Average degrees of edge-chromatic critical graphs

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\textbf{Abstract}

Let $G$ be a simple graph, and let $\Delta(G)$, $\overline{d}(G)$ and $\chi'(G)$ denote the maximum degree, the average degree and the chromatic index of $G$, respectively. We called $G$ edge-$\Delta$-critical if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \leq \Delta(G)$ for every proper subgraph $H$ of $G$. Vizing in 1968 conjectured that if $G$ is an edge-$\Delta$-critical graph of order $n$, then $\overline{d}(G) \geq \Delta(G) - 1 + \frac{3}{n}$. We prove that for any edge-$\Delta$-critical graph $G$, $\overline{d}(G) \geq \min \left\{ \frac{2\sqrt{2}\Delta - 3 - \sqrt{2}}{2\sqrt{2} + 1}, \frac{3\Delta(G)}{4} - 2 \right\}$, that is,

$$\overline{d}(G) \geq \begin{cases} \frac{3}{2}\Delta(G) - 2 & \text{if } \Delta(G) \leq 75; \\ \frac{2\sqrt{2}\Delta - 3 - \sqrt{2}}{2\sqrt{2} + 1} \approx 0.7388\Delta(G) - 1.153 & \text{if } \Delta(G) \geq 76. \end{cases}$$

This result improves the best known bound $\frac{2}{3}(\Delta(G) + 2)$ obtained by Woodall in 2007 for $\Delta(G) \geq 41$.

\textbf{Keywords:} edge-$k$-coloring; edge-critical graphs; Vizing’s Adjacency Lemma

\section{Introduction}

All graphs in this paper, unless otherwise stated, are simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\Delta(G)$ the maximum degree of $G$. An

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edge-k-coloring of a graph $G$ is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(e) \neq \varphi(f)$ for any two adjacent edges $e$ and $f$. We call $\{1, 2, \ldots, k\}$ the color set of $\varphi$. Denote by $\mathcal{C}^k(G)$ the set of all edge-$k$-colorings of $G$. The chromatic index $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$. We call $G$ class one if $\chi'(G) = \Delta(G)$. Otherwise, Vizing [13] proved $\chi'(G) = \Delta(G) + 1$ and $G$ is said to be of class two. An edge $e$ is called critical if $\chi'(G - e) < \chi'(G)$, where $G - e$ is the subgraph obtained from $G$ by removing the edge $e$. A graph $G$ is called (edge-)\Delta-critical if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \leq \Delta(G)$ for any proper subgraph $H$ of $G$. Clearly, if $G$ is \Delta-critical, then $G$ is connected and $\chi'(G - e) = \Delta(G)$ for any $e \in E(G)$. Let $\bar{d}(G)$ denote the average degree of a graph $G$. Vizing [15] made the following conjecture in 1968.

**Conjecture 1.** [Vizing’s Average Degree Conjecture] If $G$ is a \Delta-critical graph of $n$ vertices, then $\bar{d}(G) \geq \Delta(G) - 1 + \frac{3}{n}$.

The conjecture has been verified for graphs with $\Delta(G) \leq 6$, see [4, 6, 7, 9]. In general, there are a few results on the lower bound for $\bar{d}(G)$. Let $G$ be a \Delta-critical graph with maximum degree $\Delta$. Fiorini [3] showed, for $\Delta \geq 2$,

$$\bar{d}(G) \geq \begin{cases} \frac{1}{2}(\Delta + 1) & \text{if } \Delta \text{ is odd;} \\ \frac{1}{2}(\Delta + 2) & \text{if } \Delta \text{ is even.} \end{cases}$$

Haile [5] improved the bounds as follows.

$$\bar{d}(G) \geq \begin{cases} \frac{3}{5}(\Delta + 2) & \text{if } \Delta = 9, 11, 13; \\ \Delta - \frac{12}{\Delta+4} & \text{if } \Delta \geq 10, \Delta \text{ is even;} \\ \frac{\Delta+6}{15+\sqrt{29}} & \text{if } \Delta = 15; \\ \frac{\Delta+7}{2} - \frac{16}{\Delta+5} & \text{if } \Delta \geq 17, \Delta \text{ is odd.} \end{cases}$$

Sanders and Zhao [10] showed $\bar{d}(G) \geq \frac{1}{2}(\Delta + \sqrt{2\Delta - 1})$ for $\Delta \geq 2$. Woodall [17] improved the bound to $\bar{d}(G) \geq \frac{t(\Delta+t-1)}{2t-1}$, where $t = \lceil \sqrt{\Delta/2} \rceil$. Improving Vizing’s Adjacency Lemma, Woodall [16] improved the coefficient of $\Delta$ from $\frac{1}{2}$ to $\frac{2}{3}$ as follows.

$$\bar{d}(G) \geq \begin{cases} \frac{2}{3}(\Delta + 1) & \text{if } \Delta \geq 2; \\ \frac{2}{4}\Delta + 1 & \text{if } \Delta \geq 8; \\ \frac{2}{3}(\Delta + 2) & \text{if } \Delta \geq 15. \end{cases}$$

In the same paper, Woodall provided an example demonstrating that the above result cannot be improved by the use of his new adjacency Lemmas (see Lemma 2 and Lemma 3).
and Vizing’s Adjacency Lemma alone. By proving a few stronger properties of $\Delta$-critical graphs, we get the following theorem.

**Theorem 1.** If $G$ is a $\Delta$-critical graph, then $\bar{d}(G) \geq \min\{\frac{\sqrt{2}\Delta - 3 - \sqrt{\Delta}}{2\sqrt{2} + 1}, \frac{\Delta(G) - 2}{4}\}$, that is,

$$\bar{d}(G) \geq \begin{cases} \frac{\Delta(G) - 2}{4} & \text{if } \Delta(G) \leq 75; \\ \frac{2\sqrt{2}\Delta - 3 - \sqrt{\Delta}}{2\sqrt{2} + 1} & \text{if } \Delta(G) \geq 76. \end{cases}$$

We will prove a few technical lemmas in Section 2 and give the proof of Theorem 1 in Section 3. We will use the following terminology and notation. Let $G$ be a graph and $x$ be a vertex of $G$. Denote by $N(x)$ the neighborhood, and by $d(x)$ the degree of $x$, i.e., $d(x) = |N(x)|$. For any nonnegative integer $m$, we call a vertex $x$ an $m$-vertex if $d(x) = m$, a ($< m$)-vertex if $d(x) < m$, and a ($> m$)-vertex if $d(x) > m$. Similarly, we call a neighbor $y$ of $x$ an $m$-neighbor, a ($< m$)-neighbor and a ($> m$)-neighbor if $d(y) = m$, $< m$ and $> m$, respectively.

Let $G$ be a graph, $F \subseteq E(G)$ be an edge set, and let $\varphi \in \mathcal{C}^k(G - F)$ be a coloring for some integer $k \geq 0$. For a vertex $v \in V(G)$, define the two color sets $\varphi(v) = \{\varphi(e) : e \text{ is incident with } v \text{ and } e \notin F\}$ and $\bar{\varphi}(v) = \{1, 2, \ldots, k\} \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors seen by $v$ and $\bar{\varphi}(v)$ the set of colors missing at $v$. A set $X \subseteq V(G)$ is called elementary with respect to $\varphi$ if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. For any color $\alpha$, let $E_{\alpha}$ denote the set of edges assigned color $\alpha$. Clearly, $E_{\alpha}$ is a matching of $G$. For any two colors $\alpha$ and $\beta$, the components of the spanning subgraph of $G$ with edge set $E_{\alpha} \cup E_{\beta}$, named $(\alpha, \beta)$-chains, are even cycles and paths with alternating color $\alpha$ and $\beta$. For a vertex $v$ of $G$, we denote by $P_v(\alpha, \beta, \varphi)$ the unique $(\alpha, \beta)$-chain that contains the vertex $v$. Let $\varphi/P_v(\alpha, \beta, \varphi)$ be the coloring obtained from $\varphi$ by switching colors $\alpha$ and $\beta$ on the edges on $P_v(\alpha, \beta, \varphi)$. This operation is called a Kempe change. If $v$ is not incident with any edge of color $\alpha$ or $\beta$, then $P_v(\alpha, \beta, \varphi) = \{v\}$ (a path of length 0), and $\varphi/P_v(\alpha, \beta, \varphi) = \varphi$.

## 2 Lemmas

Let $G$ be a graph and $q$ be a positive number. For each edge $xy \in E(G)$, let $\sigma_q(x, y) = |\{z \in N(y) \setminus \{x\} : d(z) \geq q\}|$, the number of neighbors of $y$ (except $x$) with degree at least $q$. Vizing studied the case $q = \Delta(G)$ and obtained the following result.

**Lemma 1.** [Vizing’s Adjacency Lemma [14]] If $G$ is a $\Delta$-critical graph, then $\sigma_{\Delta(G)}(x, y) \geq \Delta(G) - d(x) + 1$ for every $xy \in E(G)$. 

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Woodall [16] studied $\sigma_q(x, y)$ for the case $q = 2\Delta(G) - d(x) - d(y) + 2$ and obtained the following two results. For convenience, we let $\sigma(x, y) = \sigma_q(x, y)$ when $q = 2\Delta(G) - d(x) - d(y) + 2$. Let $xy \in E(G)$. To study the difference between $\sigma(x, y)$ and $\Delta(G) - d(x) + 1$, Woodall defined the following two parameters.

$$p_{\min}(x) := \min_{y \in N(x)} \sigma(x, y) - \Delta(G) + d(x) - 1,$$

$$p(x) := \min \left\{ p_{\min}(x), \left\lfloor \frac{d(x)}{2} \right\rfloor - 1 \right\}.$$  

**(Lemma 2. [Woodall [16]]** Let $xy$ be an edge in a $\Delta$-critical graph $G$. Then there are at least $\Delta(G) - \sigma(x, y) \geq \Delta(G) - d(y) + 1$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma(x, z) \geq 2\Delta(G) - d(x) - \sigma(x, y)$.

**(Lemma 3. [Woodall [16]]** Every vertex $x$ in a $\Delta$-critical graph has at least $d(x) - p(x) - 1$ neighbors $y$ for which $\sigma(x, y) \geq \Delta(G) - p(x) - 1$.

Inspired by Woodall’s parameters (1) and (2), for any positive number $q$, we define the following two parameters.

$$p_{\min}(x, q) := \min_{y \in N(x)} \sigma_q(x, y) - \Delta(G) + d(x) - 1$$
and

$$p(x, q) := \min \left\{ p_{\min}(x, q), \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 \right\}.$$  

The following lemma is a generalization of Lemma 2, which serves as a key result.

**(Lemma 4.** Let $xy$ be an edge in a $\Delta$-critical graph $G$ and $q$ be a positive number. If $\Delta(G)/2 < q \leq \Delta(G) - d(x)/2 - 2$, then $x$ has at least $\Delta(G) - \sigma_q(x, y) - 2$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma_q(x, z) \geq 2\Delta(G) - d(x) - \sigma_q(x, y) - 4$.

Due to its length, the proof of Lemma 4 will be placed at the end of this section. The following is a consequence of it.

**(Lemma 5.** Let $G$ be a $\Delta$-critical graph, $x \in V(G)$ and $q$ be a positive number. If $\Delta(G)/2 < q \leq \Delta(G) - d(x)/2 - 2$, then $x$ has at least $d(x) - p(x, q) - 3$ neighbors $y$ for which $\sigma_q(x, y) \geq \Delta(G) - p(x, q) - 5$.

**Proof.** Let $y \in N(x)$ such that $p_{\min}(x, q) = \sigma_q(x, y) - \Delta(G) + d(x) - 1$, and $\Delta = \Delta(G)$.

If $p(x, q) = p_{\min}(x, q)$, by Lemma 4, $x$ has at least $\Delta - \sigma_q(x, y) - 2 - d(x) - p_{\min}(x, q) - 3$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma_q(x, z) \geq 2\Delta - d(x) - \sigma_q(x, y) - 4 = \Delta - p_{\min}(x, q) - 5$.  

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If \( p(x, q) = \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 < p_{\min}(x, q) \), then for every \( y \in N(x) \), \( \sigma_q(x, y) > \Delta - d(x) + 1 + \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 \geq \Delta - \left\lfloor \frac{d(x)}{2} \right\rfloor - 3 = \Delta - p(x, q) - 6 \). So \( \sigma_q(x, y) \geq \Delta - p(x, q) - 5 \). \( \square \)

Our approach is inspired by the recent development of the Tashkinov tree technique for multigraphs. Let \( G \) be a multigraph without loops, \( e_1 \) be an edge of \( G \) with endvertices \( y_0 \) and \( y_1 \) and \( \varphi \in C^k(G - e_1) \). A Tashkinov tree \( T \) with respect to \( G, e_1, \varphi \) is an alternating sequence \( T = (y_0, e_1, y_1, \ldots, e_p, y_p) \) with \( p \geq 1 \) consisting of edges \( e_1, e_2, \ldots, e_p \) and vertices \( y_0, y_1, \ldots, y_p \) such that the following two conditions hold.

1. The vertices \( y_0, y_1, \ldots, y_p \) are distinct and \( e_i = y_{i-1}y_i \) for each \( 1 \leq i \leq p \), where \( r < i \);

2. for every edge \( e_i \) with \( 2 \leq i \leq p \), there is a vertex \( y_h \) with \( 0 \leq h < i \) such that \( \varphi(e_i) \in \varphi(y_h) \).

Tashkinov [12] introduced this concept in his work on the well-known Goldberg’s Conjecture. In the above definition, if we replace condition (1) with the edges \( e_1, e_2, \ldots, e_p \) are distinct and \( e_i = y_0y_i \) for every \( i \), then \( T \) is a multi-fan, as defined in [11]; if, in addition, \( \varphi(e_i) \in \varphi(y_{i-1}) \) for every \( i \geq 2 \), then \( T \) is a Vizing fan. Stiebitz et al. [11] showed that all multi-fans are elementary. In the definition of Tashkinov tree, if \( e_i = y_{i-1}y_i \) for every \( i \), i.e., \( T \) is a path with end-vertices \( y_0 \) and \( y_p \), then \( T \) is a Kierstead path, which was introduced by Kierstead [8] in studying edge-colorings of multigraphs. For simple graphs, following Kierstead’s proof, Zhang [18] noticed that for a Kierstead path \( P \) the set \( V(P) \) is elementary if \( G \) is \( \Delta \)-critical and \( d(y_i) < \Delta \) for every \( i \) with \( 2 \leq i \leq p \). It is not difficult to see that every Kierstead path \( P \) with three vertices is a Vizing fan, so \( V(P) \) is elementary if \( G \) is \( \Delta \)-critical. Kostochka and Stiebitz studied Kierstead paths with four vertices and obtained the following result.

**Lemma 6.** [Kostochka and Stiebitz [11]] Let \( G \) be a graph with maximum degree \( \Delta \) and \( \chi'(G) = \Delta + 1 \). Let \( e_1 \in E(G) \) be a critical edge and \( \varphi \in C^\Delta(G - e_1) \). If \( K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3) \) is a Kierstead path with respect to \( e_1 \) and \( \varphi \), then the following statements hold:

1. \( \varphi(y_0) \cap \varphi(y_1) = \emptyset \);

2. if \( d(y_2) < \Delta \), then \( V(K) \) is elementary with respect to \( \varphi \);

3. if \( d(y_1) < \Delta \), then \( V(K) \) is elementary with respect to \( \varphi \);

4. if \( \Gamma = \varphi(y_0) \cup \varphi(y_1) \), then \( |\varphi(y_2) \cap \Gamma| \leq 1 \).
In the definition of Tashkinov tree $T = (y_0, e_1, y_1, e_2, y_2, \ldots, y_p)$, we call $T$ a broom if $e_2 = y_1y_2$ and for each $i \geq 3$, $e_i = y_2y_i$, i.e., $y_2$ is one of the end-vertices of $e_i$ for each $i \geq 3$. Moreover, we call $T$ a simple broom if $\varphi(e_i) \in \varphi(y_0) \cup \varphi(y_1)$ for each $i \geq 3$, i.e., $(y_0, e_1, y_1, e_2, y_2, e_i, y_i)$ is a Kierstead path.

**Lemma 7.** [Chen, Chen, Zhao [2]] Let $G$ be a graph with maximum degree $\Delta$ and $\chi'(G) = \Delta + 1$. Let $e_1 = y_0y_1 \in E(G)$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$, and let $B = \{y_0, e_1, y_1, e_2, y_2, \ldots, e_p, y_p\}$ be a simple broom. If $|\varphi(y_0) \cup \varphi(y_1)| \geq 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary with respect to $\varphi$.

**Lemma 8.** Let $G$ be a graph with maximum degree $\Delta$ and $\chi'(G) = \Delta + 1$, $xy \in E(G)$ be a critical edge and $\varphi \in \mathcal{C}^{\Delta}(G - xy)$. Let $q$ be a positive number with $d(x) < q \leq \Delta - 1$ and $Z = \{z \in N(x) \setminus \{y\} : d(z) > q, \varphi(xz) \in \varphi(y)\}$. Then

$$|Z| \geq \Delta - d(y) + 1 - \frac{d(x) + d(y) - \Delta - 2}{\Delta - q}.$$  \hfill (3)

$$\sum_{z \in Z} (d(z) - q) \geq (\Delta - q)(\Delta - d(y) + 1) - d(x) - d(y) + \Delta + 2.$$  \hfill (4)

and for every $z \in Z$,

$$\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) - \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q}.$$  \hfill (5)

**Proof.** Since $G$ is not $\Delta$-colorable, $\varphi(x) \cap \varphi(y) = \emptyset$. Let $Z_y := \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \varphi(y)\}$. Clearly, $Z \subseteq Z_y$ and $|Z_y| = \Delta - d(y) + 1$. Since $\{y, x\} \cup Z_y$ forms a multi-fan with center $x$, it is elementary, so $|\varphi(x)| + |\varphi(y)| + \sum_{z \in Z_y} |\varphi(z)| \leq \Delta$. Since $|\varphi(x)| = \Delta - d(x) + 1$ and $|\varphi(y)| = \Delta - d(y) + 1$, we have

$$\sum_{z \in Z_y} |\varphi(z)| \leq \Delta - |\varphi(x)| - |\varphi(y)| \leq d(x) + d(y) - \Delta - 2.$$  \hfill (6)

Since $d(z) \leq q$ for all $z \in Z_y - Z$, $\sum_{z \in Z_y} |\varphi(z)| \geq (|Z_y| - |Z|)(\Delta - q)$. Combining this with (6), we get $|Z| \geq |Z_y| - \left[\frac{d(x) + d(y) - \Delta - 2}{\Delta - q}\right] = \Delta - d(y) + 1 - \left[\frac{d(x) + d(y) - \Delta - 2}{\Delta - q}\right]$. So (3) holds.

Since $|\varphi(z)| = \Delta - d(z)$ for each $z \in Z_y$, by (6), we get

$$\sum_{z \in Z_y} d(z) \geq |Z_y|\Delta - (d(x) + d(y) - \Delta - 2).$$
Since $d(z) \leq q$ for every $z \in Z_y - Z$, we have

$$\sum_{z \in Z} (d(z) - q) \geq \sum_{z \in Z_y} (d(z) - q) \geq |Z_y| \Delta - (d(x) + d(y) - \Delta - 2) - |Z_y|q.$$ 

Substituting $|Z_y| = \Delta - d(y) + 1$ in the above inequality, we get (4).

For each $z \in Z$, let $U_z^* = \{u \in N(z) \setminus \{x, y\} : \varphi(zu) \in \varphi(x) \cup \varphi(y)\}$ and $U_z = \{u \in U_z^* : d(u) > q\}$. Since $\varphi(xz) \in \varphi(y)$, $|U_z^*| \geq 2\Delta - d(x) - d(y)$ and $\{y, x, z\} \cup U_z^*$ forms a simple broom. Since $d(x) < q \leq \Delta - 1$, we have $d(x) \leq \Delta - 2$. Thus $|\bar{\varphi}(x) \cup \bar{\varphi}(y)| \geq 4$ and $\min\{d(x), d(z)\} = d(x) < \Delta$. By Lemma 7, $\{y, x, z\} \cup U_z^*$ is elementary with respect to $\varphi$. So $\sum_{u \in U_z^*} |\bar{\varphi}(u)| + |\bar{\varphi}(x)| + |\bar{\varphi}(y)| + |\bar{\varphi}(z)| \leq \Delta$, which in turn gives $\sum_{u \in U_z^*} |\bar{\varphi}(u)| \leq d(x) + d(y) + d(z) - 2\Delta - 2$. Since $d(u) \leq q$ for every $u \in U_z^* - U_z$, $\sum_{u \in U_z^*} |\bar{\varphi}(u)| \geq (|U_z^*| - |U_z|)(\Delta - q)$. So,

$$(|U_z^*| - |U_z|)(\Delta - q) \leq d(x) + d(y) + d(z) - 2\Delta - 2.$$ 

Since $|U_z^*| \geq 2\Delta - d(x) - d(y)$, we get

$$\sigma_q(x, z) \geq |U_z| \geq 2\Delta - d(x) - d(y) - \left\lfloor \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right\rfloor.$$ 

So the inequality (5) holds. \(\square\)

2.1 Proof of Lemma 4

Lemma 4. Let $xy$ be an edge in a $\Delta$-critical graph $G$ and $q$ be a positive number. If $\Delta(G)/2 < q \leq \Delta(G) - d(x)/2 - 2$, then $x$ has at least $\Delta(G) - \sigma_q(x, y) - 2$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma_q(x, z) \geq 2\Delta(G) - d(x) - \sigma_q(x, y) - 4$.

**Proof.** Let graph $G$, edge $xy \in E(G)$ and $q$ be defined as in Lemma 4. Let $\Delta = \Delta(G)$. A vertex $z \in N(x) \setminus \{y\}$ is called feasible if there exists a coloring $\varphi \in \mathcal{C}^G(G - xy)$ such that $\varphi(xz) \in \varphi(y)$, and such a coloring $\varphi$ is called $z$-feasible. Denote by $\mathcal{C}_z$ the set of all $z$-feasible colorings. For each feasible vertex $z$ and each $z$-feasible coloring $\varphi \in \mathcal{C}_z$, let

- $Z(\varphi) = \{v \in N(z) \setminus \{x\} : \varphi(vz) \in \varphi(x) \cup \varphi(y)\}$,
- $C_z(\varphi) = \{\varphi(vz) : v \in Z(\varphi) \text{ and } d(v) < q\} \subseteq \varphi(x) \cup \varphi(y)$,
- $Y(\varphi) = \{v \in N(y) \setminus \{x\} : \varphi(vy) \in \varphi(x) \cup \varphi(z)\}$, and
- $C_y(\varphi) = \{\varphi(vy) : v \in Y(\varphi) \text{ and } d(v) < q\} \subseteq \varphi(x) \cup \varphi(z)$.

For each $\varphi \in \mathcal{C}_z$,

$$\sum_{v \in Z(\varphi)} (d(v) - q) \geq \sum_{v \in C_z(\varphi)} (d(v) - q) \geq |C_z(\varphi)| \Delta - (d(x) + d(y) - \Delta - 2) - |C_z(\varphi)|q.$$ 

Substituting $|C_z(\varphi)| = \Delta - d(z) + 1$ in the above inequality, we get (4).

By the choice of $\varphi$, $\sum_{u \in C_z(\varphi)} (d(u) - q) \leq \sum_{u \in C_z(\varphi)} (d(u) - q) \leq |C_z(\varphi)| \Delta - (d(x) + d(y) - \Delta - 2) - |C_z(\varphi)|q$. So $\sigma_q(x, z) \geq |C_z(\varphi)| \geq (|C_z(\varphi)| - |C_y(\varphi)|)(\Delta - q)$. Since $\Delta(G)/2 < q \leq \Delta(G) - d(x)/2 - 2$, we have $\sigma_q(x, z) \geq 2\Delta(G) - d(x) - 2\Delta - 2 - \sigma_q(x, y) - 4$. This completes the proof. \(\square\)
Note that $Z(\varphi)$ and $Y(\varphi)$ are vertex sets while $C_z(\varphi)$ and $C_y(\varphi)$ are color sets. For each color $\alpha \in \varphi(z)$, let $z_\alpha \in N(z)$ such that $\varphi(zz_\alpha) = \alpha$. For each color $\beta \in \varphi(y)$, let $y_\beta \in N(y)$ such that $\varphi(yy_\beta) = \beta$. Let 

$$T(\varphi) = \{ \alpha \in \varphi(x) \cap \varphi(y) \cap \varphi(z) : d(y_\alpha) < q \text{ and } d(z_\alpha) < q \}. $$

Since $(\varphi(x) \cap \varphi(y)) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y)) = \emptyset$ and $(\varphi(x) \cap \varphi(z)) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(z)) = \emptyset$, we obtain that $T(\varphi) \cap (C_z(\varphi) \cup C_y(\varphi)) = \emptyset$.

Since $G$ is $\Delta$-critical and $\varphi$ is $z$-feasible, $\{x, y, z\}$ is elementary with respect to $\varphi$. So $\bar{\varphi}(x)$, $\bar{\varphi}(y)$, $\bar{\varphi}(z)$ and $\varphi(x) \cap \varphi(y) \cap \varphi(z)$ are mutually exclusive. It is not difficult to see that 

$$|Z(\varphi)| = |\bar{\varphi}(x) \cup \bar{\varphi}(y)| - 1 \quad \text{and} \quad |Y(\varphi)| = |\bar{\varphi}(x) \cup \bar{\varphi}(z)|. \quad (7)$$

Also, 

$$\bar{\varphi}(x) \cup \bar{\varphi}(y) \cup \bar{\varphi}(z) \cup (\varphi(x) \cap \varphi(y) \cap \varphi(z)) = \{1, 2, \ldots, \Delta\}. \quad (8)$$

Recall that $\sigma_q(x, y)$ and $\sigma_q(x, z)$ are the numbers of vertices with degree at least $q$ in $N(y) \setminus \{x\}$ and $N(z) \setminus \{x\}$, respectively. So, by equations (7) and (8), we have 

$$\begin{align*}
\sigma_q(x, y) + \sigma_q(x, z) & \geq |Y(\varphi)| - |C_y(\varphi)| + |Z(\varphi)| - |C_z(\varphi)| + |\varphi(x) \cap \varphi(y) \cap \varphi(z)| - |T(\varphi)| \\
& = |\bar{\varphi}(x) \cup \bar{\varphi}(y)| - 1 - |\varphi(x) \cap \varphi(y) \cap \varphi(z)| - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)| \\
& = \Delta + |\bar{\varphi}(x)| - 1 - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)| \\
& = 2\Delta - d(x) - |C_y(\varphi)| - |C_z(\varphi)| - |T(\varphi)|.
\end{align*}$$

So, Lemma 4 follows from the three statements below.

I. For any $\varphi \in C_z$, $|C_z(\varphi)| \leq 1$ and $|C_y(\varphi)| \leq 1$;

II. there exists a $\varphi \in C_z$ such that $|T(\varphi)| \leq 2$; and

III. there are at least $\Delta - \sigma_q(x, y) - 2$ feasible vertices $z \in N(x) \setminus \{y\}$.

For every $z$-feasible coloring $\varphi \in \mathcal{C}^z(G - xy)$, let $\varphi^d \in \mathcal{C}^z(G - xz)$ be obtained from $\varphi$ by assigning $\varphi^d(xy) = \varphi(xz)$ and keeping all colors on other edges unchanged. Clearly, $\varphi^d$ is a $y$-feasible coloring and $Z(\varphi^d) = Z(\varphi)$, $Y(\varphi^d) = Y(\varphi)$, $C_z(\varphi^d) = C_z(\varphi)$ and $C_y(\varphi^d) = C_y(\varphi)$. We call $\varphi^d$ the dual coloring of $\varphi$. Considering dual colorings, we see that some properties for vertex $z$ also hold for vertex $y$.

Since $q \leq \Delta - d(x)/2 - 2$, we have
\[(\Delta - q) + (\Delta - d(x) + 1) - 5 \geq \Delta. \quad (9)\]

So, for any \(\varphi \in C^\Delta(G - xy)\) and any elementary set \(X\) with \(x \in X\),
\[X \setminus \{x\}\) contains at most one vertex with degree at most \(q\).
\[
(10)
\]

Let \(z \in N(x) \setminus \{y\}\) be a feasible vertex and \(\varphi \in C_z\). By the definition of \(Z(\varphi)\), \(\{y, x, z\} \cup Z(\varphi)\) is the vertex-set of a simple broom, and so this set is elementary with respect to \(\varphi\). Thus, \(|C_z(\varphi)| \leq 1\) by (10). By considering its dual \(\varphi^d\), we have \(|C_y(\varphi)| = |C_y(\varphi^d)| \leq 1\). Hence, I holds. The proofs of II and III are much more complicated. Let \(R(\varphi) = C_z(\varphi) \cup C_y(\varphi)\). In the remainder of the proof, we let \(Z = Z(\varphi), Y = Y(\varphi), C_z = C_z(\varphi), C_y = C_y(\varphi), R = R(\varphi)\) and \(T = T(\varphi)\) if the coloring \(\varphi\) is clear. Note that \(T \cap R = \emptyset, \varphi(xz) \notin T \cup R\) and \(|R| \leq 2\).

A coloring \(\varphi \in C_z\) is called optimal if \(|C_z(\varphi)| + |C_y(\varphi)|\) is maximum over all \(z\)-feasible colorings. Note that a \(z\)-feasible coloring \(\varphi\) is optimal if and only if its dual coloring \(\varphi^d\) is an optimal \(y\)-feasible coloring.

2.1.1 Proof of Statement II.

Suppose on the contrary that \(|T(\varphi)| \geq 3\) for every \(\varphi \in C_z\). Let \(\varphi\) be an optimal \(z\)-feasible coloring and assume, without loss of generality, \(\varphi(xz) = 1\). Since \(\chi'(G) > \Delta\) and \(1 \in \varphi(y)\), it follows that
\[
\text{for every color } i \in \varphi(x), \quad P_x(1, i, \varphi) = P_y(1, i, \varphi). \quad (11)
\]

Otherwise, an edge-\(\Delta\)-coloring of \(G\) can be gotten by \(\varphi / P_x(1, i, \varphi)\) and then coloring \(xy\) with 1, a contradiction.

Claim A. For each \(i \in \varphi(x) \setminus R\) and \(k \in T(\varphi)\), \(P_x(i, k, \varphi)\) contains both \(y\) and \(z\).

Proof. We first show that \(z \in V(P_x(i, k, \varphi))\). Otherwise, \(P_x(i, k, \varphi)\) is disjoint from \(P_x(i, k, \varphi)\). Let \(\varphi' = \varphi / P_x(i, k, \varphi)\). Since \(1 \notin \{i, k\}\), \(\varphi'\) is also \(z\)-feasible. Since colors in \(R\) are unchanged and \(d(z_k) < q\), \(C_z(\varphi') = C_z(\varphi) \cup \{i\}\) and \(C_y(\varphi') \supseteq C_y(\varphi)\), giving a contradiction to the maximality of \(|C_y(\varphi)| + |C_z(\varphi)|\). By considering the dual coloring \(\varphi^d\), we can verify that \(y \in V(P_x(i, k, \varphi))\).

Note that \(|T(\varphi)| \geq 3\). For each 3-element subset \(S(\varphi)\) of \(T(\varphi)\), we may assume
$S(\varphi) = \{k_1, k_2, k_3\}$, let
\[
V(S(\varphi)) = \{z_{k_1}, z_{k_2}, z_{k_3}\} \cup \{y_{k_1}, y_{k_2}, y_{k_3}\};
\]
\[
W(S(\varphi)) = \{u \in V(S(\varphi)) : \varphi(u) \cap \varphi(x) \setminus R = \emptyset\},
\]
\[
M(S(\varphi)) = \{u \in V(S(\varphi)) : \varphi(u) \cap \varphi(x) \setminus R \neq \emptyset\},
\]
\[
E(S(\varphi)) = \{zz_{k_1}, zz_{k_2}, zz_{k_3}, yy_{k_1}, yy_{k_2}, yy_{k_3}\},
\]
\[
E_W(S(\varphi)) = \{e \in E(S(\varphi)) : e \text{ is incident to a vertex in } W(S(\varphi))\},
\]
\[
E_M(S(\varphi)) = \{e \in E(S(\varphi)) : e \text{ is incident to a vertex in } M(S(\varphi))\}.
\]

Clearly, $V(S(\varphi)) = W(S(\varphi)) \uplus M(S(\varphi))$ and $E(S(\varphi)) = E_W(S(\varphi)) \uplus E_M(S(\varphi))$, where $\uplus$ denotes disjoint union. For convenience, we let $W = W(S(\varphi)), M = M(S(\varphi)), E_W = E_W(S(\varphi))$ and $E_M = E_M(S(\varphi))$ if $S(\varphi)$ is clear. Note that one of $y$ and $z$ has degree at least $q$ by (10), thus $\{y, z\} \not\subseteq V(S(\varphi))$.

We assume that $|E_W(S(\varphi))|$ is minimum over all optimal $z$-feasible colorings $\varphi$ and all 3-element subsets $S(\varphi)$ of $T(\varphi)$. For each $v \in M$, pick a color $\alpha_v \in \varphi(v) \cap \varphi(x) \setminus R$. Let $C_M = \{\alpha_v : v \in M\}$. Clearly, $|C_M| \leq |M|$. By (9) and the condition $q > \Delta/2$, we have
\[
|\varphi(x)| = \Delta - d(x) + 1 \geq \Delta - 2(\Delta - q) + 5 > 5.
\]
(12)

Since $|R| \leq 2$, we have
\[
\varphi(x) \setminus (R \cup C_M) \neq \emptyset \text{ if } |C_M| \leq 3.
\]
(13)

Note that $\{z_{k_1}, z_{k_2}, z_{k_3}\} \cap \{y_{k_1}, y_{k_2}, y_{k_3}\}$ may be not empty, $\frac{|E_W|}{2} \leq |W| \leq |E_W|$ and $\frac{|E_M|}{2} \leq |M| \leq |E_M|$.  

**Claim B.** Let $u$ and $v$ be two vertices of $V(S(\varphi))$. If $\varphi(u) \cap \varphi(v) \cap \varphi(x) \setminus R \neq \emptyset$, then the following statements hold.

(i) If $u, v \in W$, then $\varphi(x) \setminus (R \cup C_M) = \emptyset$.

(ii) There exists an optimal $z$-feasible coloring $\varphi^*$ with $V(S(\varphi^*)) = V(S(\varphi))$ such that $|E_W(S(\varphi^*))| \leq |E_W|$ and $\{u, v\} \cap M(S(\varphi^*)) \neq \emptyset$.

**Proof.** Noting that $\varphi(u) \cap \varphi(v) \cap \varphi(x) \setminus R \neq \emptyset$, we assume $\alpha \in \varphi(u) \cap \varphi(v) \cap \varphi(x) \setminus R$. If $\{u, v\} \cap M \neq \emptyset$, we are done (with $\varphi^* = \varphi$). So suppose $u, v \in W$. Let $\beta$ be an arbitrary color in $\varphi(x) \setminus (R \cup C_M)$ if this set is nonempty; otherwise, let $\beta$ be a color in $\varphi(x) \setminus R$, which is nonempty by (12) since $|R| \leq 2$. Since $u, v \in W$, we have $\beta \in \varphi(u) \cap \varphi(v)$. So, both $u$ and $v$ are endvertices of $(\alpha, \beta)$-chains. Assume without loss of generality that $P_u(\alpha, \beta, \varphi)$ is disjoint from $P_x(\alpha, \beta, \varphi)$. Let $\varphi^* = \varphi/P_u(\alpha, \beta, \varphi)$. Let $\varphi^*$...
First, we note that $T(\varphi^*) \supseteq T(\varphi)$. This is obvious if $\alpha \not\in T(\varphi)$, and if $\alpha \in T(\varphi)$ it holds because in this case \{z, y, z_\alpha, y_\alpha\} $\subseteq V(P_z(\alpha, \beta, \varphi))$ by Claim A and so \{z, y, z_\alpha, y_\alpha\} $\cap V(\mu_\alpha(\alpha, \beta, \varphi)) = \emptyset$. So we can choose $S(\varphi^*) = S(\varphi)$, and then $V(S(\varphi^*)) = V(S(\varphi))$.

Next, we claim that $\varphi^*$ is an optimal $z$-feasible coloring. Note that $\varphi^*(xz) = 1 \in \tilde{\varphi}^*(y)$. This is obvious if $\alpha \neq 1$ (since $\beta \neq 1$), and if $\alpha = 1$ it holds because in this case $P_\alpha(\alpha, \beta, \varphi) = P_y(\alpha, \beta, \varphi)$ by (11) and so $V(P_\alpha(\alpha, \beta, \varphi)) \cap \{x, y, z\} = \emptyset$. So $\varphi^*$ is z-feasible.

Since $\alpha, \beta \notin R = C_y \cup C_z$, it follows that $C_y \subset C_y(\varphi^*)$ and $C_z \subset C_z(\varphi^*)$. Since $\varphi$ is optimal, we have $C_y(\varphi^*) = C_y, C_z(\varphi^*) = C_z, R(\varphi^*) = R$ and so $\varphi^*$ is optimal.

Since $\alpha \in \varphi(u)$, it follows that $\beta \in \varphi^*(u) \cap \varphi^*(x) \setminus R(\varphi^*)$, so $u \in M(S(\varphi^*))$ (that is, $u$ has moved from $W$ to $M(S(\varphi^*))$). To avoid the contradiction that $|E_W(S(\varphi^*))| < |E_W|$, it is necessary that $P_u(\alpha, \beta, \varphi)$, which starts with an edge of color $\beta$ at $u$, must end with an edge of color $\alpha$ at a vertex $w \in M$ such that $\beta$ is the unique color in $\varphi(w) \cap \varphi(x) \setminus R$, thus $\varphi^*(w) \cap \varphi^*(x) \setminus R(\varphi^*) = \emptyset$ (that is, $w$ has moved from $M$ to $W(S(\varphi^*))$). But then we must have chosen $\alpha_w = \beta$, so $\beta \in C_M$. By the choice of $\beta$, we have $\varphi(x) \setminus (R \cup C_M) = \emptyset$. Thus (i) holds. By (13), we have $|C_M| \geq 4$. Since $u, v \in W$, $|V(S(\varphi))| \leq 6$ and $|C_M| \leq |M|$, we have $|V(S(\varphi))| = 6, |W| = 2$ and $|M| = 4$. Thus $|E_W(S(\varphi^*))| = |E_W|$. Hence, (ii) holds.

Claim C. There exist a color $k \in \{k_1, k_2, k_3\}$, three distinct colors $i, j, l$ and an optimal z-feasible coloring $\varphi^*$ with $\varphi^*(xz) = 1$ such that $i \in \varphi^*(z_k) \cap \varphi^*(x) \setminus R(\varphi^*)$, $j \in \varphi^*(y_k) \cap \varphi^*(x) \setminus R(\varphi^*)$ and $l \in (\varphi^*(z_k) \cup \varphi^*(y_k)) \cap (\varphi^*(x) \cup \{1\} \setminus R(\varphi^*))$.

Proof. First we show that there exists a color $k \in \{k_1, k_2, k_3\}$ such that

\[
\varphi(z_k) \cap \varphi(x) \setminus R \neq \emptyset \quad \text{and} \quad \varphi(y_k) \cap \varphi(x) \setminus R \neq \emptyset.
\]  

(14)

If $|E_M| \geq 4$, by the definition of $M$ and $E_M$, it is not difficult to see that the above statement holds. Thus we may assume that $|E_M| \leq 3$. So $|E_W| \geq 3$ and then we have $|W| \geq 2$ as $|W| \geq |E_W|/2$. Let $u, v \in W$. Since $|C_M| \leq |M| \leq |E_M| \leq 3$, we have $\varphi(x) \setminus (R \cup C_M) \neq \emptyset$ by (13). By Claim B (i), we have

\[
\varphi(u) \cap \varphi(v) \cap \varphi(x) \setminus R = \emptyset.
\]  

(15)

On the other hand, since $u, v \in W \subseteq V(S(\varphi))$ and $S(\varphi) \subseteq T(\varphi)$, we note that $d(u) < q$ and $d(v) < q$. Since $|R| \leq 2$, by (9), we have

\[
|\varphi(u) \setminus R| + |\varphi(v) \setminus R| + |\varphi(x)| > 2(\Delta - q - 2) + \Delta - d(x) + 1 > \Delta.
\]

Moreover, by the definition of $W$ and $u, v \in W$, we have $(\varphi(u) \setminus R) \cap \varphi(x) = (\varphi(v) \setminus R) \cap \varphi(x) = \emptyset$. This implies that $(\varphi(u) \setminus R) \cap (\varphi(v) \setminus R) \cap \varphi(x) \neq \emptyset$, which contradicts (15).
This contradiction shows that there exists a color \( k \in \{ k_1, k_2, k_3 \} \) such that (14) holds. Now we claim that

\[
\bar{\varphi}(z_k) \cap \bar{\varphi}(y_k) \cap \bar{\varphi}(x) \setminus R = \emptyset.
\]  

(16)

For otherwise, we assume \( i \in \bar{\varphi}(z_k) \cap \bar{\varphi}(y_k) \cap \bar{\varphi}(x) \setminus R \), then by Claim A, the path \( P_x(i, k, \varphi) \) contains three endvertices \( x, z_k \) and \( y_k \), a contradiction.

By (14) and (16), we can find two distinct colors \( i, j \) such that \( i \in \bar{\varphi}(z_k) \cap \bar{\varphi}(x) \setminus R \) and \( j \in \bar{\varphi}(y_k) \cap \bar{\varphi}(x) \setminus R \). If there still exists another color \( \ell \in (\bar{\varphi}(z_k) \cup \bar{\varphi}(y_k)) \cap (\bar{\varphi}(x) \cup \{1\} \setminus R) \), then Claim C holds with \( \varphi^* = \varphi \). Thus color \( \ell \) does not exist, that is,

\[
\bar{\varphi}(z_k) \cap \bar{\varphi}(x) \setminus R = \{i\}, \mbox{ } \bar{\varphi}(y_k) \cap \bar{\varphi}(x) \setminus R = \{j\} \mbox{ and } 1 \notin \bar{\varphi}(z_k) \cup \bar{\varphi}(y_k),
\]

which implies that

\[
((\bar{\varphi}(z_k) \setminus (R \cup \{i\})) \cup (\bar{\varphi}(y_k) \setminus (R \cup \{j\}))) \cap (\bar{\varphi}(x) \cup R \cup \{1\}) = \emptyset.
\]  

(17)

By the definition of \( T(\varphi) \), we have \( d(z_k) < q \) and \( d(y_k) < q \). Since \( |R| \leq 2 \), by (9), we have

\[
|\bar{\varphi}(z_k) \setminus (R \cup \{i\})| + |\bar{\varphi}(y_k) \setminus (R \cup \{j\})| + |\bar{\varphi}(x) \cup (R \cup \{1\})| > 2(|R| - q - |R| - 1) + |R| - d(x) + 2 \geq \Delta.
\]

So there exists a color \( \alpha \) in the intersection of sets \( \bar{\varphi}(z_k) \setminus (R \cup \{i\}) \) and \( \bar{\varphi}(y_k) \setminus (R \cup \{j\}) \) by (17). Thus \( \alpha \in \bar{\varphi}(z_k) \cap \bar{\varphi}(y_k) \cap \varphi(x) \setminus (R \cup \{i, j, 1\}) \).

Since \( |R| \leq 2 \), by (12), there exists a color \( \beta \in \bar{\varphi}(x) \setminus (R \cup \{i, j\}) \). By (17), we have \( \beta \in \varphi(\bar{z}_k) \cap \varphi(y_k) \). We may assume that \( P_{zk}(\alpha, \beta, \varphi) \) is disjoint from \( P_z(\alpha, \beta, \varphi) \). Let \( \varphi^* = \varphi/P_{zk}(\alpha, \beta, \varphi) \). Similar to the proof of Claim B, \( \varphi^* \) is an optimal \( z \)-feasible coloring with \( R(\varphi^*) = R \) and we can choose \( V(S(\varphi^*)) = V(S(\varphi)) \). Note that \( \bar{\varphi}^*(x) = \bar{\varphi}(x) \).

Since \( \alpha \in \bar{\varphi}(z_k) \), we have \( \beta \in \bar{\varphi}^*(z_k) \cap \bar{\varphi}^*(x) \setminus (R(\varphi^*) \cup \{i, j\}) \). Moreover, we have \( i \in \bar{\varphi}^*(z_k) \cap \bar{\varphi}^*(x) \setminus R(\varphi^*) \) and \( j \in \bar{\varphi}^*(y_k) \cap \bar{\varphi}^*(x) \setminus R(\varphi^*) \) as \( \alpha, \beta \notin \{i, j\} \). Thus \( i, j, \beta \) are the required colors and then Claim C holds.

Let \( k, i, j, \ell \) and \( \varphi^* \) be as stated in Claim C. If \( \ell \neq 1 \), we consider the coloring obtained from \( \varphi^* \) by interchanging colors \( 1 \) and \( \ell \) for edges not on the path \( P_x(1, \ell, \varphi^*) \), and rename it as \( \varphi^* \). This is valid, by the argument in the proof of Claim B, since \( 1, l \notin R(\varphi^*) \cup T(\varphi^*) \), \( 1 \in \varphi^*(x) \) and \( l \in \varphi^*(x) \). So we may assume \( 1 \in \varphi^*(y_k) \cup \varphi^*(z_k) \).

We first consider the case of \( 1 \in \varphi^*(y_k) \). Note that \( i, j \in \varphi^*(x) \setminus R(\varphi^*) \). By Claim A, the paths \( P_x(i, k, \varphi^*) \) and \( P_x(j, k, \varphi^*) \) both contain \( y, z \). Since \( \varphi^*(yy_k) = \varphi^*(zz_k) = k \), these two paths also contain \( y_k, z_k \). Since \( i \in \varphi^*(z_k) \), it follows that \( x \) and \( z_k \) are the two endvertices of \( P_x(i, k, \varphi^*) \). So, \( i \in \varphi^*(y) \cap \varphi^*(z) \cap \varphi^*(y_k) \). Similarly, we have \( j \in \varphi^*(y) \cap \varphi^*(z) \cap \varphi^*(z_k) \). We now consider the following sequence of colorings of \( G - xy \).
Let $\varphi_1$ be obtained from $\varphi^*$ by assigning $\varphi_1(yy_k) = 1$. Since 1 was missing at both $y$ and $y_k$, $\varphi_1$ is an edge-$\Delta$-coloring of $G - xy$. Now $k$ is missing at $y$ and $y_k$, and $i$ is still missing at $x$ and $z_k$. Since $G$ is not $\Delta$-colorable, $P_x(i, k, \varphi_1) = P_y(i, k, \varphi_1)$; otherwise $\varphi_1/P_y(i, k, \varphi_1)$ can be extended to an edge-$\Delta$-coloring of $G$ by coloring $xy$ with $i$, giving a contradiction. Furthermore, $z_k, y_k \notin V(P_x(i, k, \varphi_1))$ since either $i$ or $k$ is missing at these two vertices, which in turn shows that $z \notin V(P_x(i, k, \varphi_1))$ since $\varphi_1(zz_k) = k$.

Let $\varphi_2 = \varphi_1/P_x(i, k, \varphi_1)$. We have $k \in \bar{\varphi}_2(x), i \in \bar{\varphi}_2(y) \cap \bar{\varphi}_2(z_k)$ and $j \in \bar{\varphi}_2(x) \cap \bar{\varphi}_2(y_k)$. Since $G$ is not edge-$\Delta$-colorable, we have $P_x(i, j, \varphi_2) = P_y(i, j, \varphi_2)$, which contains neither $y_k$ nor $z_k$.

Let $\varphi_3 = \varphi_2/P_x(i, j, \varphi_2)$. Then $k \in \bar{\varphi}_3(x)$ and $j \in \bar{\varphi}_3(y) \cap \bar{\varphi}_3(y_k)$.

Let $\varphi_4$ be obtained from $\varphi_3$ by recoloring $yy_k$ by $j$. Then $1 \in \bar{\varphi}_4(y)$, $\varphi_4(xz) = 1$, $k \in \bar{\varphi}_4(x)$, $\varphi_4(zz_k) = k$. Since $\varphi_4(xz) = 1 \in \varphi_4(y)$, $\varphi_4$ is $z$-feasible. Since $i, j, k \notin R = C_y \cup C_z$, the colors in $R$ are unchanged during this sequence of re-colorings, so $C_y(\varphi_4) \supseteq C_y$ and $C_z(\varphi_4) \supseteq C_z$. Since $\varphi_4(zz_k) = k \in \bar{\varphi}_4(x)$ and $d(z_k) < q$, we have $k = \varphi_4(zz_k) \in C_z(\varphi_4)$. So, $C_z(\varphi_4) \supseteq C_z \cup \{k\}$. We therefore have $|C_y(\varphi_4)| + |C_z(\varphi_4)| \geq |C_y| + |C_z| + 1$, giving a contradiction.

For the case of $1 \in \bar{\varphi}^*(z_k)$, we consider the dual coloring $\varphi^{*d}$ of $G - xz$ obtained from $\varphi^*$ by uncoloring $xz$ and coloring $xy$ with color 1. Following the exact same argument above, we can reach a contradiction to the maximum of $|C_y| + |C_z|$. This completes the proof of Statement II.

2.1.2 Proof of Statement III.

For a coloring $\varphi \in C^\Delta(G - xy)$, let $X(\varphi) = \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \bar{\varphi}(y)\}$ and $S(\varphi) = \{z \in N(x) \setminus (X(\varphi) \cup \{y\}) : d(y_{\varphi(xz)}) < q\}$, where $y_j \in N(y)$ with $\varphi(yy_{y_j}) = j$ for any color $j$. We call vertices in $S(\varphi)$ semi-feasible vertices of $\varphi$. Note that the vertices in $X(\varphi)$ are feasible. Statement III clearly follows from the following claim.

Claim 2.1. For any coloring $\varphi \in C^\Delta(G - xy)$, the following two statements hold.

a. $|X(\varphi) \cup S(\varphi)| \geq \Delta - \sigma_q(x, y) - 1$;

b. With one possible exception, for each $z \in S(\varphi)$ there exists a coloring $\varphi^* \in C^\Delta(G - xy)$ such that $\varphi^*(xz) \in \bar{\varphi}^*(y)$.

Proof. Let $\varphi \in C^\Delta(G - xy)$. Since $G$ is $\Delta$-critical, it is easy to see that $\bar{\varphi}(y) \subseteq \varphi(x)$ and $\bar{\varphi}(x) \subseteq \varphi(y)$. To prove a, we divide $\varphi(y)$ into two subsets:

$\varphi(y, \geq q) = \{i \in \varphi(y) : d(y_i) \geq q\}$ and $\varphi(y, < q) = \{i \in \varphi(y) : d(y_i) < q\}$.
Clearly, $\sigma_q(x, y) = |\varphi(y, \geq q)|$ and $|\bar{\varphi}(y)| = |\varphi(y, < q)| = \Delta - \sigma_q(x, y)$. Also, $|X(\varphi)| = |\bar{\varphi}(y) \cap \varphi(x)| = |\bar{\varphi}(y)|$ since $\bar{\varphi}(y) \subseteq \varphi(x)$, and $|S(\varphi)| = |\varphi(y, < q) \cap \varphi(x)|$. Since edge $xy$ and the edges incident to $y$ with colors in $\varphi(x)$ form a multi-fan $F$, the vertex set $V(F)$ is elementary with respect to $\varphi$. By (10), $V(F) \setminus \{x\}$ contains at most one vertex with degree less than $q$. Thus $|\bar{\varphi}(x) \cap \varphi(y, < q)| \leq 1$ and so $|S(\varphi)| \geq |\varphi(y, < q)| - 1$, and this proves \textbf{a}.

To prove \textbf{b}, we show that for any two distinct vertices $z_k, z_\ell \in S(\varphi)$, there is a coloring $\varphi^* \in \mathcal{C}^\Delta(G - xy)$ such that at least one of $\varphi^*(xz_k)$ and $\varphi^*(xz_\ell)$ is in $\bar{\varphi}^*(y)$. We assume $\varphi(xz_k) = k$ and $\varphi(xz_\ell) = \ell$. Let $y_k, y_\ell \in N(y) \setminus \{x\}$ such that $\varphi(yy_k) = k$ and $\varphi(yy_\ell) = \ell$.

By the definition of $S(\varphi)$, we have $d(y_k) < q$ and $d(y_\ell) < q$. It follows from (9) that

\begin{equation}
|\bar{\varphi}(x)| + |\bar{\varphi}(y_k)| + |\bar{\varphi}(y_\ell)| > \Delta. \tag{18}
\end{equation}

Let $z$ be an arbitrary vertex in $X(\varphi)$, which exists since $|X(\varphi)| = |\bar{\varphi}(y)| > 0$, and assume $\varphi(xz) = 1$, so that (11) holds. We claim that there exists a coloring $\varphi' \in \mathcal{C}^\Delta(G - xy)$ such that $\varphi'(xz) = 1 \in \bar{\varphi}'(y)$, $\varphi'(xz_k) = \varphi'(yy_k) = k$, $\varphi'(xz_\ell) = \varphi'(yy_\ell) = \ell$, and

\begin{equation}
\bar{\varphi}'(x) \cap (\bar{\varphi}'(y_k) \cup \bar{\varphi}'(y_\ell)) \neq \emptyset. \tag{19}
\end{equation}

Suppose (19) does not hold with $\varphi' = \varphi$. Then $\bar{\varphi}(y_k) \cup \bar{\varphi}(y_\ell) \subseteq \varphi(x)$, and so by (18) there exists a color $r \in \varphi(x) \cap \bar{\varphi}(y_k) \cap \bar{\varphi}(y_\ell)$. Choose a color $i \in \varphi(x)$. Note that $\{i, r\} \cap \{1, k, l\} = \emptyset$ unless $r = 1$. Since at least one of colors $i$ and $r$ is missing at each of $x$, $y_k$ and $y_\ell$, we may assume $P_{y_k}(i, r, \varphi)$ is disjoint from $P_x(i, r, \varphi)$, and also from $P_y(i, r, \varphi)$ if $r = 1$, by (11). Then (19) holds with $\varphi' = \varphi/P_{y_k}(i, r, \varphi)$, since in this coloring $i$ is missing at both $x$ and $y_k$.

By (19), we may assume that there exists a color $i \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y_k)$. Applying (11) to $\varphi'$, we have $P_x(1, i, \varphi') = P_y(1, i, \varphi')$, and so $P_{y_k}(1, i, \varphi')$ is disjoint from $P_x(1, i, \varphi')$ and $P_y(1, i, \varphi')$. Thus in the coloring $\varphi'' = \varphi'/P_{y_k}(1, i, \varphi')$, color 1 is missing at both $y$ and $y_k$.

Form $\varphi^*$ from $\varphi''$ by changing the color of $yy_k$ from $k$ to $1$. Then $\varphi^*(xz_k) = k \in \bar{\varphi}^*(y)$. This proves \textbf{b}, and so completes the proofs of both Claim 2.1 and \textbf{III}. \hfill \Box

3 Proof of Theorem 1

Let $G$ be a $\Delta$-critical graph with maximum degree $\Delta$. Let $n = |V(G)|$ and $m = |E(G)|$.

Clearly, $\bar{d}(G) = 2m/n$. Let $q$ be a positive number less than $\Delta$. Note that $2\Delta/3 \geq 3\Delta/4 - 1$ if $\Delta < 12$. By Woodall’s result [16], Theorem 1 holds if $\Delta < 12$. In this section, we
assume $\Delta \geq 12$. We initially assign to each vertex $x$ of $G$ a charge $M(x, q) = d(x)$ and redistribute the charge according to the following 2-step discharging rule:

**Step 1:** each $(> q)$-vertex $y$ distributes its surplus charge of $d(y) - q$ equally among all $(< q)$-neighbors of $y$.

Assume each vertex $x$ now have charge $M_1(x, q)$, and call $x$ deficient if $M_1(x, q) < q$.

**Step 2:** For every vertex $x$ such that $d(x) \geq \Delta - 1$ and every neighbor of $x$ has degree at least $q$, if $x$ is at distance 2 from a 2-vertex or a deficient 3-vertex: $x$ sends $\frac{1}{4}$ to each such 2-vertex and $\frac{1}{8}$ to each such deficient 3-vertex.

Denote by $M_2(x, q)$ the resulting charge on each vertex $x$. Clearly, $\sum_{x \in V(G)} M_2(x, q) = \sum_{x \in V(G)} M_1(x, q) = \sum_{x \in V(G)} M(x, q) = 2m$, and for each $x$ with $d(x) \leq q_1$ we have $M_2(x, q_1) \geq M_2(x, q_2)$ if $q_1 \leq q_2$. We show that $M_2(x, q) \geq q$ for all vertices $x \in V(G)$ if $q = \min\{\frac{2\sqrt{2\Delta-3}\sqrt{\Delta}}{2\sqrt{\Delta+1}}, \frac{3}{4}\Delta - 2\}$, which gives $d(G) \geq \min\{\frac{2\sqrt{2\Delta-3}\sqrt{\Delta}}{2\sqrt{\Delta+1}}, \frac{3}{4}\Delta - 2\}$. Denote by $d_{<q}(y)$ the number of $(< q)$-neighbors of $y$.

Claim 4.2 in [1] shows that if $4 \leq d(x) \leq \frac{\Delta}{4}$, then $M_1(x, \frac{3\Delta}{4} - 8) \geq \frac{3\Delta}{4} - 8$. Actually, in the proof, it showed that $M_1(x, \frac{3\Delta}{4} - 1) \geq \frac{3\Delta}{4} - 1$. By using Step 2, Claim 4.3 and Claim 4.4 in [1] showed that $M_2(x, \frac{3\Delta}{4} - 1) \geq \frac{3\Delta}{4} - 1$ if $x$ is a 2-vertex, a deficient 3-vertex, or a $(\geq \Delta - 1)$ vertex. In the remainder of this proof, let $q = \min\{\frac{2\sqrt{2\Delta-3}\sqrt{\Delta}}{2\sqrt{\Delta+1}}, \frac{3}{4}\Delta - 2\}$. If $4 \leq d(x) < \Delta - 1$, we only use Step 1, it is easy to check that $M_2(x, q) = M_1(x, q)$ and $M_2(x, q) = M_1(x, q) \geq q$ if $q \leq d(x) < \Delta - 1$. We show that for all vertices $x$ with $\frac{\Delta}{4} < d(x) < q$, $M_2(x, q) = M_1(x, q) \geq q$.

**Claim 3.1.** If $\frac{\Delta}{4} < d(x) \leq \Delta - q + 1$, then $M_2(x, q) = M_1(x, q) \geq q$.

**Proof.** Since $q > \frac{\Delta+1}{2}$, $d(x) \leq \Delta - q + 1 < q$. Let $y$ be an arbitrary neighbor of $x$. Since $2\Delta - d(x) - d(y) + 2 \geq \Delta - d(x) + 2 > q$, we have $\sigma_q(x, y) \geq \sigma(x, y)$. We will use lower bounds of $\sigma(x, y)$ to estimate $\sigma_q(x, y)$. By (1) and (2), we have

$$1 \leq d_{<q}(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - d(x) + p(x) + 1).$$

(20)

By Lemma 3, $x$ has at least $d(x) - p(x) - 1$ neighbors $y$ for which $\sigma(x, y) \geq \Delta - p(x) - 1$, so for these neighbors $y$ we have

$$1 \leq d_{<q}(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - p(x) - 1).$$

(21)

By the hypothesis of Claim 3.1 and (20), we have $q \leq \Delta - d(x) + 1 \leq \Delta - d(x) + p(x) + 1 < d(y)$. Since $\frac{d(y) - a}{d(y) - b}$ with $a \leq b$ is a decreasing function of $d(y)$, for each $y \in N(x)$, $x$
receives at least
\[
\frac{d(y) - q}{d(y) - (\Delta - d(x) + p(x) + 1)} \geq \frac{\Delta - q}{d(x) - p(x) - 1}.
\]

And there are at least \(d(x) - p(x) - 1\) neighbors \(y\) of \(x\) giving \(x\) at least
\[
\frac{d(y) - q}{d(y) - (\Delta - p(x) - 1)} \geq \frac{\Delta - q}{p(x) + 1},
\]
where the inequality holds because \(q \leq \Delta - d(x) + 1 \leq \Delta - p(x) - 1\) as \(p(x) \leq \lfloor \frac{d(x)}{2} \rfloor - 1\).

Thus \(x\) receives at least
\[
(d(x) - p(x) - 1) \frac{\Delta - q}{p(x) + 1} + (p(x) + 1) \frac{\Delta - q}{d(x) - p(x) - 1} = (\theta + \theta^{-1})(\Delta - q) \geq 2(\Delta - q),
\]
where \(\theta = \frac{d(x) - p(x) - 1}{p(x) + 1}\). It follows that \(M_1(x, q) \geq M(x, q) + 2(\Delta - q) \geq \frac{\Delta}{4} + 2(\Delta - q) > q\). \(\Box\)

**Claim 3.2.** If \(\Delta - q + 1 < d(x) < q\), then \(M_2(x, q) = M_1(x, q) \geq q\).

**Proof.** Let \(y\) be a neighbor of \(x\). Then there exists a coloring \(\varphi \in C^\Delta(G - xy)\) as \(G\) is \(\Delta\)-critical. Let \(Z_q = \{z \in N(x) : d(z) > q\}\), \(Z_y = \{z \in N(x) \setminus \{y\} : \varphi(xz) \in \varphi(y)\}\) and \(Z_y^* = Z_q \cap Z_y\). Note that \(Z_y^*\) was called \(Z\) in Lemma 8, thus (3), (4) and (5) in Lemma 8 also hold for \(Z_y^*\). Moreover, by the definition of \(Z_q\) and the discharging rule, for each \(z \in Z_q\), \(x\) receives charge at least \(\frac{d(z) - q}{d(z) - \sigma_q(x, z)}\). Thus \(M_1(x, q) \geq d(x) + \sum_{z \in Z_q} \frac{d(z) - q}{d(z) - \sigma_q(x, z)}\).

Since \(\Delta \geq 12\), we have \(\Delta - q \geq 5\) and so \(\Delta - q + 1 \leq 2(\Delta - q) - 4\). We consider the following three cases to complete the proof.

**Case 1.** \(\Delta - q + 1 < d(x) < q\) and \(x\) has a neighbor \(y\) such that \(d(y) \leq q\).

It is easy to see that, for any vertex \(z\),
\[
d(x) + d(y) + d(z) - 2\Delta - 2 < 2q - \Delta < 2(\Delta - q),
\]
since \(q < \frac{3}{4}\Delta\). It follows from this and (5) in Lemma 8 that, for each vertex \(z \in Z_y^*\), \(\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) - 1\), and so \(d(z) - \sigma_q(x, z) \leq d(x) + d(y) - \Delta + 1\). Also, by (4) in Lemma 8, \(\sum_{z \in Z_y^*} (d(z) - q) \geq (\Delta - q)(\Delta - d(y) + 1) + 3 - (d(x) + d(y) - \Delta + 1)\).
So, using \( d(y) \leq q \) in the second line, we have

\[
M_1(x, q) \geq d(x) + \sum_{z \in Z_y^*} \frac{d(z) - q}{d(z) - \sigma_q(x, z)} \\
> d(x) + \frac{(\Delta - q)(\Delta - d(y) + 1)}{d(x) + d(y) - \Delta + 1} - 1 \\
\geq d(x) + q - \Delta + 1 + \frac{(\Delta - q)(\Delta - q + 1)}{d(x) + q - \Delta + 1} + \Delta - q - 2 \\
\geq 2\sqrt{(\Delta - q)(\Delta - q + 1)} + \Delta - q - 2 \\
> 3(\Delta - q) - 2 > q,
\]

where in the penultimate line we used \( a + b \geq 2\sqrt{ab} \) for all \( a, b > 0 \), and in the final line that \( 3(\Delta - q) \geq q + 8 \) since \( q \leq \frac{3}{4}\Delta - 2 \).

**Case 2.** \( 2(\Delta - q) - 4 < d(x) < q \) and \( d(y) > q \) for every \( y \in N(x) \).

Let \( y \in N(x) \) such that the degree of \( y \) is minimum in \( N(x) \). Let \( u \) be an arbitrary vertex in \( N(x) \). Then \( d(u) \geq d(y) \) by the choice of \( y \). By Lemma 1, we have \( \sigma_q(x, u) \geq \sigma_\Delta(x, u) \geq \Delta - d(x) + 1 \). Since \( d(u) \geq d(y) \) and \( q > \Delta - d(x) + 1 \), we have

\[
\frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq \frac{d(u) - q}{d(u) - (\Delta - d(x) + 1)} \geq \frac{d(y) - q}{d(y) - (\Delta - d(x) + 1)},
\]

where the inequality holds because \( \frac{d(u) - a}{d(u) - b} \) with \( a > b > 0 \) is a increasing function of \( d(u) \).

Note that \( |N(x) \setminus Z_y^*| \geq |N(x) \setminus Z_y| = d(x) - (\Delta - d(y) + 1) \). Combining this with the above inequality, we have

\[
\sum_{u \in N(x) \setminus Z_y^*} \frac{d(u) - q}{d(u) - \sigma_q(x, u)} \geq d(y) - q. \tag{22}
\]

For any vertex \( z \), since \( d(y) \leq \Delta, d(z) \leq \Delta \) and \( q < \frac{3}{4}\Delta \),

\[
d(x) + d(y) + d(z) - 2\Delta - 2 < d(x) < q < 3(\Delta - q).
\]

For each \( z \in Z_y^* \), by (5) in Lemma 8,

\[
\sigma_q(x, z) \geq 2\Delta - d(x) - d(y) - \left[ \frac{d(x) + d(y) + d(z) - 2\Delta - 2}{\Delta - q} \right] \\
\geq 2\Delta - d(x) - d(y) - 2,
\]

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and so $d(z) - \sigma_q(x, z) \leq d(x) + d(y) - \Delta + 2$. Combining this with (4) in Lemma 8, (22) and using $\Delta - d(y) + 1 = d(x) + 3 - (d(x) + d(y) - \Delta + 2)$ in the next line, gives

$$M_1(x, q) \geq d(x) + \sum_{z \in Z^*_y} \frac{d(z) - q}{d(z) - \sigma_q(x, y)} + \sum_{u \in N(x) \setminus Z^*_y} \frac{d(u) - q}{d(u) - \sigma_q(x, u)}$$

$$\geq d(x) + \frac{(\Delta - q)(\Delta - d(y) + 1) + 4}{d(x) + d(y) - \Delta + 2} - 1 + d(y) - q$$

$$= d(x) + d(y) - \Delta + 2 + \frac{(\Delta - q)(d(x) + 3) + 4}{d(x) + d(y) - \Delta + 2} - 3$$

$$\geq 2\sqrt{\Delta - q}(d(x) + 3) + 4 - 3$$

$$> 2\sqrt{2}(\Delta - q - \frac{1}{2}) - 3 \geq q.$$

where in the penultimate line we used $a + \frac{b}{a} \geq 2\sqrt{b}$ for all $a, b > 0$, and in the final line we used $(\Delta - q)(d(x) + 3) + 4 > 2(\Delta - q - \frac{1}{2})^2$ since $d(x) > 2(\Delta - q) - 4$ and we also used $q \leq \frac{2\sqrt{2}\Delta - 3 - \sqrt{2}}{2\sqrt{2} + 1}$.

**Case 3.** $\Delta - q + 1 < d(x) \leq 2(\Delta - q) - 4$ and $d(y) > q$ for every $y \in N(x)$.

In this case, we let $p' := p(x, q)$ since it will be used heavily here. So, $p' = \min\{ p_{\min}(x, q), \lfloor \frac{d(x)}{2} \rfloor - 3 \}$, where $p_{\min}(x, q) = \min_{y \in N(x)} \sigma_q(x, y) - \Delta + d(x) - 1$. So $\sigma_q(x, y) \geq \Delta - d(x) + p' + 1$ for every $y \in N(x)$.

Since $d(x) \leq 2(\Delta - q) - 4$ and $p' \leq \lfloor \frac{d(x)}{2} \rfloor - 3$, we have $q \leq \Delta - \frac{d(x)}{2} - 2 \leq \Delta - p' - 5$.

By Lemma 5, $x$ has at least $d(x) - p' - 3$ neighbors $y$ for which $\sigma_q(x, y) \geq \Delta - p' - 5$, so for these neighbors $y$,

$$\frac{d(y) - q}{d(y) - \sigma_q(x, y)} \geq \frac{d(y) - q}{d(y) - (\Delta - p' - 5)} \geq \frac{\Delta - q}{p' + 5}. \quad (23)$$
If \( q > \Delta - d(x) + p' + 1 \), i.e., \( p' < d(x) + q - \Delta - 1 \), then

\[
M_1(x, q) \geq d(x) + (d(x) - p' - 3)\frac{\Delta - q}{p' + 5} \geq d(x) + \frac{(\Delta - q - 2)(\Delta - q)}{d(x) + q - \Delta + 4}.
\]

\[
= (d(x) + q - \Delta + 4) + \frac{(\Delta - q)(\Delta - q - 2)}{d(x) + q - \Delta + 4} - (q - \Delta + 4) \geq 2\sqrt{(\Delta - q)(\Delta - q - 2)} + \Delta - q - 4 \geq 3(\Delta - q) - 8 \geq q,
\]

since \( q \leq \frac{3}{4}\Delta - 2 \).

We now consider the case \( q \leq \Delta - d(x) + p' + 1 \). Since \( \sigma_q(x, y) \geq \Delta - d(x) + p' + 1 \) for every \( y \in N(x) \),

\[
\frac{d(y) - q}{d(y) - \sigma_q(x, y)} \geq \frac{d(y) - q}{d(y) - (\Delta - d(x) + p' + 1)} \geq \frac{\Delta - q}{d(x) - p' - 1}.
\]  \hspace{1cm} (24)

By (23), (24), the fact that \( d(x) > \Delta - q + 1 \) and the inequality \( \theta + \theta^{-1} \geq 2 (\theta = \frac{d(x) - p' - 3}{p' + 5}) \), we have

\[
M_1(x, q) \geq d(x) + (d(x) - p' - 3)\frac{\Delta - q}{p' + 5} + (p' + 3)\frac{\Delta - q}{d(x) - p' - 1}
\geq \Delta - q + 1 + (\Delta - q)\left(\frac{d(x) - p' - 3}{p' + 5} + \frac{p' + 3}{d(x) - p' - 1}\right)
\geq \Delta - q + 1 + (\Delta - q)(2 - \left(\frac{p' + 5}{d(x) - p' - 3} - \frac{p' + 3}{d(x) - p' - 1}\right)).
\]

Since \( d(x) - p' - 1 \geq \frac{d(x)}{2} + 2 \) and \( d(x) - p' - 3 \geq \frac{d(x)}{2} \geq \frac{\Delta - q + 1}{2} \), we get

\[
\frac{p' + 5}{d(x) - p' - 3} - \frac{p' + 3}{d(x) - p' - 1} \leq \frac{2d(x) + 4}{(d(x) - p' - 3)(d(x) - p' - 1)} \leq \frac{8(d(x) + 2)}{(\Delta - q + 1)(d(x) + 4)} < \frac{8}{\Delta - q}.
\]

Thus,

\[
M_1(x, q) > \Delta - q + 1 + (\Delta - q)(2 - \frac{8}{\Delta - q}) = 3(\Delta - q) - 7 > q.
\]

\[ \square \]
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References


