A characterization of multigraphs attaining maximum chromatic index

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Abstract

Let $G$ be a multigraph with maximum degree $\Delta$ and chromatic index $\chi'$. If $G$ is bipartite then $\chi' = \Delta$. Otherwise, by a theorem of Goldberg, $\chi' \leq \Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor$, where $g_o$ denotes the odd girth of $G$. Stiebitz, Scheide, Toft, and Favrholdt in their book conjectured that if $\chi' = \Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor$ then $G$ contains as a subgraph a ring graph $R$ with the same chromatic index. Vizing’s characterization of graphs with chromatic index attaining the Shannon’s bound showed the above conjecture holds for $g_o = 3$. Stiebitz et al. verified the conjecture for graphs with $g_o = 5$ and $\Delta \geq 10$. McDonald proved the conjecture when $\Delta - 2$ is divisible by $g_o - 1$. In this paper, we show that the chromatic index condition alone is not sufficient to give the conclusion in the conjecture. On the positive side, we show that the conjecture holds for every $g_o \geq 3$ with $\Delta \geq \frac{g_o^2 - 2g_o + 5}{2}$, and the maximum degree condition is best possible.

Keywords. chromatic index; maximum degree; critical chromatic graph; odd girth

1 Introduction

Graphs in this paper may contain multiple edges but no loops. We will generally follow Stiebits et al. in [8] for notations and terminologies. Let $G = (V(G), E(G))$ be a (multi)graph with maximum degree $\Delta(G)$. If $G$ is not a bipartite, denote by $g_o(G)$ the length of smallest odd cycles of $G$ and call it odd girth of $G$. For two distinct vertices $x, y \in V(G)$, let the multiplicity of $x, y$ be the number of edges with ends $x$ and $y$ in $G$, denoted by $\mu_{x,y}$. A graph is called a ring graph if its underlying simple graph is a cycle. A $k$-edge-coloring of graph $G$ is a mapping $\varphi$: $E(G) \rightarrow \{1, 2, \ldots, k\}$ that assigns to every edge $e$ of $G$ a color $\varphi(e) \in \{1, 2, \ldots, k\}$ such that no two adjacent edges of $G$ receive the same color. Denote by $C^k(G)$ the set of all $k$-edge-colorings of $G$. The chromatic index $\chi'(G)$ of $G$ is the least integer $k \geq 0$ such that $C^k(G) \neq \emptyset$. Clearly,
χ′(G) ≥ Δ(G). It is well-known that the equality holds for bipartite graphs. In this paper, we will only consider nonbipartite graphs, so the odd girth g_o is well-defined. Shannon [7] in 1949 showed that χ′(G) ≤ ⌊3/2Δ(G)⌋, known as Shannon’s bound. Vizing [9] in 1965 gave a characterization of the extremal graphs attaining Shannon’s bound in the following theorem.

**Theorem 1.1.** A graph G satisfies χ′(G) = ⌊3/2Δ(G)⌋ if and only if G contains as a subgraph a ring graph of order three with ⌊3/2Δ(G)⌋ edges distributed evenly among three pairs of vertices, i.e., with ⌊Δ(G)/2⌋, ⌊Δ(G)/2⌋ and ⌊(Δ(G)+1)/2⌋ many multiply edges between pairs of vertices.

Goldberg [1, 3] extended Shannon’s result and established the following bound for the chromatic index in terms of the maximum degree and odd girth:

\[ χ′(G) \leq Δ(G) + 1 + \left\lfloor \frac{Δ(G) - 2}{g_o(G) - 1} \right\rfloor. \] (1.1)

Note that Goldberg’s bound is exactly Shannon’s bound when \( g_o = 3 \). Therefore it is natural to consider the extremal graphs attaining Goldberg’s bound. For an integer \( Δ \geq 2 \) and an odd integer \( g_o \geq 3 \), denote by \( GO(Δ, g_o) \) the set of all graphs \( G \) with maximum degree \( Δ \) and odd girth \( g_o \) attaining Goldberg’s bound. In the book [8] page 249, Stiebitz, Scheide, Toft, and Favrholdt made the following conjecture.

**Conjecture 1.1.** Let \( Δ, g_o \) be integers such that \( g_o \) is odd and \( Δ \geq g_o + 1 \geq 4 \). Then every graph \( G \in GO(Δ, g_o) \) contains as a subgraph a ring graph \( R \in GO(Δ, g_o) \).

It follows from Theorem 1.1 that Conjecture 1.1 holds when \( g_o = 3 \). McDonald [5] showed that Conjecture 1.1 is true when \( Δ(G) - 2 \) is divisible by \( g_o - 1 \). In the same book ([8], Theorem 9.1), Stiebitz et al. verified Conjecture 1.1 when \( g_o = 5 \) and \( Δ(G) \geq 10 \). It seems that Conjecture 1.1 should be true in general based on these results. However, we in Section 4 will provide a series of counterexamples showing that Conjecture 1.1 fails for every \( g_o \geq 5 \) when \( Δ(G) = \frac{1}{2}(g_o^2 - 2g_o + 5) - 1 \) and \( Δ(G) = \frac{1}{2}(g_o^2 - 2g_o + 5) - 2 \). Recall that McDonald verified the conjecture is true when \( Δ(G) - 2 \) is divisible by \( g_o - 1 \), so it is impossible to find a counterexample for every given maximum degree below \( \frac{1}{2}(g_o^2 - 2g_o + 5) \). On the positive side, we will prove the following result which shows that Conjecture 1.1 holds if \( Δ(G) \geq \frac{1}{2}(g_o^2 - 2g_o + 5) \). The examples mentioned above demonstrate that the maximum degree condition in Theorem 1.2 is best possible.

**Theorem 1.2.** Let \( Δ, g_o \) be integers such that \( g_o \) is odd, \( Δ \geq g_o + 1 \geq 4 \) and \( Δ \geq \frac{g_o^2 - 2g_o + 5}{2} \). Let \( G \) be a graph with maximum degree \( Δ \) and odd girth \( g_o \). If \( χ′(G) = Δ + 1 + \left\lfloor \frac{Δ - 2}{g_o - 1} \right\rfloor \), then \( G \) contains as a subgraph a ring graph \( R \) with the same maximum degree and odd girth such that \( χ′(R) = Δ + 1 + \left\lfloor \frac{Δ - 2}{g_o - 1} \right\rfloor \).

Note that \( \frac{1}{2}(g_o^2 - 2g_o + 5) \) is an integer since \( g_o \) is odd. In particular, \( \frac{1}{2}(g_o^2 - 2g_o + 5) = 10 \) when \( g_o = 5 \). Thus Theorem 1.2 implies the case \( g_o = 5 \) and \( Δ(G) \geq 10 \) proved in book [8].
2 Preliminaries

For a $\chi'(G)$-edge-coloring of $G$, since the edges in each color class of $G$ form a matching, we have $|E(H)| \leq \chi'(G)|V(H)|/2$ for any $H \subseteq G$. Therefore for an arbitrary graph $G$, apart from the maximum degree there is another trivial lower bound for the chromatic index:

$$\chi'(G) \geq \rho(G) = \max \left\{ \frac{2|E(H)|}{|V(H)| - 1} : H \subseteq G, |V(H)| \geq 3 \text{ odd} \right\},$$

where $\rho(G)$ is called the **density** of $G$. Clearly, $\rho$ is a monotone parameter of $G$. In the 1970s, Goldberg [2] and Seymour [6] independently made the following conjecture, which is known as the Goldberg-Seymour conjecture.

**Conjecture 2.1.** For any graph $G$, if $\chi'(G) > \Delta(G) + 1$ then $\chi'(G) = \rho(G)$.

In joint work with Wenan Zang, two authors of this paper, Guantao Chen and Guangming Jing gave a proof of the Goldberg-Seymour conjecture currently available at https://math.gsu.edu/gchen/research.html. We assume that the Goldberg-Seymour conjecture is true in this paper.

Let us introduce some notations and terminologies before proceeding. A graph $G$ is called **critical** if $\chi'(H) < \chi'(G)$ for every proper subgraph $H$ of $G$, and $k$-critical if it is critical and $\chi'(G) = k + 1$. Let $G$ be a $k$-critical graph and $e$ be an arbitrary edge in $E(G)$. Then $\chi'(G - e) = k$. Let $\varphi$ be a $k$-edge-coloring of $G - e$. For a vertex $v \in V(G)$, denote by $\overline{\varphi(v)}$ the sets of colors not-assigned to edges incident $v$. $V(G)$ is called **elementary** if $\overline{\varphi(u)} \cap \overline{\varphi(v)} = \emptyset$ for any distinct vertices $u, v \in V(G)$.

**Lemma 2.1.** [8] Let $G$ be a $k$-critical graph with $k > \Delta(G)$ and let $\varphi$ be a $k$-edge coloring of $G - e$ with $e \in E(G)$. Then $V(G)$ is elementary under $\varphi$ and $|V(G)|$ is odd if the Goldberg-Seymour conjecture is true.

Using Lemma 2.1, we have the following Lemma.

**Lemma 2.2.** Let $G$ be a $k$-critical graph with $k > \Delta(G)$. Then $\chi'(G) = k + 1 \leq \Delta(G) + 1 + \left\lfloor \frac{\Delta(G) - 2}{|V(G)| - 1} \right\rfloor.$

*Proof.* Let $G = (V(G), E(G))$ be a $k$-critical graph with $k > \Delta(G)$ and let $\varphi$ be a $k$-edge coloring of $G - e$ with $e \in E(G)$. By Lemma 2.1, $V(G)$ is $\varphi$-elementary. Note that $|\overline{\varphi(v)}| \geq k - \Delta(G)$ for each $v \in V$. Moreover, since $e$ is uncolored under $\varphi$, we have $|\overline{\varphi(v)}| \geq k + 1 - \Delta(G)$ for each vertex $v$ incident $e$. Because $V(G)$ is elementary, we can see that $|\cup_{v \in V(G)} \overline{\varphi(v)}| = \sum_{v \in V(G)} |\overline{\varphi(v)}| \leq k$. Therefore we have $(k - \Delta(G))|V(G)| + 2 \leq \sum_{v \in V(G)} |\overline{\varphi(v)}| \leq k$. As a result, we have $k + 1 \leq \Delta(G) + 1 + \left\lfloor \frac{\Delta(G) - 2}{|V(G)| - 1} \right\rfloor$ as desired.

**Lemma 2.3.** [8] Let $\tau$ be one monotone graph parameter. Then every graph $G$ contains an $\tau$-critical subgraph $H$ with $\tau(H) = \tau(G)$. 

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Noting that the chromatic index is a monotone parameter of $G$, every graph $G$ contains a critical subgraph $H$ such that $\chi'(H) = \chi'(G)$.

3 Proof of Theorem 1.2

Let $\Delta$ and $g_o$ be two integers such that $\Delta \geq \frac{g_o^2 - 2g_o + 5}{2}$ and $g_o \geq 3$ is odd; and let $G$ be a graph with maximum degree $\Delta$ and odd girth $g_o$ such that chromatic index $\chi'(G) = \Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor$. Let $H$ be a critical subgraph of $G$ such that $\chi'(H) = \chi'(G)$. Thus the following equality holds.

$$\chi'(H) = \Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor$$

We will show that $\Delta(H) = \Delta$, $g_o(H) = g_o$ and $H$ is a ring graph.

Combining (3.2) and (1.1), we have

$$\Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor = \chi'(H) \leq \Delta(H) + 1 + \lfloor \frac{\Delta(H) - 2}{g_o(H) - 1} \rfloor,$$

(3.3)

Since $H$ is a subgraph of $G$, $g_o(H) \geq g_o$ and $\Delta(H) \leq \Delta$, which in turn gives $\lfloor \frac{\Delta - 2}{g_o - 1} \rfloor \geq \lfloor \frac{\Delta(H) - 2}{g_o(H) - 1} \rfloor$. So (3.3) implies that $\Delta(H) = \Delta$.

If $|V(H)| = g_o$, then $g_o(H) = g_o$ since $|V(H)| \geq g_o(H) \geq g_o$, and hence $H$ is a ring graph as desired. Suppose $|V(H)| \neq g_o$. Then, $|V(H)| \geq g_o + 2$ since $|V(H)|$ is odd by Lemma 2.1. We claim the following inequality

$$\lfloor \frac{\Delta - 2}{g_o - 1} \rfloor - \lfloor \frac{\Delta - 2}{|V(H)| - 1} \rfloor \geq 1,$$

(3.4)

which implies that

$$\chi'(H) \leq \Delta + 1 + \lfloor \frac{\Delta - 2}{|V(H)| - 1} \rfloor < \Delta + 1 + \lfloor \frac{\Delta - 2}{g_o - 1} \rfloor,$$

giving a contradiction to (3.2). Since $|V(H)| \geq g_o + 2$, (3.4) follows immediately from the inequality below.

$$\lfloor \frac{\Delta - 2}{g_o - 1} \rfloor - \lfloor \frac{\Delta - 2}{g_o + 1} \rfloor \geq 1.$$

(3.5)

We first assume that $\frac{g_o^2 - 2g_o + 5}{2} \leq \Delta < \frac{g_o^2 + 3}{2}$. In this case, $\frac{g_o^2 - 2g_o + 1}{2} \leq \Delta - 2 < \frac{g_o^2 - 1}{2}$, which gives that $\frac{g_o - 1}{2} \leq \frac{\Delta - 2}{g_o - 1} < \frac{g_o + 1}{2}$. Hence, $\lfloor \frac{\Delta - 2}{g_o - 1} \rfloor = \frac{g_o - 1}{2}$. On the other hand, $\frac{g_o^2 - 2g_o + 1}{2} < \frac{\Delta - 2}{g_o + 1} < \frac{g_o - 1}{2}$. Since $g_o^2 - 2g_o + 1 \geq (g_o - 3)(g_o + 1)$, we have that $\frac{\Delta - 2}{g_o + 1} \geq \frac{g_o - 3}{2}$. Thus, $\lfloor \frac{\Delta - 2}{g_o + 1} \rfloor = \frac{g_o - 3}{2}$, and therefore (3.5) holds.

We now assume $\Delta \geq \frac{g_o^2 + 3}{2}$. In this case, $\frac{\Delta - 2}{g_o - 1} - \frac{\Delta - 2}{g_o + 1} \geq \frac{g_o^2 + 3}{2} - 2 \cdot \frac{2}{(g_o - 1)(g_o + 1)} = 1$ which implies $\lfloor \frac{\Delta - 2}{g_o - 1} \rfloor - \lfloor \frac{\Delta - 2}{g_o + 1} \rfloor \geq 1$. Thus (3.5) holds in general.
4 Counterexamples

Let \( g_o \geq 5 \) be an odd integer and \( G_c = (V_c, E_c) \) be a simple graph obtained from a cycle \( C^* = v_1v_2 \ldots v_{g_o}v_{g_o+1}v_{g_o+2}v_1 \) by adding a chord \( v_1v_{g_o} \) (such graphs are called \( \Theta \)-graphs). Let \( C \) denote the induced \( g_o \)-cycle \( v_1v_2 \ldots v_{g_o}v_1 \). Graph \( G_c \) is depicted in Figure 1. We will use \( G_c \) as the underlying simple graph to construct counterexamples \( G_1 \) and \( G_2 \) as below.

\[
\Delta = \frac{1}{2}(g_o^2 - 2g_o + 3).
\]

Let \( g_o \) is odd, \( \Delta = \frac{1}{2}(g_o^2 - 2g_o + 3) = \frac{1}{2}(g_o - 1)^2 + 1 \) is an odd integer. We construct \( G_1 \) by replacing each edge of \( G_c \) by a set of parallel edges with the following rule for multiplicities:

- \( \mu_{v_i, v_{i+1}} = \frac{\Delta - 1}{2} \) if \( 1 \leq i \leq g_o - 1 \) and \( i \) is odd;
- \( \mu_{v_i, v_{i+1}} = \frac{\Delta + 1}{2} \) if \( 2 \leq i \leq g_o - 1 \) and \( i \) is even;
- \( \mu_{v_1, v_{g_o}} = \frac{\Delta - g_o}{2}, \mu_{v_1, v_{g_o+2}} = \frac{g_o + 1}{2}, \mu_{v_{g_o+1}, v_{g_o+2}} = \Delta - \frac{g_o + 1}{2} \) and \( \mu_{v_{g_o}, v_{g_o+1}} = \frac{g_o - 1}{2} \).

Graph \( G_1 \) is depicted in Figure 2. Since \( g_o \geq 5 \), the multiplicities defined above are positive and \( \Delta \geq g_o + 1 \geq 4 \). The following theorem shows that \( G_1 \) is a counterexample to Conjecture 1.1 when \( \Delta = \frac{1}{2}(g_o^2 - 2g_o + 3) \).

**Theorem 4.1.** Let \( G_1 \) be defined as above. Then \( G_1 \in GO(\Delta, g_o) \) and \( G_1 \) does not contain as a subgraph a ring graph \( R \in GO(\Delta, g_o) \).
Proof. Clearly, $g_o(G_1) = g_o(G_c) = g_o$ and $\Delta(G_1) = \Delta$. Plugging $\Delta = \frac{1}{2}(g_o^2 - 2g_o + 3)$, we get the following equalities.

\[
\frac{\Delta - 2}{g_o - 1} = \frac{g_o^2 - 2g_o - 1}{2(g_o - 1)} = \frac{(g_o - 1)^2 - 2}{2(g_o - 1)} = \frac{g_o - 1}{2} - 1
\]

\[
\frac{\Delta - 1}{g_o + 1} = \frac{g_o^2 - 2g_o + 1}{2(g_o + 1)} = \frac{(g_o - 1)^2 - 2(g_o - 1)}{2(g_o + 1)} = \frac{g_o - 1}{2} - 1
\]

\[
\frac{\Delta - g_o}{g_o - 1} = \frac{g_o^2 - 4g_o + 3}{2(g_o - 1)} = \frac{(g_o - 1)^2 - 2(g_o - 1)}{2(g_o - 1)} = \frac{g_o - 1}{2} - 1
\]

By the definition, every vertex of $G_1$ has degree $\Delta$ except for $v_{g_o+1}$ which has degree $\Delta - 1$. Thus $|E(G_1)| = \frac{\Delta(g_o + 2) - 1}{2}$. Since $|V(G_1)| = g_o + 2$, we have

\[
\frac{2|E(G_1)|}{|V(G_1)| - 1} = \frac{\Delta(g_o + 2) - 1}{g_o + 1} = \Delta + \frac{\Delta - 1}{g_o + 1} = \Delta + \frac{g_o - 1}{2}.
\]
Thus, \( \chi'(G_1) \geq \rho(G_1) \geq \left\lceil \frac{2|E(G_1)|}{|V(G_1)| - 1} \right\rceil = \Delta + \frac{g_o}{2} - 1 \). On the other hand, (1.1) gives the following inequality.

\[
\chi'(G_1) \leq \Delta + 1 + \left\lfloor \frac{\Delta - g_o}{g_o - 1} \right\rfloor = \Delta + \frac{g_o - 1}{2}.
\]

Thus \( \chi'(G_1) = \Delta + \frac{g_o - 1}{2} \) and \( G_1 \in \mathcal{G}\mathcal{O}(\Delta, g_o) \).

Let \( R \subset G_1 \) be a ring graph, and \( H_1 = G_1[V(C)] \) be the subgraph induced by \( V(C) \). We will show \( \chi'(R) < \chi'(G_1) \) which completes the proof. Since \( C \) is the only induced odd cycle of the underlying simple graph \( G_c \), \( R \) is a spanning subgraph of \( H_1 \). Thus \( \chi'(R) \leq \chi'(H_1) \), and hence it is sufficient to show \( \chi'(H_1) < \chi'(G_1) \). In fact, since the Goldberg-Seymour conjecture holds for all ring graphs (see Theorem 6.3 in [8]), \( \chi'(H_1) < \chi'(G_1) \) is equivalent to \( \rho(H_1) < \chi'(G_1) = \Delta + \frac{g_o - 1}{2} \).

Note that every odd subgraph of \( H_1 \) is contained in an odd induced subgraph of \( H_1 \). Thus to compute \( \rho(H_1) \), we only need to consider \( \left\lceil \frac{2|E(H_1)|}{|V(H_1)| - 1} \right\rceil \) for every odd induced subgraph \( H \) of \( H_1 \).

By the definition of \( G_1 \), the degree sequence of \( H_1 \) is \( \{\Delta, \ldots, \Delta, \Delta - \frac{g_o}{2}, \Delta - \frac{g_o + 1}{2}\} \). Thus \( |E(H_1)| = \frac{\Delta g_o - g_o}{2} \). Hence we have

\[
\left\lceil \frac{2|E(H_1)|}{|V(H_1)| - 1} \right\rceil = \frac{\Delta g_o - g_o}{g_o - 1} = \Delta + \frac{\Delta - g_o}{g_o - 1} = \Delta + \frac{g_o - 1}{2} - 1.
\]

Let \( H'_1 \) be an induced subgraph of \( H_1 \) with \( V(H'_1) \neq V(H_1) \). Then \( H'_1 \) is a bipartite graph. By König’s Theorem [4], \( \chi'(H'_1) = \Delta(H'_1) \leq \Delta \). Recall that density \( \rho \) is a lower bound for chromatic index, hence \( \left\lceil \frac{2|E(H'_1)|}{|V(H'_1)| - 1} \right\rceil \leq \rho(H'_1) \leq \chi'(H'_1) \leq \Delta \). Therefore, \( \rho(H_1) \leq \max\{\Delta + \frac{g_o - 1}{2} - 1, \Delta\} < \Delta + \frac{g_o - 1}{2} \). \( \blacksquare \)

Let \( \Delta = \frac{1}{2}(g_o^2 - 2g_o + 3) \). Since \( g_o \) is odd, \( \Delta \) is even. We construct \( G_2 \) by replacing each edge of \( G_c \) by a set of parallel edges with the following rule for multiplicities:

- \( \mu_{v_i, v_{i+1}} = \frac{\Delta}{2} \) if \( 1 \leq i \leq g_o - 1 \);
- \( \mu_{v_i, v_{g_o}} = \frac{\Delta - g_o + 1}{2} \);
- \( \mu_{v_1, v_{g_o+2}} = \frac{g_o - 1}{2}, \mu_{v_{g_o+1}, v_{g_o+2}} = \Delta - \frac{g_o - 1}{2} \) and \( \mu_{v_{g_o}, v_{g_o+1}} = \frac{g_o - 1}{2} \).

Graph \( G_2 \) is depicted in Figure 3. Similar to the proof of Theorem 4.1, we can show that \( G_2 \) is a counterexample to Conjecture 1.1 as stated below. (The proof is in appendix.)

**Theorem 4.2.** Let \( G_2 \) be defined as the above. Then \( G_2 \in \mathcal{G}\mathcal{O}(\Delta, g_o) \) and \( G_2 \) does not contain as a subgraph a ring graph \( R \in \mathcal{G}\mathcal{O}(\Delta, g_o) \).

Finally, it is worth pointing out that both \( G_1 \) and \( G_2 \) are critical. Additionally, for large \( g_o \), counterexamples with smaller maximum degree exist in a similar flavor.
Figure 3. $G_2$

References


5 Appendix

Proof of Theorem 4.2:

Clearly, \( g_o(G_2) = g_o(G_c) = g_o \) and \( \Delta(G_2) = \Delta \). Plugging \( \Delta = \frac{1}{2}(g_o^2 - 2g_o + 1) \), we get the following equalities.

\[
\begin{align*}
\left\lfloor \frac{\Delta - 2}{g_o - 1} \right\rfloor &= \left\lfloor \frac{g_o^2 - 2g_o - 3}{2(g_o - 1)} \right\rfloor = \left\lfloor \frac{(g_o - 1)^2 - 4}{2(g_o - 1)} \right\rfloor = \frac{g_o - 1}{2} - 1 \\
\left\lfloor \frac{\Delta}{g_o + 1} \right\rfloor &= \left\lfloor \frac{g_o^2 - 2g_o + 1}{2(g_o + 1)} \right\rfloor = \left\lfloor \frac{(g_o - 1)^2 - 2(g_o - 1)}{2(g_o + 1)} \right\rfloor = \frac{g_o - 1}{2} \\
\left\lfloor \frac{\Delta - g_o + 1}{g_o - 1} \right\rfloor &= \left\lfloor \frac{g_o^2 - 4g_o + 3}{2(g_o - 1)} \right\rfloor = \left\lfloor \frac{(g_o - 1)^2 - 2(g_o - 1)}{2(g_o - 1)} \right\rfloor = \frac{g_o - 1}{2} - 1
\end{align*}
\]

By the definition, every vertex of \( G_2 \) has degree \( \Delta \). Thus \( |E(G_2)| = \frac{\Delta(g_o + 2)}{2} \). Since \( |V(G_2)| = g_o + 2 \), we have

\[
\left\lfloor \frac{2|E(G_2)|}{|V(G_2)| - 1} \right\rfloor = \left\lfloor \frac{\Delta(g_o + 2)}{g_o + 1} \right\rfloor = \Delta + \left\lfloor \frac{\Delta}{g_o + 1} \right\rfloor = \Delta + \frac{g_o - 1}{2}.
\]

Thus, \( \chi'(G_2) \geq \rho(G_2) \geq \left\lfloor \frac{2|E(G_2)|}{|V(G_2)| - 1} \right\rfloor = \Delta + \frac{g_o - 1}{2} \). On the other hand, (1.1) gives the following inequality.

\[
\chi'(G_2) \leq \Delta + 1 + \left\lfloor \frac{\Delta - 2}{g_o - 1} \right\rfloor = \Delta + \frac{g_o - 1}{2}.
\]

Thus \( \chi'(G_2) = \Delta + \frac{g_o - 1}{2} \) and \( G_2 \in \mathcal{GO}(\Delta, g_o) \).

Let \( R \subset G_2 \) be a ring graph, and \( H_2 = G_2[V(C)] \) be the subgraph induced by \( V(C) \). We will show \( \chi'(R) < \chi'(G_2) \) which completes the proof. Since \( C \) is the only induced odd cycle of the underlying simple graph \( G_c \), \( R \) is a spanning subgraph of \( H_2 \). Thus \( \chi'(R) \leq \chi'(H_2) \), and hence it is sufficient to show \( \chi'(H_2) < \chi'(G_2) \). Since the Goldberg-Seymour conjecture holds for all ring graphs, \( \chi'(H_2) < \chi'(G_2) \) is equivalent to \( \rho(H_2) < \chi'(G_2) = \Delta + \frac{g_o - 1}{2} \). Similarly we only need to consider \( \left\lfloor \frac{2|E(H')|}{|V(H')| - 1} \right\rfloor \) for every odd induced subgraph \( H' \) of \( H_2 \).

By the definition of \( G_2 \), the degree sequence of \( H_2 \) is \( \{\Delta, \ldots, \Delta, \Delta - \frac{g_o - 1}{2}, \Delta - \frac{g_o - 1}{2}\} \). Thus \( |E(H_2)| = \frac{\Delta g_o - g_o + 1}{2} \). Hence we have

\[
\left\lfloor \frac{2|E(H_2)|}{|V(H_2)| - 1} \right\rfloor = \left\lfloor \frac{\Delta g_o - g_o + 1}{g_o - 1} \right\rfloor = \Delta + \left\lfloor \frac{\Delta - g_o + 1}{g_o - 1} \right\rfloor = \Delta + \frac{g_o - 1}{2} - 1.
\]

Let \( H'_2 \) be a subgraph of \( H_2 \) with \( V(H'_2) \neq V(H_2) \). Then \( H'_2 \) is a bipartite graph. Consequently, \( \chi'(H'_2) = \Delta(H'_2) \leq \Delta \). Recall that density \( \rho \) is a lower bound for chromatic index, hence \( \left\lfloor \frac{2|E(H'_2)|}{|V(H'_2)| - 1} \right\rfloor \leq \rho(H'_2) \leq \chi'(H'_2) \leq \Delta \). Therefore, \( \rho(H_2) \leq \max\{\Delta + \frac{g_o - 1}{2} - 1, \Delta\} < \Delta + \frac{g_o - 1}{2} \).

\[\blacksquare\]