup D_{n,q}| \geq 3 \), because \( T_{y_2} \) contains precisely two edges. Thus there exists a color \( \beta \in \varphi(T_{y_2}) - \varphi(T_{y_2} - V(T_{y_2})) - \varphi(T_{y_2} - D_{n,q}) \) such that \( \beta \not\in \Gamma^\varphi \).

So we assume that \( \beta \in \varphi(T_{y_2}) - \varphi(T_{y_2} - V(T_{y_2})) - \varphi(T_{y_2} - D_{n,q}) \). By (16), \( \beta \not\in \varphi(T_{y_2} - V(T_{y_2})) \). It follows from these two observations that

(17) if \( q \geq 1 \), then \( \beta \not\in \varphi(T_{y_2}) \) or \( \beta \not\in \Gamma^\varphi \); if \( q = 0 \), then \( \beta \not\in \varphi(T_{y_2}) \) or \( \beta \not\in \varphi(T_{y_2}) \).

By (9), (17) and Lemma 6.6, we obtain \( P_{v_0}(\alpha, \beta, \varphi_n) = P_{v_0}(\alpha, \beta, \varphi_n) \), which is disjoint from \( P_{v_0}(\alpha, \beta, \varphi_n) \). Let \( \mu_3 = \mu_3/P_{v_0}(\alpha, \beta, \varphi_n) \). By Lemma 6.7, \( \mu_3 \) satisfies all the properties described in (7.3) (with \( \mu_3 \) in place of \( \varphi_n \)). By (9) and (16), we have \( \alpha, \beta \not\in \varphi(T_{y_2}) \).

Thus we can
obtain $T$ from $T_{n,q}^{*}$ by using TAA under $\mu_3$, and hence $T$ satisfies MP under $\mu_3$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_3$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore, $(T, \mu_3)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta$ is missing at two vertices.

Since $\theta \in \varphi_n(T_{n-1})$, it follows from (6.6) that $\theta \notin \varphi_n(T_{n,q})$. By assumption, $\theta \notin D_{n,q}$. As $\varphi_n(T_n) \cup D_n \subseteq \varphi_n(T_{n,q}) \cup D_{n,q}$, we obtain

\[ (19) \quad \theta \notin D_n \text{ and hence } \theta \notin \mu_3(T(y_{p-1}) - T_{n,q}^{*}) \] by TAA.

By (17)-(19) and Lemma 6.6, we obtain $P_{y_p}(\theta, \beta, \varphi_n) = P_{y_{p-1}}(\theta, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\theta, \beta, \varphi_n)$. Let $\mu_4 = \mu_3/P_{y_p}(\theta, \beta, \mu_3)$. By Lemma 6.7, $\mu_4$ satisfies all the properties described in (7.3) (with $\mu_4$ in place of $\sigma_\lambda$). By (18) and (19), we have $\mu_4(f) = \mu_3(f)$ for each $f \in E(T(y_{p-1}))$ and $\varphi_n(u) = \varphi_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain $T$ from $T_{n,q}^{*}$ by using TAA under $\mu_4$, and hence $T$ satisfies MP under $\mu_4$. Since either $\beta \notin \Gamma^\theta$ or $\beta \in \Gamma^\theta_h$ for some $\eta \in D_{n,q} \cap \varphi_n(T_{n,q+1})$ by (16), it follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_4$, with the same $\Gamma$-sets as those under $\mu_3$. Therefore, $(T, \mu_4)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta \in \varphi_n(T_{n,1}) \cap \varphi_n(T_{n,q-1})$, $\theta \notin D_{n,q}$, and $\mu_4(e_p) = \beta \notin \varphi_n(y_{p-1})$. Thus the present subcase reduces to Subcase 2.1.

**Case 3.** $\alpha \in \varphi_n(y_{p-1}) \cap \varphi_n(v)$ for some vertex $v < y_{p-1}$.

Set $T(y_0) = T_{n,q}^{*}$. Let us first impose some restrictions on $\alpha$.

(20) We may assume that $\alpha \in \varphi_n(T(y_{p-2})) - \varphi_n(T - T_{n,q}^{*})$, such that either $\alpha \notin D_{n,q} \cup \Gamma^\theta$ if $q \geq 1$ and $\alpha \notin D_{n,q} \cup \Gamma^\theta$ if $q = 0$, or $\alpha$ is some $\eta \in D_{n,q}$ satisfying $\Gamma^\theta_h \cap \varphi_n(T - T_{n,q}^{*}) = \emptyset$.

To justify this, note that if $[\varphi_n(T(y_{p-2}))] - [\varphi_n(T_{n,0} - V(T))] - [\varphi_n(T(y_{p-2}) - T_{n,q}^{*})] - [\Gamma^\theta \cup D_{n,q}] \geq 3$, then $[\varphi_n(T(y_{p-2}))] - [\varphi_n(T_{n,0} - V(T))] - [\varphi_n(T(y_{p-2}) - T_{n,q}^{*})] - [\Gamma^\theta \cup D_{n,q}] \geq 3$, because $T - T(y_{p-2})$ contains precisely two edges. Thus there exists a color $\beta \in \varphi_n(T(y_{p-2})) - \varphi_n(T_{n,0} - V(T)) - \varphi_n(T(y_{p-2}) - T_{n,q}^{*}) - (\Gamma \cup D_{n,q})$. Clearly, $\beta \in \varphi_n(T(y_{p-2})) - \varphi_n(T - T_{n,q}^{*})$ and $\beta \notin D_{n,q} \cup \Gamma^\theta$ if $q \geq 1$ and $\beta \notin D_{n,q} \cup \Gamma^\theta$ if $q = 0$.

If $[\varphi_n(T(y_{p-2}))] - [\varphi_n(T_{n,0} - V(T))] - [\varphi_n(T(y_{p-2}) - T_{n,q}^{*})] - [\Gamma \cup D_{n,q}] \leq 4$, then, by Lemma 6.4, there exist 7 distinct colors $\eta_1 \in D_{n,q} \cap \varphi_n(T(y_{p-2}))$ such that $(\Gamma^\theta_h \cup \{\eta_1\}) \cap \varphi_n(T(y_{p-2}) - T_{n,q}^{*}) = \emptyset$. Since $T - T(y_{p-2})$ contains precisely two edges, there exists one of these $\eta_1$, denoted by $\beta$, such that $(\Gamma^\theta_h \cup \{\eta_1\}) \cap \varphi_n(T - T_{n,q}^{*}) = \emptyset$.

Combining the above observations, we conclude that

(21) there exists $\beta \in \varphi_n(T(y_{p-2})) - \varphi_n(T - T_{n,q}^{*})$, such that either $\beta \notin D_{n,q} \cup \Gamma^\theta$ if $q \geq 1$ and $\beta \notin D_{n,q} \cup \Gamma^\theta$ if $q = 0$, or $\beta$ is some $\eta \in D_{n,q}$ satisfying $\Gamma^\theta_h \cap \varphi_n(T - T_{n,q}^{*}) = \emptyset$.

If $\beta \notin \varphi_n(y_{p-1})$, then (20) holds by replacing $\alpha$ with $\beta$. So we assume hereafter that $\beta \notin \varphi_n(y_{p-1})$.

Let $Q = P_{y_p}(\alpha, \beta, \varphi_n)$ and let $\sigma_n = \varphi_n / Q$. We propose to show that one of the following statements (a) and (b) holds:

(a) $\sigma_n$ is a $(T_{n,q}^{*}, D_n, \varphi_n)$-strongly stable coloring, $T$ is also an ETT satisfying MP with respect to $\sigma_n$, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of $T$ under $\sigma_n$, with the same $\Gamma$-sets (see Definition 5.2) as those under $\varphi_n$. Moreover, (20) holds with respect to $(T, \sigma_n)$.

(b) There exists an ETT $T'$ satisfying MP with respect to $\varphi_n$, such that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of $T'$ under $\varphi_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Moreover, $V(T')$ is not elementary with respect to $\varphi_n$ and $p(T') < p(T)$.
Note that if (b) holds, then \((T', \varphi_n)\) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on \((T, \varphi_n)\).

Let us first assume that \(Q\) is vertex-disjoint from \(T(y_{p-1})\). By Lemma 5.8, \(\sigma_n\) is both \((T(y_{p-1}), D_n, \varphi_n)\)-stable and \((T(y_{p-1}), \varphi_n)\)-invariant. If \(\Theta_n = PE\), then \(\sigma_n\) is also \((T_n \oplus R_n, D_n, \varphi_n)\)-stable. Furthermore, \(T(y_{p-1})\) is an ETT satisfying MP with respect to \(\sigma_n\), and \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_{p-1})\) is a good hierarchy of \(T(y_{p-1})\), with the same \(\Gamma\)-sets as \(T\) under \(\sigma_n\). By definition, \(\sigma_n\) is a \((T_n, D_n, \varphi_n)\)-strongly stable coloring. By the hypothesis of Case 3 and assumption on \(\beta\), we have \(\varphi_n(e_p) \neq \alpha, \beta\). Thus it is clear that (a) is true, and (20) follows if we replace \(\varphi_n\) by \(\sigma_n\) and \(\alpha\) by \(\beta\).

Next we assume that \(Q\) and \(T(y_{p-1})\) have vertices in common. Let \(u\) be the first vertex of \(Q\) contained in \(T(y_{p-1})\) as we traverse \(Q\) from \(y_p\). Define \(T' = T(y_{p-1}) \cup Q[u, y_p]\) if \(u = y_{p-1}\) and \(T' = T(y_{p-2}) \cup Q[u, y_p]\) otherwise. By the hypothesis of Case 3 and (21), we have \(\alpha, \beta \in \overline{\varphi_n}(T(y_{p-2}))\). So \(T'\) can be obtained from \(T(y_{p-2})\) by using TAA under \(\varphi_n\), with \(p(T') < p(T)\). It follows that \(T'\) is an ETT satisfying MP with respect to \(\varphi_n\).

By Definition 5.2, we have \(D_{n,q} \cap \Gamma^q = \emptyset\). Thus (22) \(\alpha \notin \Gamma^q\) by (21).

Let us proceed by considering three possibilities for \(\alpha\).

- \(\alpha \notin \Gamma^q\). Since both \(\alpha\) and \(\beta\) are outside \(\Gamma^q\) (see (22)), it is easy to see that \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'\) is a good hierarchy of \(T'\) under \(\varphi_n\), with the same \(\Gamma\)-sets as \(T\) under \(\varphi_n\). Hence (b) holds.

- \(\alpha \in \Gamma^q \cap \varphi_n(T - T_{n,q}^\ast)\). Let \(\alpha \in \Gamma^q_h\) for some \(\eta_i \in D_{n,q}\). Since \(\varphi(e_p) \neq \alpha\), we have \(\alpha \in \varphi_n(T(y_{p-1}) - T_{n,q}^\ast)\). Hence \(\eta_i \in \overline{\varphi_n}(T(y_{p-2}))\) by Definition 5.2(i). Furthermore, \(\beta \in \overline{\varphi_n}(T(y_{p-2}))\) and \(\beta \notin \Gamma^q\) by (21) and (22). Therefore, \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'\) is a good hierarchy of \(T'\) under \(\varphi_n\), with the same \(\Gamma\)-sets as \(T\) under \(\varphi_n\). Hence (b) holds.

- \(\alpha \in \Gamma^q \setminus \varphi_n(T - T_{n,q}^\ast)\). By the definition of \(\Gamma^q\), we have \(\alpha \in \overline{\varphi_n}(T_{n,q})\) if \(q \geq 1\) and \(\alpha \in \overline{\varphi_n}(T)\) if \(q = 0\). It follows from Lemma 6.6 that \(P_{\eta}(\alpha, \beta, \varphi_n) = P_{\eta}(\alpha, \beta, \varphi_n)\), which is disjoint from \(Q\). By Lemma 6.7, \(\sigma_n = \varphi_n/Q\) satisfies all the properties described in (7.3). Since \(\alpha, \beta \notin \varphi_n(T - T_{n,q}^\ast)\) by the assumption on \(\alpha\) and (21), we have \(\sigma_n(f) = \varphi_n(f)\) for each \(f \in E(T)\) and \(\varphi_n(u) = \overline{\varphi_n}(u)\) for each \(u \in V(T(y_{p-1}))\). So we can obtain \(T\) from \(T_{n,q}^\ast\) by using TAA under \(\sigma_n\), and hence \(T\) satisfies MP under \(\sigma_n\). Furthermore, \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T\) remains to be a good hierarchy of \(T\) under \(\sigma_n\), with the same \(\Gamma\)-sets as those under \(\varphi_n\). Therefore, \((T, \sigma_n)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \(\beta\) is missing at two vertices. So (a) holds and therefore (20) is established by replacing \(\varphi_n\) with \(\sigma_n\) and \(\alpha\) by \(\beta\).

Let \(\alpha\) be a color as specified in (20). Recall that \(\theta = \varphi_n(e_p)\). We consider two subcases according to whether \(\theta \in \overline{\varphi_n}(y_{p-1})\).

**Subcase 3.1.** \(\theta \notin \overline{\varphi_n}(y_{p-1})\).

Consider the tree sequence \(T^- = (T_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)\). As stated before, \(T^-\) is obtained from \(T\) by deleting \(y_{p-1}\). Clearly, \(T^-\) is obtained from \(T_{n,q}^\ast\) by using TAA under \(\varphi_n\), so it is an ETT and satisfy MP with respect to \(\varphi_n\). Observe that (23) \(T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-\) is a good hierarchy of \(T^-\) under \(\varphi_n\), unless \(\theta \in \Gamma^q_m\) for some \(\eta_m \in D_{n,q}\) and \(\eta_m \in \overline{\varphi_n}(y_{p-1})\).

It follows that the exceptional case stated in (23) must occur, for otherwise, \((T^-, \varphi_n)\) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality as-
remains to be a good hierarchy of $T \varphi_n(T(y_p-1))$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore, $(T, \mu_1)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta$ is missing at two vertices.

Subcase 3.2. $\theta \in \varphi_n(y_p-1)$.

We first assume that $\theta \in D_{n,q}$. Let $\theta = \eta_m \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma_{m_1}$ and $\gamma_{m_2}$ in $\Gamma_n$ (see Definition 5.2) to $\gamma_1$ and $\gamma_2$, respectively. By (20) and Definition 5.2(i), we have

$$\{\alpha, \gamma_1, \gamma_2\} \cap \varphi_n(T - T_{n,q}^*) = \emptyset.$$

By (26) (see Definition 5.2(i)), we obtain $P_{\nu}^\alpha(\gamma_1, \gamma_2) = P_{\nu_{\gamma_1}}^\alpha(\gamma_1, \gamma_2) = \emptyset$, which is disjoint from $P_{y_p}(\gamma_1, \gamma_2)$. Let $\mu_1 = \varphi_n/P_{\nu_{\gamma_1}}^\alpha(\gamma_1, \gamma_2)$. By Lemma 6.7, $\mu_1$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). Therefore, $(T, \mu_1)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta$ is missing at two vertices.

By (27) and Lemma 6.6, we obtain $P_{\nu_{\gamma_1}}^\alpha(\gamma_1, \gamma_2) = P_{y_p}(\gamma_1, \gamma_2)$. Let $\mu_2 = \mu_1/P_{\nu_{\gamma_1}}^\alpha(\gamma_1, \gamma_2)$. By Lemma 6.7, $\mu_2$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_m \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that $e_1$ is outside $P_{y_p}(\gamma_1, \eta_m, \mu_1)$. Since $\mu_2(f) = \mu_1(f)$ for each
f \in E(T(y_{p-1}))$ by (27), and $p_2(u) = p_1(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain $T$ from $T^*_{n,q}$ by using TAA under $\mu_2$ and hence $T$ satisfies MP under $\mu_2$. Furthermore, $T_0 = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_2$, with the same $\Gamma$-sets as those under $\mu_1$. Therefore, $(T, \mu_2)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\eta_n \in p_2(y_p) \cap p_2(y_{p-1})$, $\eta_n \in D_{n,q}$, and $\mu_2(e_p) = \gamma \notin p_2(y_{p-1})$, the present subcase reduces to Subcase 1.1.

Next we assume that $\theta \notin D_{n,q}$. By (6.6) and the hypothesis of the present subcase, we have $\theta \notin p_2(T^*_{n,q})$. So $\theta \notin p_2(T^*_{n,q}) \cup D_{n,q}$, which implies $\theta \notin p_2(T_n) \cup D_n$. In particular, (28) $\theta \notin D_{n,q} \cup \Gamma^0$ if $q \geq 1$ and $\theta \notin D_n \cup \Gamma^0$ if $q = 0$. Furthermore, $\theta$ is not used by any edge in $T(y_{p-1}) - T_{n,q}$ by TAA (see, for instance, (1)).

We proceed by considering two possibilities for $\alpha$.

- $\alpha \notin D_{n,q}$. Now it follows from (20) that (29) $\alpha \notin D_{n,q} \cup \Gamma^0$ if $q \geq 1$ and $\alpha \notin D_n \cup \Gamma^0$ if $q = 0$.

By (20) and Lemma 6.6, we obtain $P_{v_2}(\alpha, \theta, \varphi_n) = P_{y_{p-1}}(\alpha, \theta, \varphi_n)$, which is disjoint from $P_{y_{p}}(\alpha, \theta, \varphi_n)$. Let $\sigma_n = \varphi_{y_{p}} / P_{y_{p}}(\alpha, \theta, \varphi_n)$. By Lemma 6.7, $\sigma_n$ satisfies all the properties described in (7.3). Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{p-1}))$ by (20) and (28), and $\bar{p}_n(u) = \bar{p}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain $T$ from $T^*_{n,q}$ by using TAA under $\sigma_n$ and hence $T$ satisfies MP under $\sigma_n$. In view of (28) and (29), $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\sigma_n$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore, $(T, \sigma_n)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in p_2(y_p) \cap p_2(y_{p-1})$, $\theta \notin D_{n,q}$, and $\sigma_n(e_p) = \alpha \notin p_2(y_{p-1})$, the present subcase reduces to Subcase 2.1.

- $\alpha \in D_{n,q}$. Let $\alpha = \eta_n \in D_{n,q}$. For simplicity, we use $\varepsilon_1$ and $\varepsilon_2$ to denote the two colors $\gamma_{h_1}$ and $\gamma_{h_2}$ in $\Gamma^0$ (see Definition 5.2), respectively. By (20), we have (30) $\{\alpha, \varepsilon_1, \varepsilon_2\} \cap \varphi_n(T - T^*_{n,q}) = \emptyset$.

By (30) and Lemma 6.6, we obtain $P_{v_2}(\alpha, \varepsilon_1, \varphi_n) = P_{v_2}(\alpha, \varepsilon_1, \varphi_n)$, which is disjoint from $P_{y_{p}}(\alpha, \varepsilon_1, \varphi_n)$. Let $\mu_1 = \varphi_{y_{p}} / P_{y_{p}}(\alpha, \varepsilon_1, \varphi_n)$. By Lemma 6.7, $\mu_1$ satisfies all the properties described in (7.3) (with $\mu_1$ in place of $\sigma_n$). Since $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ by (30), and $\bar{p}_n(u) = \bar{p}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain $T$ from $T^*_{n,q}$ by using TAA under $\mu_1$ and hence $T$ satisfies MP under $\mu_1$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_1$, with the same $\Gamma$-sets as those under $\varphi_n$. Therefore, $(T, \mu_1)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\varepsilon_1$ is missing at two vertices. From (30) and Definition 5.2(i) we see that (31) $\varepsilon_1 \notin \mu_1(T - T^*_{n,q})$.

By (31) and Lemma 6.6, we obtain $P_{v_2}(\theta, \varepsilon_1, \mu_1) = P_{y_{p-1}}(\theta, \varepsilon_1, \mu_1)$, which is disjoint from $P_{y_{p}}(\theta, \varepsilon_1, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_{p}}(\theta, \varepsilon_1, \mu_1)$. By Lemma 6.7, $\mu_2$ satisfies all the properties described in (7.3) (with $\mu_2$ in place of $\sigma_n$). In view of (28) and (31), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$ and $\bar{p}_n(u) = \bar{p}_n(u)$ for each $u \in V(T(y_{p-1}))$. So $T$ can be obtained from $T^*_{n,q}$ by using TAA and hence satisfies MP under $\mu_2$. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of $T$ under $\mu_2$, with the same $\Gamma$-sets as those under $\mu_1$. Therefore, $(T, \mu_2)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in p_2(y_p) \cap p_2(y_{p-1})$, $\theta \notin D_{n,q}$, and $\mu_2(e_p) = \varepsilon \notin p_2(y_{p-1})$, the present subcase reduces to Subcase 2.1. This completes our discussion about Situation 7.2.

**Situation 7.3.** $2 \leq p(T) \leq p - 1$. 
Recall that $T = T^*_{n,q} \cup \{e_1, y_1, e_2, \ldots, e_p, y_p\}$, and the path number $p(T)$ of $T$ is the smallest subscript $t \in \{1, 2, \ldots, p\}$ such that the sequence $(y_t, e_{t+1}, \ldots, e_p, y_p)$ corresponds to a path in $G$. Set $I_{\varphi_n} = \{1 \leq t \leq p - 1 : \varphi_n(y_t) \cap \varphi_n(y_{t+1}) \neq \emptyset\}$. We use $\max(I_{\varphi_n})$ to denote the maximum element of $I_{\varphi_n}$ if $I_{\varphi_n} \neq \emptyset$. For convenience, set $\max(I_{\varphi_n}) = -1$ if $I_{\varphi_n} = \emptyset$.

If $\max(I_{\varphi_n}) \geq p(T)$, then we may assume that $\max(I_{\varphi_n}) = p - 1$ (the proof is exactly the same as that of Claim 7.2). Let $\alpha = \varphi_n(y_{p-1}) \cap \varphi_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let $\sigma_n$ be obtained from $\varphi_n$ by recoloring $e_p$ with $\alpha$ and let $T' = T(y_{p-1})$. Then $\beta \in \varphi_n(y_{p-1}) \cap \varphi_n(T')$ and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of $T'$ under $\sigma_n$. So $(T', \sigma_n)$ is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on $(T, \varphi_n)$.

So we may assume hereafter that $\max(I_{\varphi_n}) < p(T)$. Let $i = \max(I_{\varphi_n})$ if $I_{\varphi_n} \neq \emptyset$, and let $j = p(T)$. Then $e_j$ is not incident to $y_{j-1}$. In our proof $y_0$ is the maximum vertex (in the order $\prec$) in $T^*_{n,q}$.

Claim 7.4. We may assume that there exists $\alpha = \varphi_n(y_p) \cap \varphi_n(T(y_{j-2}))$, such that either $\alpha \notin \Gamma^q \cup \varphi_n(T'_{n,0} - V(T_n))$ or $\alpha \in \Gamma^m_n$ for some $\eta_n \in D_{n,q}$ with $v_{\eta_n} \leq y_{j-2}$.

To establish this statement, we consider two cases, depending on whether $I_{\varphi}$ is nonempty.

Case 1. $I_{\varphi} \neq \emptyset$. By assumption, $\max(I_{\varphi}) < p(T)$. So $i \leq j - 1$. Let $\alpha = \varphi_n(y_p) \cap \varphi_n(y_i)$. By (6.6), we obtain

1. $\alpha \notin \varphi_n(T_{n,\emptyset})$. So $\alpha \in \Gamma^q \cup \varphi_n(T_{n,0} - V(T_n))$.

If $i \leq j - 2$, then $\alpha \in \varphi_n(T(y_{j-2}))$, as desired. Thus we may assume that $i = j - 1$.

2. There exists a color $\beta \in \varphi_n(T(y_{j-2} - \varphi_n(T_{n,0} - V(T_n))) \cap \varphi_n(T(y_{j-1} - T_{n,0}^*)) - (\Gamma^q \cup D_{n,q})$ or a color $\beta \in \Gamma^q \cap \varphi_n(T_{n,0} - V(T_n))$ for some $\eta_n \in D_{n,q}$ with $v_{\eta_n} \leq y_{j-2}$ and $(\Gamma^q \cup \varphi_n(T_{n,0} - V(T_n))) \cap \varphi_n(T(y_{j-1} - T_{n,0}^*)) = \emptyset$.

To justify this, note that if $|\varphi_n(T(y_{j-2}))| - |\varphi_n(T_{n,0} - V(T_n))| - |\varphi_n(T(y_{j-1} - T_{n,0}^*))| - |\Gamma^q \cup D_{n,q}| \geq 5$, then there exists a color $\beta \in \varphi_n(T(y_{j-2}) - T_{n,0}^* - V(T_n)) - \varphi_n(T(y_{j-1} - T_{n,0}^*)) - (\Gamma^q \cup D_{n,q})$, because $T(y_{j-1} - T(y_{j-2})$ contains only one edge.

If $|\varphi_n(T(y_{j-2}))| - |\varphi_n(T_{n,0} - V(T_n))| - |\varphi_n(T(y_{j-2} - T_{n,0}^*))| - |\Gamma^q \cup D_{n,q}| \leq 4$, then, by Lemma 6.4, there exist 7 distinct colors $\eta_n \in D_{n,q} \cap \varphi_n(T_{n,0} - V(T_n))$ such that $(\Gamma^q \cup \varphi_n(T(y_{j-2} - T_{n,0}^*) = \emptyset$. Since $T(y_{j-1}) - T(y_{j-2})$ contains only one edge, there exists at least one of these $\eta_n$, say $\eta_n$, such that $(\Gamma^q \cup \varphi_n(T_{n,0} - V(T_n)) \cap \varphi_n(T(y_{j-1} - T_{n,0}^*)) = \emptyset$. (So 2) is true.

Depending on whether $\alpha$ is contained in $D_{n,q}$, we distinguish between two subcases.

Subcase 1.1. $\alpha \in D_{n,q}$. In this subcase, let $\alpha = \eta_n \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{\alpha}$ and $\gamma^q_{\eta}$ in $\Gamma^q_h$ (see Definition 5.2) to $\gamma_1$ and $\gamma_2$, respectively. Since $\eta_n \in \varphi_n(y_{j-1})$, by Definition 5.2(i) and TAA, we have

3. $\gamma_1, \gamma_2 \notin \varphi_n(T(y_{j-1} - T_{n,0}^*)$, and $\eta_n$ is not used by any edge in $T(y_{j-1}) - T_{n,0}^*$, except possibly $e_1$ when $q = 0$ and $T_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_n \in D_{n,q} \subseteq D_n$).

By (3) and Lemma 6.6, we obtain $P_{\gamma_1}(\gamma_1, \eta_n, \varphi_n) = P_{\gamma_1}(\gamma_1, \eta_n, \varphi_n)$, which is disjoint from $P_{\gamma_1}(\gamma_1, \eta_n, \varphi_n)$. Let $\mu_1 = \varphi_n/P_{\gamma_1}(\gamma_1, \eta_n, \varphi_n)$. By Lemma 6.7, $\mu_1$ satisfies all the properties described in (7.3) (with $\mu_1$ in place of $\sigma_n$). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_n \in D_{n,q}$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that $e_1$ is outside $P_{\gamma_1}(\gamma_1, \eta_n, \varphi_n)$. Using (3) and (6.6), we get $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$ and $T_{n,1} = \varphi_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain $T$ from $T_{n,0}$ by using TAA under $\mu_1$, and hence $T$ satisfies MP under $\mu_1$. Furthermore, since $\eta_n \in \varphi_n(y_{p-1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be
good under \( \mu_1 \), with the same \( \Gamma \)-sets as those under \( \mu_1 \). Therefore, \((T, \mu_1)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \gamma_1 \) is missing at two vertices.

From (3) we see that

(4) \( \gamma_1, \gamma_2 \notin \mu_1(T(y_{j-1}) - T_{n,q}^*) \), and \( \eta_h \) is not used by any edge in \( T(y_{j-1}) - T_{n,q}^* \) under \( \mu_1 \), except possibly \( \epsilon_1 \) when \( q = 0 \) and \( T_{n,0}^* = T_n \) (now \( \epsilon_1 = f_n \) in Algorithm 3.1 and \( \mu_1(\epsilon_1) = \eta_h \in D_{n,q} \subseteq D_n \)).

Let \( \beta \) be a color as specified in (2). Note that

(5) \( \beta \notin \mu_1(T(y_{j-1}) - T_{n,q}^*) \), \( \beta \notin D_{n,q} \), and \( \beta \neq \eta_h = \alpha \).

Since \( \gamma_1 \in \mathcal{P}_1(T_{n,q}) \) if \( q \geq 1 \) and \( \gamma_1 \in \mathcal{P}_1(T_n) \) if \( q = 0 \), from Lemma 6.6 we deduce that

\[
P_{v_1}(\gamma_1, \beta, \mu_1) = P_{y_1}(\gamma_1, \beta, \mu_1),
\]

which is disjoint from \( P_{y_1}(\gamma_1, \beta, \mu_1) \). Let \( \mu_2 = \mu_1 / P_{y_1}(\gamma_1, \beta, \mu_1). \)

By Lemma 6.7, \( \mu_2 \) satisfies all the properties described in (7.3) (with \( \mu_2 \) in place of \( \sigma_n \)). By (4), (5) and (6.6), we have \( \mu_2(f) = \mu_1(f) \) for each \( f \in E(T(y_{j-1})) \), and \( \mu_2(u) = \mu_1(u) \) for each \( u \in V(T(y_{j-1})) \). So we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \mu_2 \) and hence \( T \) satisfies MP under \( \mu_2 \). If \( \beta \notin \Gamma^q \), then clearly \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be a good hierarchy of \( T \) under \( \mu_2 \), with the same \( \Gamma \)-sets as those under \( \mu_1 \). So we assume that \( \beta \in \Gamma^q \). By (2), we have \( \beta \in \Gamma^q_n \) for some \( \eta_m = D_{n,q} \) with \( v_{m} \neq y_{j-2} \) and \( (\Gamma^q_n \cup \{ \eta_m \}) \cap \phi_n(T(y_{j-1}) - T_{n,q}^*) = \emptyset \). It follows that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) is still a good hierarchy of \( T \) under \( \mu_2 \), with the same \( \Gamma \)-sets as those under \( \mu_1 \). Therefore, \((T, \mu_2)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \beta \in \mathcal{P}_2(y_p) \cap \mathcal{P}_2(T(y_{j-2})) \).

From (2) and the definitions of \( \mu_1 \) and \( \mu_2 \), we see that either \( \beta \notin \Gamma^q \) or \( \beta \notin \mathcal{P}_n(T_{n,0}^* - V(T)) \) or \( \beta \notin \mathcal{P}_n^\eta \) for some \( \eta_m \in D_{n,q} \) with \( v_{m} \neq y_{j-2} \). Thus Claim 7.4 holds by replacing \( \varphi_n \) with \( \mu_2 \) and \( \alpha \) with \( \beta \).

**Subcase 1.2.** \( \alpha \notin D_{n,q} \). In this subcase, using (1) and the set inclusion \( \mathcal{P}_n(T_n) \cup D_n \subseteq \mathcal{P}_n(T_{n,q}^* \cup D_{n,q}) \), we get

(6) \( \alpha \notin D_n \). So \( \alpha \) is not used by any edge in \( T(y_{j-1}) - T_{n,q}^* \) by TAA.

Let \( \beta \) be a color as specified in (2). Then there are two possibilities for \( \beta \).

- \( \beta \in \mathcal{P}_n(T(y_{j-2})) - \mathcal{P}_n(T_{n,0}^* - V(T_n)) - \phi_n(T(y_{j-1}) - T_{n,q}^*) - (\Gamma^q \cup D_{n,q}) \). Now it follows from Lemma 6.6 that \( P_{y_1}(\alpha, \beta, \varphi_n) = P_{y_1}(\alpha, \beta, \varphi_n) \), so this path is disjoint from \( P_{y_1}(\alpha, \beta, \varphi_n) \). Let \( \sigma_n = \varphi_n / P_{y_1}(\alpha, \beta, \varphi_n) \). By Lemma 6.7, \( \sigma_n \) satisfies all the properties described in (7.3). By (6), the assumption on \( \beta \) and (6.6), we have \( \sigma_n(f) = \sigma_n(f) \) for each \( f \in E(T(y_{j-1})) \), and \( \sigma_n(u) = \sigma_n(u) \) for each \( u \in V(T(y_{j-1})) \). So we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \sigma_n \) and hence \( T \) satisfies MP under \( \sigma_n \). Since \( \alpha, \beta \notin \Gamma^q \) (see (1)), the hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be good under \( \sigma_n \), with the same \( \Gamma \)-sets as those under \( \varphi_n \).

Therefore, \((T, \sigma_n)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \beta \in \mathcal{P}_n(y_p) \cap \mathcal{P}_n(T(y_{j-2})) \). Thus Claim 7.4 holds by replacing \( \varphi_n \) with \( \sigma_n \) and \( \alpha \) with \( \beta \).

- \( \beta \in \Gamma^q_m \) for some \( \eta_m \in D_{n,q} \) with \( v_{m} \neq y_{j-2} \) and \( (\Gamma^q_n \cup \{ \eta_m \}) \cap \phi_n(T(y_{j-1}) - T_{n,q}^*) = \emptyset \). Note that \( \eta_m \in \mathcal{P}_n(T(y_{j-2})) \) and hence \( \alpha \neq \eta_m \) by (6.6). In view of Lemma 6.6, we obtain \( P_{y_1}(\alpha, \beta, \varphi_n) = P_{y_1}(\alpha, \beta, \varphi_n) \), which is disjoint from \( P_{y_1}(\alpha, \beta, \varphi_n) \). Let \( \sigma_n = \varphi_n / P_{y_1}(\alpha, \beta, \varphi_n) \). By Lemma 6.7, \( \sigma_n \) satisfies all the properties described in (7.3). By (6), the assumption on \( \beta \) and (6.6), we have \( \sigma_n(f) = \sigma_n(f) \) for each \( f \in E(T(y_{j-1})) \), and \( \sigma_n(u) = \sigma_n(u) \) for each \( u \in V(T(y_{j-1})) \). So we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \sigma_n \) and hence \( T \) satisfies MP under \( \sigma_n \). Since \( \alpha, \beta \notin \Gamma^q \) (see (1)) and \( \eta_m \in \mathcal{P}_n(T(y_{j-2})) \), the hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be good under \( \sigma_n \), with the same \( \Gamma \)-sets as those under \( \varphi_n \).

Therefore, \((T, \sigma_n)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which
\[ \beta \in \mathcal{P}_n(y_p) \cap \mathcal{P}_n(T(y_{j-2})). \] Thus Claim 7.4 holds by replacing \( \varphi_n \) with \( \sigma_n \) and \( \alpha \) with \( \beta \).

**Case 2.** \( I_\varphi = \emptyset. \)

Let \( \alpha \in \mathcal{P}_n(y_p) \cap \mathcal{P}_n(T(y_{p-1})). \) By the hypothesis of the present case, we have \( \alpha \in \mathcal{P}_n(T_{n,q}^*). \)

If \( \alpha \notin \Gamma^q \cup \mathcal{P}_n(T_{n,0}^* - V(T_n)), \) we are done. So we assume that \( \alpha \in \Gamma^q \cup \mathcal{P}_n(T_{n,0}^* - V(T_n)). \)

**Subcase 2.1.** \( \alpha \in \mathcal{P}_n(T_{n,0}^* - V(T_n)) \subseteq \Gamma^q. \) Let us first show that (7) there exists a color \( \beta \in \mathcal{P}_n(T_{n,q}^*) - \mathcal{P}_n(T_{n,0}^* - V(T_n)) - \Gamma^q. \)

Indeed, since \( V(T_{n,q}^*) \) is elementary with respect to \( \varphi_n, \) we have |\( \mathcal{P}_n(T_{n,q}^*) \) - |\( \mathcal{P}_n(T_{n,0}^* - V(T_n)) \) - |\( \Gamma^q \). In view of (7.2), we obtain |\( \mathcal{P}_n(T_n) \) | \( \geq 2n+11 \) and |\( \mathcal{P}_n(T_n) \) | \( \leq 2|D_n| \leq 2n. \) So |\( \mathcal{P}_n(T_{n,q}^*) \) - |\( \mathcal{P}_n(T_{n,0}^* - V(T_n)) \) - |\( \Gamma^q \) | \( \geq 11, \) which implies (7).

By (7) and Lemma 6.6, we obtain \( P_{v_s}(\alpha, \beta, \varphi_n) = P_{v_0}(\alpha, \beta, \varphi_n), \) which is disjoint from \( P_{v_p}(\alpha, \beta, \varphi_n). \) Let \( \sigma_n = \varphi_n/P_{v_p}(\alpha, \beta, \varphi_n). \) By Lemma 6.7, \( \sigma_n \) satisfies all the properties described in (7.3). Since \( \alpha, \beta \in \mathcal{P}_n(T_{n,q}^*), \) we have \( \sigma_n(f) = \varphi_n(f) \) for each \( f \in E(T_{n,q}^*), \) and \( \mathcal{P}_n(u) = \mathcal{P}_n(u) \) for each \( u \in V(T(y_{p-1})). \) So we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \sigma_n \) and hence \( T \) satisfies MP under \( \sigma_n. \) As \( \alpha, \beta \notin \Gamma^q, \) the hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be good under \( \sigma_n, \) with the same \( \Gamma \)-sets as those under \( \varphi_n. \) Therefore, \( (T, \sigma_n) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \beta \in \mathcal{P}_n(y_p) \cap \mathcal{P}_n(T(y_{j-2})). \) Thus Claim 7.4 holds by replacing \( \varphi_n \) with \( \sigma_n \) and \( \alpha \) with \( \beta \).

**Subcase 2.2.** \( \alpha \in \Gamma^q. \) Let \( \alpha \in \Gamma_m^q \) for some \( \eta_m \in D_{n,q}. \) Depending on whether \( \eta_m \) is contained in \( \mathcal{P}_n(T(y_{p-1})), \) we consider two possibilities.

- **\( \eta_m \notin \mathcal{P}_n(T(y_{p-1})). \)** By Definition 5.2(i), we have \( \alpha \notin \varphi_n(T - T_{n,q}^*). \) Since \( T - T(y_{p-2}) \) contains precisely two edges, Lemma 6.4 guarantees the existence of a color \( \beta \in \mathcal{P}_n(T(y_{p-2})) - \mathcal{P}_n(T_{n,0}^* - V(T_n)) - \varphi_n(T - T_{n,q}^*) - (\Gamma^q \cap D_{n,q}) \) or a color \( \beta = \eta_v \in D_{n,q} \cap \mathcal{P}_n(T(y_{p-2})) \) such that \( (\Gamma_m^q \cup \{ \eta_v \}) \cap \varphi_n(T - T_{n,q}^*) = \emptyset. \) Note that \( \beta \in \mathcal{P}_n(T(y_{p-2})) - \varphi_n(T - T_{n,q}^*). \) By Lemma 6.6, we obtain \( P_{v_s}(\alpha, \beta, \varphi_n) = P_{v_0}(\alpha, \beta, \varphi_n), \) which is disjoint from \( P_{v_p}(\alpha, \beta, \varphi_n). \) Let \( \sigma_n = \varphi_n/P_{v_p}(\alpha, \beta, \varphi_n). \) By Lemma 6.7, \( \sigma_n \) satisfies all the properties described in (7.3). Since \( \alpha, \beta \in \mathcal{P}_n(T_{n,q}^*), \) we have \( \sigma_n(f) = \varphi_n(f) \) for each \( f \in E(T), \) and \( \mathcal{P}_n(u) = \mathcal{P}_n(u) \) for each \( u \in V(T(y_{p-1})). \) So we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \sigma_n \) and hence \( T \) satisfies MP under \( \sigma_n. \) As \( \alpha, \beta \notin \Gamma^q \) and \( \alpha, \beta \notin \sigma_n(T - T_{n,q}^*), \) the hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be good under \( \sigma_n, \) with the same \( \Gamma \)-sets as those under \( \varphi_n. \) Therefore, \( (T, \sigma_n) \) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \beta \in \mathcal{P}_n(y_p) \cap \mathcal{P}_n(T(y_{j-2})). \) Thus Claim 7.4 holds if \( y_{j-2} \leq y_j \), the present subcase reduces to the case when max(\( I_{\sigma_n} \)) \( \geq p(T) \) if \( y_{j-2} \leq y_j \) (see the paragraphs above Claim 7.4), and the present subcase reduces to Case 1 (where \( I_{\sigma_n} \neq \emptyset \)) if \( y_{j-1} = y_{j}. \)

- **\( \eta_m \in \mathcal{P}_n(T(y_{p-1})). \)** Note that \( \eta_m \notin \mathcal{P}_n(T_{n,q}^*), \) because \( \eta_m \in D_{n,q}. \) So \( \eta_m \in \mathcal{P}_n(y_p) \) for some \( 1 \leq t \leq p - 1. \) If \( t \leq j - 2, \) then Claim 7.4 holds. Thus we may assume that \( t \geq j - 1. \) Since \( \eta_m \in \mathcal{P}_n(y_p), \) it is not used by any edge in \( T(y_t) - T_{n,q}^* \), except possibly \( e_1 \) when \( q = 0 \) and \( T_{n,0}^* = T_n \) (now \( e_1 = f_n \) in Algorithm 3.1 and \( \varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n \)). Since \( \alpha \in \Gamma_m^q, \) by Definition 5.2(i), \( \alpha \) is not used by any edge in \( T(y_t) - T_{n,q}^* \). It follows from Lemma 6.6 that \( P_{v_s}(\alpha, \eta_m, \varphi_n) = P_{v_0}(\alpha, \eta_m, \varphi_n), \) which is disjoint from \( P_{v_p}(\alpha, \eta_m, \varphi_n). \) Let \( \sigma_n = \varphi_n/P_{v_p}(\alpha, \eta_m, \varphi_n). \) By Lemma 6.7, \( \sigma_n \) satisfies all the properties described in (7.3). In particular, if \( e_1 = f_n \) and \( \varphi_n(e_1) = \eta_m \in D_n, \) then \( \sigma_n(e_1) = \varphi_n(e_1), \) which implies that \( e_1 \) is outside \( P_{v_p}(\alpha, \eta_m, \varphi_n). \) Since \( \sigma_n(f) = \varphi_n(f) \) for each \( f \in E(T(y_t)) \) and \( \mathcal{P}_n(u) = \mathcal{P}_n(u) \) for each \( u \in V(T(y_{p-1})), \) we can obtain \( T \) from \( T_{n,q}^* \) by using TAA under \( \sigma_n, \) so \( T \) satisfies MP under \( \sigma_n. \)
Furthermore, as \( \alpha, \eta_m \in \sigma_n(T(y_j)) \), the hierarchy \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T \) remains to be good under \( \sigma_n \), with the same \( \Gamma \)-sets as those under \( \varphi_n \). Therefore, \((T, \sigma_n)\) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which \( \eta_m \in \sigma_n(y_p) \cap \sigma_n(y_j) \). Thus the present subcase reduces to the case when \( \max(I_{\sigma_n}) \geq p(T) \) if \( j \leq t \) (see the paragraphs above Claim 7.4), and reduces to Case 1 (where \( I_{\sigma_n} \neq \emptyset \)) if \( t = j - 1 \). This proves Claim 7.4.

Let \( \alpha \) be a color as described in Claim 7.4; that is, \( \alpha \in \mathcal{F}_n(y_p) \cap \mathcal{F}_n(T(y_j-2)) \), such that either \( \alpha \notin \Gamma^q \cup \mathcal{F}_n(T_{n,0} - V(T_n)) \) or \( \alpha \in \Gamma^q_m \) for some \( \eta_m \in D_{n,q} \) with \( v_{\eta_m} \leq y_j - 2 \). Since \( T(y_j) - T(y_j-2) \) contains precisely two edges, Lemma 6.4 guarantees the existence of a color \( \beta \) in \( \mathcal{F}_n(T(y_j-2)) \) such that \( \alpha \notin \mathcal{F}_n(T_{n,0} - V(T_n)) \) or \( \alpha \in \Gamma^q \cup D_{n,q} \). If \( \beta = \eta_h \in D_{n,q} \cap \mathcal{F}_n(T(y_j-2)) \), then \( (\gamma^h \cup \{\eta_h\}) \cap \mathcal{F}_n(T(y_j) - T_{n,q}^*) = \emptyset \). Note that \( (8) \beta \notin \mathcal{F}_n(T(y_j) - T_{n,q}^*) \cup \Gamma^q \).

Let \( Q = P_{\mathcal{F}_n}(\alpha, \beta, \varphi_n) \). We consider two cases, depending on whether \( Q \) intersects \( T(y_j-1) \).

**Case 1.** \( Q \) and \( T(y_j-1) \) have vertices in common. Let \( u \) be the first vertex of \( Q \) contained in \( T(y_j-1) \) as we traverse \( Q \) from \( y_p \). Define \( T' = T(y_j-1) \cup Q[u, y_p] \) if \( u = y_j-1 \) and \( T' = T(y_j-2) \cup Q[u, y_p] \) otherwise. By the choices of \( \alpha \) and \( \beta \), we have \( \alpha, \beta \not\in \mathcal{F}_n(T(y_j-2)) \). Then \( T' \) can be obtained from \( T(y_j-2) \) by using TAA under \( \varphi_n \). It follows that \( T' \) is an ETT satisfying MP with respect to \( \varphi_n \), with \( p(T') < p(T) \). If \( \alpha \notin \Gamma^q \), then both \( \alpha \) and \( \beta \) are outside \( \Gamma^q \) (see (8)), so \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T' \) is a good hierarchy of \( T' \) under \( \varphi_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \). If \( \alpha \in \Gamma^q \), then \( \alpha \in \Gamma^q_m \) for some \( \eta_m \in D_{n,q} \) with \( v_{\eta_m} \leq y_j - 2 \) by Claim 7.4. Since \( \alpha, \eta_m \in \mathcal{F}_n(T(y_j-2)) \) and \( \beta \notin \Gamma^q \), it is clear that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T' \) is also a good hierarchy of \( T' \) under \( \varphi_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \). So \( (T', \varphi_n) \) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on \((T, \varphi_n)\).

**Case 2.** \( Q \) is vertex-disjoint from \( T(y_j-1) \). Let \( \sigma_n = \varphi_n/Q \). By Lemma 5.8, \( \sigma_n \) is both \((T(y_j-1), D_{n,\varphi_n})\)-stable and \((T(y_j-1), \varphi_n)\)-invariant. In particular, if \( \Theta_n = PE \), then \( \sigma_n \) is also \((T_n \oplus R_n, D_{n,\varphi_n})\)-stable. Furthermore, \( T(y_j-1) \) is an ETT satisfying MP with respect to \( \sigma_n \), and \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_j-1) \) is a good hierarchy of \( T(y_j-1) \), with the same \( \Gamma \)-sets as \( T \) under \( \sigma_n \). By definition, \( \sigma_n \) is a \((T_{n,0}^*, D_{n,\varphi_n})\)-strongly stable coloring. If \( \alpha \notin \Gamma^q \), then both \( \alpha \) and \( \beta \) are outside \( \Gamma^q \) (see (8)), so \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T \) is a good hierarchy of \( T \) under \( \varphi_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \). If \( \alpha \in \Gamma^q \), then \( \alpha \in \Gamma^q_m \) for some \( \eta_m \in D_{n,q} \) with \( v_{\eta_m} \leq y_j - 2 \) by Claim 7.4. Since \( \alpha, \eta_m \in \mathcal{F}_n(T(y_j-2)) \) and \( \beta \notin \Gamma^q \), it is clear that \( T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T \) is also a good hierarchy of \( T \) under \( \varphi_n \), with the same \( \Gamma \)-sets as \( T \) under \( \varphi_n \). So \( (T, \sigma_n) \) is a counterexample to Theorem 5.3, in which \( \beta \) is missing at two vertices.

From the choice of \( \beta \) and the definition of \( \sigma_n \), we see that

(9) either \( \beta \notin \sigma_n(T_{n,0}^* - V(T_n)) \cup \mathcal{F}_n(T_{n,y_j}) \cup \mathcal{F}_n(T_{n,q}) \cup \Gamma^q \cup D_{n,q} \) or \( \beta = \eta_h \in D_{n,q} \cap \sigma_n(T(y_j-2)) \), such that \( (\gamma^h \cup \{\eta_h\}) \cap \sigma_n(T(y_j) - T_{n,q}^*) = \emptyset \).

Let \( \theta \in \sigma_n(y_p) \). Then \( \theta \notin \Gamma^q \). We proceed by considering two subcases.

**Subcase 2.1.** \( \theta \notin D_{n,q} \). In this subcase, using (6.6) and the set inclusion \( \mathcal{F}_n(T_n) \subset D_n \subset \mathcal{F}_n(T_{n,q}^*) \cup D_{n,q} \), we obtain

(10) \( \theta \notin \sigma_n(T(y_j-1)) \) and \( \theta \notin D_n \). So \( \theta \) is not assigned to any edge in \( T(y_j) - T_{n,q}^* \) by TAA.

As described in (9), there are two possibilities for \( \beta \).

- \( \beta \notin \sigma_n(T_{n,0}^* - V(T_n)) \cup \mathcal{F}_n(T_{n,y_j}) \cup \mathcal{F}_n(T_{n,q}) \cup \Gamma^q \cup D_{n,q} \). Observe that \( \beta \notin D_n \) if \( q = 0 \).
By Lemma 6.6, we obtain $P_{y_j} (\beta, \theta, \sigma_n) = P_{y_j} (\beta, \theta, \sigma_n)$, which is disjoint from $P_{y_p} (\beta, \theta, \sigma_n)$. Let $\mu_1 = \sigma_n / P_{y_p} (\beta, \theta, \sigma_n)$. By Lemma 6.7, $\mu_1$ satisfies all the properties described in (7.3). By (10), the assumption on $\beta$ and (6.6), we have $\mu_1 (f) = \sigma_n (f)$ for each $f \in E(T(y_j))$ and $\overline{p}_1 (u) = \sigma_n (u)$ for each $u \in V(T(y_p-1))$. So we can obtain $T$ from $T_{n,q}^*$ by using TAA under $\mu_1$ and hence $T$ satisfies MP under $\mu_1$. As $\beta, \theta \notin \Gamma^0$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under $\mu_1$, with the same $\Gamma$-sets as those under $\sigma_n$. Therefore, $(T, \mu_1)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta \in \overline{p}_1 (y_j) \cap \overline{p}_1 (y_j)$. Thus the present subcase reduces to the case when $\max (I_{\mu_1}) \geq p(T)$ (see the paragraphs above Claim 7.4).

- $\beta = \eta_1 \in D_{n,q} \cap \sigma_n (T(y_j-2))$, such that $(\pi_{\eta_1} \cup \{ \eta_1 \}) \cap \sigma_n (T(y_j) - T_{n,q}^*) = \emptyset$. For simplicity, we abbreviate the two colors $\gamma_{n_1}^q$ and $\gamma_{n_2}^q$ in $\Gamma_n^q$ (see Definition 5.2) to $\gamma_1$ and $\gamma_2$, respectively. By Lemma 6.6, we obtain $P_{v_j} (\beta, \gamma_1, \sigma_n) = P_{v_j} (\beta, \gamma_1, \sigma_n)$, which is disjoint from $P_{y_p} (\beta, \gamma_1, \sigma_n)$. Let $\mu_2 = \sigma_n / P_{y_p} (\beta, \gamma_1, \sigma_n)$. By Lemma 6.7, $\mu_2$ satisfies all the properties described in (7.3). By the assumption on $\beta$ and the definition of $\mu_2$, we deduce that

$$(11) \beta = \eta_1 \in D_{n,q} \cap \overline{p}_2 (T(y_j-2)), \text{ such that } (\pi_{\eta_1} \cup \{ \eta_1 \}) \cap \mu_2 (T(y_j) - T_{n,q}^*) = \emptyset.$$

By (11) and Lemma 6.6, we obtain $P_{v_j} (\theta, \gamma_1, \mu_2) = P_{y_j} (\theta, \gamma_1, \mu_2)$, which is disjoint from $P_{y_p} (\theta, \gamma_1, \mu_2)$. Let $\mu_3 = \mu_2 / P_{y_p} (\theta, \gamma_1, \mu_2)$. By Lemma 6.7, $\mu_3$ satisfies all the properties described in (7.3). By (11), (11) and (6.6), we have $\mu_3 (f) = \mu_2 (f)$ for each $f \in E(T(y_j))$ and $\overline{p}_3 (u) = \overline{p}_2 (u)$ for each $u \in V(T(y_p-1))$. So we can obtain $T$ from $T_{n,q}^*$ by using TAA under $\mu_3$ and hence $T$ satisfies MP under $\mu_3$. Furthermore, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under $\mu_3$, with the same $\Gamma$-sets as those under $\sigma_n$. Therefore, $(T, \mu_3)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta$ is missing at both $y_p$ and $v_{\gamma_1}$.

Thus the present subcase reduces to the case when $\max (I_{\mu_3}) \geq p(T)$ (see the paragraphs above Claim 7.4).

Subcase 2.2. $\theta \in D_{n,q}$. Let $\theta = \eta \in D_{n,q}$. For simplicity, we use $\varepsilon_1$ and $\varepsilon_2$ to denote the two colors $\gamma_{n_1}^q$ and $\gamma_{n_2}^q$ in $\Gamma_n^q$ (see Definition 5.2), respectively.

$$(12) \varepsilon_1, \varepsilon_2 \notin \sigma_n (T(y_j) - T_{n,q}^*) \text{ and } \eta_1 \text{ is not used by any edge in } T(y_j) - T_{n,q}^* \text{ under } \sigma_n, \text{ except possibly on } \varepsilon_1 \text{ when } q = 0 \text{ and } T_{n,0} = T_n \text{ (now } \varepsilon_1 = f_n \text{ in Algorithm 3.1 and } \sigma_n (e_1) = \eta \in D_{n,q} \subseteq D_n).$$

By (9), (12) and Lemma 6.6, we obtain $P_{v_4} (\varepsilon_1, \beta, \sigma_n) = P_{v_4} (\varepsilon_1, \beta, \sigma_n)$, which is disjoint from $P_{y_p} (\varepsilon_1, \beta, \sigma_n)$. Let $\mu_4 = \sigma_n / P_{y_p} (\varepsilon_1, \beta, \sigma_n)$. By Lemma 6.7, $\mu_4$ satisfies all the properties described in (7.3). By (9), we have $\beta \notin \sigma_n (T(y_j) - T_{n,q}^*)$, which together with (12) and (6.6) implies $\mu_4 (f) = \sigma_n (f)$ for each $f \in E(T(y_j))$ and $\overline{p}_4 (u) = \sigma_n (u)$ for each $u \in V(T(y_p-1))$. So we can obtain $T$ from $T_{n,q}^*$ by using TAA under $\mu_4$ and hence $T$ satisfies MP under $\mu_4$. Since $\beta \notin \Gamma^0$ by (9) and $\eta \in \overline{p}_4 (y_j)$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under $\mu_4$, with the same $\Gamma$-sets as those under $\sigma_n$. Therefore, $(T, \mu_4)$ is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\varepsilon_1$ is missing at both $y_p$ and $v_{\varepsilon_2}$.

From (12) and (6.6) it can be seen that
We propose to show that edge of \( P_3 \) and \((\_\_\_\_\_\_\_n)\) with respect to \(j_n\) De

**Definition 3.7** With stable coloring, and \( T \) of \( Q \) secting \( T \phi \) the interchangeability property with respect to \(n\). In the preceding subsection we have proved Theorem 5.3 and hence Theorem 3.10(i). To complete 7.2 Proof of Theorem 3.10(ii)

In the preceding subsection we have proved Theorem 5.3 and hence Theorem 3.10(i). To complete the proof of Theorem 3.10, we still need to establish the interchangeability property as described in Theorem 3.10(ii).

**Lemma 7.1.** Suppose Theorem 3.10(i), (iii)-(vi) hold for all ETTs with \( n \) rungs and satisfying MP, and suppose Theorem 3.10(ii) holds for all ETTs with \( n - 1 \) rungs and satisfying MP. Then Theorem 3.10(ii) holds for all ETTs with \( n \) rungs and satisfying MP; that is, \( T_{n+1} \) has the interchangeability property with respect to \( \varphi_n \).

**Proof.** Let \( T = T_{n+1} \), let \( \sigma_n \) be a \((T,D_n,\varphi_n)\)-stable coloring, and let \( \alpha \) and \( \beta \) be two colors in \([k]\) with \( \alpha \in \sigma_n(T) \) (equivalently \( \alpha \in \varphi_n(T) \)). We aim to prove that \( \alpha \) and \( \beta \) are \( T \)-interchangeable under \( \sigma_n \). Recalling (5.2), we may assume that \( T_{n+1} \) is a closure of \( T \cap R_n \) under \( \varphi_n \), which is a special closure of \( T_n \) under \( \varphi_n \), if \( \Theta_n = PE \). As introduced in Section 5, \( T_{n,0} = T_n \cap R_n \) if \( \Theta_n = PE \) and \( T_{n,0} = T_n \) otherwise.

Assume the contrary: there are at least two \((\alpha,\beta)\)-paths \( Q_1 \) and \( Q_2 \) with respect to \( \sigma_n \) intersecting \( T \). By Theorem 3.10(i), \( V(T) \) is elementary with respect to \( \varphi_n \), so it is also elementary with respect to \( \sigma_n \). Since \( T = T_{n+1} \) is closed with respect to \( \varphi_n \), it is also closed with respect to \( \sigma_n \). As \( \alpha \in \sigma_n(T) \), it follows that \( |V(T)| \) is odd and \( \beta \) is outside \( \sigma_n(T) \). From the existence of \( Q_1 \) and \( Q_2 \), we see that \( |\partial_{\sigma_n,\beta}(T)| \) is odd and at least three. Thus \( G \) contains at least three \((T,\sigma_n,\{\alpha,\beta\})\)-exit paths \( P_1, P_2, P_3 \).

We call the tuple \((\sigma_n,T,\alpha,\beta,P_1,P_2,P_3)\) a counterexample if \( \sigma_n \) is a \((T_{n,0},D_n,\varphi_n)\)-strongly stable coloring, and \( T \) is a closed ETT corresponding to \((\sigma_n,T_n)\) (see Theorem 3.10(vi) and Definition 3.7) with \( n \) rungs and satisfying MP under \( \sigma_n \). Moreover, \( P_1, P_2, P_3 \) are three \((T,\sigma_n,\{\alpha,\beta\})\)-exit paths. We use \( K \) to denote the set of all such counterexamples. With a slight abuse of notation, let \((\sigma_n,T,\alpha,\beta,P_1,P_2,P_3)\) be a counterexample in \( K \) with the minimum \( |P_1| + |P_2| + |P_3| \). For \( i = 1, 2, 3 \), let \( a_i \) and \( b_i \) be the ends of \( P_i \) with \( b_i \in V(T) \), and \( f_i \) be the edge of \( P_i \) incident to \( b_i \). Renaming subscripts if necessary, we may assume that \( b_1 \prec b_2 \prec b_3 \). We propose to show that
Theorem 3.10(vi), stable and a \((T; \varphi)\) with respect to \(Q\), which is a \((T; D, n, \varphi_n)\)-stable coloring. Since \(\Theta_n = SE\) or \(RE\), from Algorithm 3.1 we see that \(\mu_1\) is also a \((T, D, n, \varphi_n)\)-stable coloring. By Theorem 3.10(vi), \(T_n\) is an ETT corresponding to \(\mu_1\) (see Theorem 3.10(vi) and Definition 3.7) and satisfies MP under \(\mu_1\), with \(n - 1\) rungs. Since \(P_1, P_2, P_3\) are three \((T_n, \mu_1, \{\gamma, \beta\})\)-exit paths, there are at least two \((\gamma, \beta)\)-paths with respect to \(\mu_1\) intersecting \(T_n\). Hence \(\gamma\) and \(\beta\) are not \(T_n\)-interchangeable under \(\mu_1\), contradicting Theorem 3.10(ii) because \(T_n\) has \(n - 1\) rungs. So (1) is established.

(2) \(b_3 \notin V(T_n \vee R_n)\) if \(\Theta_n = PE\).

The proof is similar to that of (1). Assume the contrary: \(b_3 \in V(T_n \vee R_n)\). Let \(\gamma \in \sigma_n(T_n)\) and \(\mu_1 = \sigma_n/(G - T, \alpha, \gamma)\). By Lemma 5.8, \(\mu_1\) is both a \((T \cup D, n, \varphi_n)\)-stable and a \((T, \varphi_n)\)-invariant coloring. By definition, \(\mu_1\) is a \((T, D, n, \varphi_n)\)-stable coloring. Since \(\Theta_n = SE\) or \(RE\), from Algorithm 3.1 we see that \(\mu_3\) is also a \((T, D, n, \varphi_n)\)-stable coloring. By Theorem 3.10(vi), \(T_n\) is an ETT corresponding to \(\mu_3\) (see Theorem 3.10(vi) and Definition 3.7) and satisfies MP under \(\mu_3\), with \(n - 1\) rungs. Since \(P_1, P_2, P_3\) are three \((T_n, \mu_1, \{\gamma, \beta\})\)-exit paths, there are at least two \((\gamma, \beta)\)-paths with respect to \(\mu_1\) intersecting \(T_n \vee R_n\), contradicting Lemma 6.1(iii). So (2) holds.

Let \(\gamma \in \sigma_n(b_3)\) and \(\mu_2 = \sigma_n/(G - T, \alpha, \gamma)\). By Lemma 5.8, \(\mu_2\) is both a \((T \cup D, n, \varphi_n)\)-stable and a \((T, \varphi_n)\)-invariant coloring. So \(\mu_2\) is a \((T^*, D, n, \varphi_n)\)-strongly stable coloring. By Theorem 3.10(vi), \(T\) is an ETT corresponding to \(\mu_2\) (see Definition 3.7) and satisfies MP under \(\mu_2\). Note that \(f_i\) is colored by \(\beta\) under both \(\mu_2\) and \(\sigma_n\) for \(i = 1, 2, 3\).

Consider \(\mu_3 = \mu_2/P_{b_3}(\beta, \gamma, \mu_2)\). Clearly, \(\beta \in \overline{p_3}(b_3)\). By (1), (2) and Lemma 5.8, \(\mu_3\) is a \((T^*_{\mu_0}, D, n, \mu_3)\)-strongly stable coloring. It follows from Lemma 2.4 that \(\mu_3\) is a \((T^*_{\mu_0}, D, n, \varphi_n)\)-strongly stable coloring. By Theorem 3.10(vi), \(T(b_3)\) is an ETT corresponding to \(\mu_3\) (see Definition 3.7) and satisfies MP under \(\mu_3\). Let \(T'\) be obtained from \(T(b_3)\) by adding \(f_1\) and \(f_2\) and let \(T''\) be a closure of \(T'\) under \(\mu_3\). Obviously, both \(T'\) and \(T''\) are ETTs corresponding to \(\mu_3\) and MP under \(\mu_3\). By Theorem 3.10(i), \(V(T'')\) is elementary with respect to \(\mu_3\), because \(T''\) has \(n\) rungs.

Observe that none of \(a_1, a_2, a_3\) is contained in \(T''\), for otherwise, let \(a_i \in V(T''\) for some \(i\) with \(1 \leq i \leq 3\). Since \(\{\beta, \gamma\} \cap \overline{p_3}(a_i) \neq \emptyset\) and \(\beta \in \overline{p_3}(b_3)\), we obtain \(\gamma \in \overline{p_3}(a_i)\). Hence from TAA we see that \(P_1, P_2, P_3\) are all entirely contained in \(G[T'']\), which in turn implies \(\gamma \in \overline{p_3}(a_j)\) for \(j = 1, 2, 3\). So \(V(T'')\) is not elementary with respect to \(\mu_3\), a contradiction. Each \(P_i\) contains a subpath \(Q_i\), which is a \(T''\)-exit path with respect to \(\mu_3\). Since \(f_1\) is not contained in \(Q_1\), we obtain \(|Q_1| + |Q_2| + |Q_3| < |P_1| + |P_2| + |P_3|\). In view of (1) and (2), we have \(T_n \subseteq T''\) if \(\Theta_n \neq PE\) and \(T_n \vee R_n \subseteq T''\) if \(\Theta_n = PE\). Thus the existence of the counterexample \((\mu_3, T'', \gamma, \beta, Q_1, Q_2, Q_3)\) violates the minimality assumption on \((\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)\).

This completes our proof of Lemma 7.1 and hence of Theorem 3.10.

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