

THE LOWER SEMICONTINUITY OF THE FROBENIUS SPLITTING NUMBERS

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ABSTRACT. We show that, under mild conditions, the (normalized) Frobenius splitting numbers of a local ring of prime characteristic are lower semicontinuous.

1. INTRODUCTION AND TERMINOLOGY

Throughout this paper, all rings are assumed to be commutative Noetherian of positive characteristic p with p prime (unless stated otherwise explicitly). Let $q = p^e$ denote a power of the characteristic of the ring with $e \geq 0$. By a local ring (R, \mathfrak{m}, k) we mean a Noetherian ring R with only one maximal ideal \mathfrak{m} and the residue field k .

In recent years, a number of authors have studied a sequence of numbers associated to a local ring (R, \mathfrak{m}, k) of prime characteristic $p > 0$, called here the Frobenius splitting numbers, that arise naturally in connection to the Frobenius homomorphism $F : R \rightarrow R$, $F(r) = r^p$, for all $r \in R$.

Let us assume that R is reduced and denote by $R^{1/q}$ the ring of q^{th} roots of elements in R where $q = p^e$ with $e \geq 0$. Further assume that R is F-finite, which by definition means that $R^{1/q}$ is module finite over R for all q . Write

$$R^{1/q} \simeq R^{\oplus a_e} \oplus M_e,$$

which is a direct sum decomposition of $R^{1/q}$ over R such that M_e has no free direct summands. The number a_e is called the e^{th} Frobenius splitting number of R and much work has been dedicated to investigating the size of these numbers as in [6, 1, 2, 3, 11, 13, 14]. They are intimately connected to the notions of F-purity and strong F-regularity and in fact they can be defined more generally for local rings of prime characteristic, F-finite or not.

Our main result of the paper states that under mild conditions these numbers are lower semicontinuous, and therefore they exhibit a natural geometric behavior. In fact, we conjecture that the lower semicontinuity of these numbers holds for all excellent locally equidimensional rings.

We will now proceed to define the (normalized) Frobenius splitting numbers of a (not necessarily F-finite or reduced) Noetherian local ring (R, \mathfrak{m}, k) .

For any $e \geq 0$, we let $R^{(e)}$ be the R -algebra defined as follows: as a ring $R^{(e)}$ equals R while the R -algebra structure is defined by $r \cdot s = r^q s$, for all $r \in R$, $s \in R^{(e)}$. Note that when R is reduced we have that $R^{(e)}$ is isomorphic to $R^{1/q}$ as R -algebras. Also, $R^{(e)}$ as an $R^{(e)}$ -algebra

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is simply R as an R -algebra. For example, given an ideal I of R , we have $R/I \otimes_R R^{(e)}$ is (naturally isomorphic to) $R/I^{[q]}$, in which $I^{[q]}$ is the ideal of R generated by $\{x^q : x \in I\}$.

Let $E = E_R(k)$ denote the injective hull of the residue field $k = R/\mathfrak{m}$. We have a natural short exact sequence of R -modules:

$$0 \rightarrow k \xrightarrow{\psi} E \xrightarrow{\phi} E/k \rightarrow 0,$$

which induces an exact sequence

$$R^{(e)} \otimes_R k \xrightarrow{1 \otimes \psi} R^{(e)} \otimes_R E \xrightarrow{1 \otimes \phi} R^{(e)} \otimes_R E/k \rightarrow 0.$$

One can see that $K_e := \text{Im}(1 \otimes_R \psi)$ is finitely generated over $R^{(e)}$ and killed by $\mathfrak{m}^{[q]}$. Therefore it has finite length as an $R^{(e)}$ -module.

Let u be a socle generator of E . The reader can note that K_e is in fact the $R^{(e)}$ -submodule of $R^{(e)} \otimes_R E$ generated by $1 \otimes u$.

Definition 1.1. Let (R, \mathfrak{m}, k) be a local ring of positive prime characteristic $p > 0$ and let $e \geq 0$. Also let $E = E(k)$, ψ , ϕ and K_e be as above.

(1) the e^{th} normalized Frobenius splitting number of R , denoted by $s_e(R)$, is defined as

$$s_e(R) := \frac{\lambda_{R^{(e)}}(K_e)}{q^{\dim(R)}} = \frac{\lambda_{R^{(e)}}(\text{Im}(1 \otimes_R \psi))}{q^{\dim(R)}}.$$

(2) in the particular case when R is F-finite, let $\alpha(R) = \log_p[k : k^p]$ so that $[k : k^q] = q^{\alpha(R)}$ for all $q = p^e$. Then the e^{th} Frobenius splitting number of R , denoted by $a_e(R)$, is defined as

$$a_e(R) = s_e(R) \cdot q^{\dim(R) + \alpha(R)}.$$

(3) also, for a prime ideal P in R , we will use $s_e(P)$ and $a_e(P)$ to denote $s_e(R_P)$ and $a_e(R_P)$ respectively.

The reader should note that Yao has shown in [14, Lemma 2.1] that when R is F-finite the numbers $a_e(R)$ are exactly the numbers a_e (mentioned earlier in the introduction) such that

$$R^{(e)} = R^{\oplus a_e} \oplus M_e,$$

is a direct sum decomposition of $R^{(e)}$ over R where M_e has no free direct summands.

Yao has showed the following characterization of regular local rings which emphasizes the importance of these number in full generality, see [14, Lemma 2.5]. The corresponding result for F-finite rings was known from work of Huneke and Leuschke [6].

Theorem 1.2. *Let (R, \mathfrak{m}, k) be a local ring of positive characteristic p , where p is prime. Then R is regular if and only if*

$$s_e(R) = 1 \quad \text{for some (equivalently, for all) } e \geq 1.$$

One important point to make is that if R is F-finite reduced then the Frobenius splitting numbers can be defined directly as we did at the beginning of our introduction, while in the general case it is necessary to first consider the normalized Frobenius splitting numbers.

We remark that the limit of the sequence of normalized Frobenius splitting numbers, when exists, is a remarkable invariant of the ring, namely the *F-signature*. That is, the F-signature of R , denoted $s(R)$, equals

$$s(R) := \lim_{e \rightarrow \infty} s_e(R) \quad \text{if it exists.}$$

In fact, this is why $s_e(R)$ is called the e^{th} normalized Frobenius splitting number.

In general, we have an *upper F-signature* (respectively, *lower F-signature*) of R defined by $s^+(R) = \limsup_{e \rightarrow \infty} s_e(R)$ (respectively, $s^-(R) = \liminf_{e \rightarrow \infty} s_e(R)$). An important result states that for an excellent local ring R , $s^+(R) > 0$ if and only if $s^-(R) > 0$ if and only if R is strongly F-regular (see [4, Theorem 0.2] and [14, Theorem 1.3 (2)]). For more information on the Frobenius splitting numbers, the F-signature and related concepts, we refer the reader to [6, 1, 2, 3, 4, 11, 13, 14].

2. THE CONJECTURE

We are now in position to state the aim of our paper. We remind the reader that a function $f : X \rightarrow \mathbb{R}$, with X a topological space, is *lower semicontinuous* if the set $X_{\leq r} := \{x \in X : f(x) \leq r\}$ is closed or, equivalently, $X_{> r} := \{x \in X : f(x) > r\}$ is open for all $r \in \mathbb{R}$.

A ring (R, \mathfrak{m}, k) is *equidimensional* if $\dim(R/P) = \dim(R)$ for all minimal primes P of R . A ring R is *locally equidimensional* if R_P is equidimensional for all $P \in \text{Spec}(R)$. If $P \subseteq Q$ are prime ideals in an equidimensional and catenary ring R , then $\text{ht}(Q) = \text{ht}(P) + \text{ht}(Q/P)$ (see [9, Lemma 2 on page 250]).

It is also helpful to remind the reader the following notations: for any ideal I in R , we denote $V(I) := \{P \in \text{Spec}(R) : I \subseteq P\}$ which is a closed subset in $\text{Spec}(R)$. For $x \in R$, we denote $D(x) := \{Q \in \text{Spec}(R) : x \notin Q\}$ which is an open subset of $\text{Spec}(R)$.

Conjecture 2.1. Let R be a Noetherian of positive characteristic p with p prime and fix $e \geq 0$. Let $s_e : \text{Spec}(R) \rightarrow \mathbb{Q}$ be defined by

$$s_e(P) := s_e(R_P), \quad \forall P \in \text{Spec}(R).$$

If R is excellent and locally equidimensional, then s_e is lower semicontinuous.

In this paper we will show that Conjecture 2.1 holds true in many significant cases. In light of the fact that the F-signature and the Hilbert-Kunz multiplicity of a ring exhibit at times parallel behavior, it is perhaps interesting to note here that Shepherd-Barron proved in [10] that the Hilbert-Kunz functions are upper semicontinuous for an excellent and locally equidimensional ring of prime characteristic.

First we note the following rather general fact.

Proposition 2.2. *Let R be a Noetherian ring (not necessarily of prime characteristic p) and M a finitely generated R -module. For a prime ideal P of R , let $\#_P(M)$ equal the maximal number of free copies of R_P as direct summands in M_P . Consider the function*

$$\text{Spec}(R) \rightarrow \mathbb{Q} \quad \text{defined by} \quad P \mapsto \#_P(M).$$

Then this function is lower semicontinuous.

Proof. Let $r \in \mathbb{R}$ and $P \in \text{Spec}(R)$ such that $\#_P(M) > r$. Let $n = \#_P(M)$. Then there exists a R -linear surjection

$$M_P \rightarrow R_P^{\oplus n} \rightarrow 0.$$

Note that $(\text{Hom}_R(M, R^n))_P \simeq \text{Hom}_{R_P}(M_P, R_P^{\oplus n})$ and so one can lift the above surjection to $\phi \in \text{Hom}_R(M, R^{\oplus n})$, which gives the following exact sequence

$$M \xrightarrow{\phi} R^{\oplus n} \longrightarrow C \longrightarrow 0,$$

in which C is the cokernel of ϕ . This forces $C_P = 0$ as ϕ_P is surjective.

But $C_P = 0$ implies that there exists $x \notin P$ such that $C_x = 0$, which gives the following exact sequence

$$M_x \xrightarrow{\phi_x} R_x^{\oplus n} \longrightarrow 0.$$

Thus, for all $Q \in D(x) = \{Q \in \text{Spec}(R) : x \notin Q\}$, there is an exact sequence

$$M_Q \longrightarrow R_Q^{\oplus n} \longrightarrow 0,$$

which implies that $\#_Q(M) \geq n > r$ for all Q in the open set $D(x)$. \square

Lemma 2.3. *Let R be a ring of prime characteristic p , F -finite and locally equidimensional.*

Then, on a connected component of $\text{Spec}(R)$, the number $\dim(R_P) + \alpha(R_P)$ is constant, in which $\alpha(R_P)$ is as described in Definition 1.1 (2).

Proof. This was essentially proved by Kunz in [7, Corollary 2.7]. \square

Remark 2.4. The reader should be aware that Kunz states his result under the hypothesis that R is equidimensional. In the generality stated in [7] the result is not correct as Shepherd-Barron showed in [10].

The error in the proof of Kunz is in the last line of his proof where he assumes without proof that $\text{ht}(Q) = \text{ht}(P) + \text{ht}(Q/P)$ for prime ideals $P \subseteq Q$ in R . For our Lemma stated above, this follows from the condition that R is locally equidimensional: if we localize at Q we get an equidimensional ring R_Q and in an equidimensional excellent local ring the relation holds as remarked at the beginning of this section (cf. [9, Lemma 2 on page 250]).

As an immediate consequence we obtain:

Corollary 2.5. *Let R be a F -finite locally equidimensional ring of positive characteristic p , p prime. Define $a_e : \text{Spec}(R) \rightarrow \mathbb{Q}$ by $a_e(P) = a_e(R_P)$ (the F -finite property localizes, so the definition is possible).*

Then the Frobenius splitting numbers and the normalized Frobenius splitting numbers are lower semicontinuous, i.e., both s_e and a_e are lower semicontinuous functions for all e . Moreover, these functions are proportional on each connected component of $\text{Spec}(R)$ (with a possibly different factor of proportionality on each component).

Proof. As remarked earlier, we have the relationship

$$a_e(R_P) = s_e(R_P) \cdot q^{\dim(R_P) + \alpha(R_P)}, \quad \forall P \in \text{Spec}(R).$$

Since lower semicontinuity can be checked on each connected component of $\text{Spec}(R)$, we may assume $\text{Spec}(R)$ is connected without loss of generality.

But then $a_e(P) = a_e(R_P)$ and $s_e(P) = s_e(R_P)$ are proportional, as $\dim(R_P) + \alpha(R_P)$ is constant by Lemma 2.3. Hence it suffices to show a_e is lower semicontinuous.

Also note that $a_e(R_P)$ is simply $\#_P(R^{(e)})$ as in Proposition 2.2. Since $R^{(e)}$ is finitely generated over R because R is F-finite, we can apply Proposition 2.2 and conclude that a_e is lower semicontinuous. This implies that s_e is lower semicontinuous as well by the fact that they are related by proportionality. \square

This Corollary answers our Conjecture 2.1 in the F-finite case. Much more work will be needed if we are not under the presence of the F-finite condition as our next sections will show.

We will need the following criteria attributed to Nagata; see [8].

Proposition 2.6. *Let R be a Noetherian ring and U be a subset of $\text{Spec}(R)$. Then U is open if and only if both of the following statements hold*

- (i) *if $P \in U$, $Q \in \text{Spec}(R)$ and $Q \subseteq P$, then $Q \in U$.*
- (ii) *for all $P \in U$, $U \cap V(P)$ contains a nonempty open subset of $V(P)$; or equivalently, for all $P \in U$, there exists $x \in R \setminus P$ such that $D(x) \cap V(P) \subseteq U$.*

3. HOMOMORPHIC IMAGES OF REGULAR RINGS

Throughout this section we let $R = S/I$ where S is a regular ring of prime characteristic p and I an ideal of S . As always $q = p^e$.

Proposition 3.1. *Let $R = S/I$ be a homomorphic image of regular local ring (S, \mathfrak{n}, k) . Then*

$$s_e(R) \cdot q^{\dim(R)} = \lambda_S \left(\frac{S}{\mathfrak{n}^{[q]} : (I^{[q]} : I)} \right) = \lambda_S \left(\frac{(I^{[q]} : I) + \mathfrak{n}^{[q]}}{\mathfrak{n}^{[q]}} \right),$$

for any nonnegative integer e .

Proof. We can make a faithfully flat extension of S (and hence R) by first completing and then enlarging the residue field of S (and hence R) to its algebraic closure. This flat local extension has its closed fiber equal to a field. Note that the rings are F-finite after extension.

In [14, Remark 2.3 (3)], it is shown that the normalized Frobenius splitting number is unchanged under such extensions. Also, $\lambda_S \left(\frac{S}{\mathfrak{n}^{[q]} : (I^{[q]} : I)} \right)$ is not affected under such an extension.

So, it is enough to check the equality $s_e(R) \cdot q^{\dim(R)} = \lambda_S \left(\frac{S}{\mathfrak{n}^{[q]} : (I^{[q]} : I)} \right)$ under the additional hypothesis that R is F-finite and this is observed in [2] (see the remarks immediately after [2, Theorem 4.2]).

Finally, the equality $\lambda_S \left(\frac{S}{\mathfrak{n}^{[q]} : (I^{[q]} : I)} \right) = \lambda_S \left(\frac{(I^{[q]} : I) + \mathfrak{n}^{[q]}}{\mathfrak{n}^{[q]}} \right)$ holds true by Matlis Duality as shown in [2, page 11]. \square

We will also need the following lemma. A proof is included for completeness.

Lemma 3.2. *Let R be a Noetherian ring, P a prime ideal in R and $L \subseteq M$ be R -modules such that M/L is finitely generated over R . Assume that $\lambda_{R_P}(M_P/L_P) = n < \infty$.*

Then there exists $x \in R \setminus P$ and a filtration of R_x -modules

$$L_x = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M_x$$

such that $M_i/M_{i-1} \simeq (R/P)_x$ for all $i = 1, \dots, n$.

Proof. Let $L = N_0 \subseteq N_1 \subseteq \dots \subseteq N_r = M$ be a prime filtration of $L \subseteq M$ such that $N_i/N_{i-1} \simeq R/Q_i$, where $Q_i \in \text{Spec}(R)$ for $i = 1, \dots, r$.

For each of the indices i such that $Q_i \not\subseteq P$, choose $x_i \in Q_i$ but not in P . Let x equal to the product of all these elements x_i . Tensoring the original filtration with R_x , we get $(N_i/N_{i-1})_x \simeq (R/Q_i)_x = 0$ for all these i (such that $Q_i \not\subseteq P$).

For the rest indices j (so that $Q_j \subseteq P$), since $(M/L)_P$ has finite length over R_P , we conclude that $\lambda_{R_P}((R/Q_j)_P) < \infty$, which forces $Q_j = P$ in this case. Moreover, we see that there are precisely n many indices j such that $Q_j = P$.

Thus, by tensoring $L = N_0 \subseteq N_1 \subseteq \dots \subseteq N_r = M$ with R_x , removing the repeated terms and relabeling everything properly, we obtain a required filtration over R_x . \square

Lemma 3.3. *Let $A \rightarrow B$ be a flat homomorphism of Noetherian rings and consider*

$$M_0 \subseteq M_1 \subseteq \dots \subseteq M_n,$$

which is a filtration of A -modules. Write $M_i/M_{i-1} = N_i$ for $i = 1, \dots, n$. Assume that \mathbf{x} is a sequence in B that is regular on $N_i \otimes_A B$ for all $i = 2, \dots, n$.

(1) *there is the following natural isomorphism*

$$\frac{M_1 \otimes B + (\mathbf{x})(M_n \otimes B)}{M_0 \otimes B + (\mathbf{x})(M_n \otimes B)} \simeq \frac{M_1}{M_0} \otimes \frac{B}{(\mathbf{x})B}.$$

(2) *assume furthermore that $\lambda_B \left(\frac{N_i \otimes_A B}{(\mathbf{x})(N_i \otimes_A B)} \right) < \infty$ for $i = 1, \dots, n$. Then*

$$\lambda_B \left(\frac{M_n}{M_0} \otimes \frac{B}{(\mathbf{x})B} \right) = \sum_{i=1}^n \lambda_B \left(N_i \otimes \frac{B}{(\mathbf{x})B} \right).$$

Proof. (1) Note that \mathbf{x} is a regular sequence on $M_n/M_1 \otimes_A B$ by a routine short exact sequence argument. Next, consider the short exact sequence

$$0 \longrightarrow M_1/M_0 \longrightarrow M_n/M_0 \longrightarrow M_n/M_1 \longrightarrow 0.$$

Tensoring with B and then with $B/(\mathbf{x})B$, we get

$$0 \longrightarrow \frac{M_1}{M_0} \otimes \frac{B}{(\mathbf{x})B} \longrightarrow \frac{M_n}{M_0} \otimes \frac{B}{(\mathbf{x})B} \longrightarrow \frac{M_n}{M_1} \otimes \frac{B}{(\mathbf{x})B} \longrightarrow 0,$$

since the Koszul homology $H_1(\mathbf{x}, \frac{M_n}{M_1} \otimes B) = 0$. But also note the natural isomorphisms

$$\begin{aligned} \frac{M_n}{M_0} \otimes \frac{B}{(\mathbf{x})B} &\simeq \frac{M_n \otimes B}{M_0 \otimes B + (\mathbf{x})(M_n \otimes B)} \quad \text{and} \\ \frac{M_n}{M_1} \otimes \frac{B}{(\mathbf{x})B} &\simeq \frac{M_n \otimes B}{M_1 \otimes B + (\mathbf{x})(M_n \otimes B)}. \end{aligned}$$

So using these natural isomorphisms, one finishes the proof of this part.

(2) Consider a piece of the filtration, which gives a short exact sequence

$$0 \longrightarrow \frac{M_{i-1}}{M_0} \longrightarrow \frac{M_i}{M_0} \longrightarrow N_i \longrightarrow 0, \quad i = 2, \dots, n,$$

and tensor with B which is A -flat.

Note that since \mathbf{x} are a regular sequence on $N_i \otimes B$, we get that the Koszul homology $H_1(\mathbf{x}, N_i \otimes B) = 0$. Applying the long exact sequence of Koszul homology with respect to \mathbf{x} , we get an exact sequence as follows

$$0 \longrightarrow \frac{M_{i-1}}{M_0} \otimes_A \frac{B}{(\mathbf{x})B} \longrightarrow \frac{M_i}{M_0} \otimes_A \frac{B}{(\mathbf{x})B} \longrightarrow N_i \otimes_A \frac{B}{(\mathbf{x})B} \longrightarrow 0.$$

Using the additivity of the length function and summing over all such short exact sequences for $2 = 1, \dots, n$, we obtain that

$$\lambda_B \left(\frac{M_n}{M_0} \otimes_A \frac{B}{(\mathbf{x})B} \right) = \sum_{i=1}^n \lambda_B \left(N_i \otimes \frac{B}{(\mathbf{x})B} \right). \quad \square$$

Theorem 3.4. *Let R be a homomorphic image of a regular ring of prime characteristic p and assume that R is excellent and locally equidimensional.*

Then the normalized Frobenius splitting numbers are lower semicontinuous, i.e., s_e is lower semicontinuous for every $e \geq 0$.

Proof. Using the notations introduced at the beginning of the section we let $R = S/I$. Let $\overline{P} = P/I$ be a prime ideal of R , in which P is a prime ideal of S containing I . Quite generally, denote $\overline{Q} := Q/I$ for all $Q \in \text{Spec}(S) \cap V(I)$. Let $K = (I^{[q]} :_S I)$, an ideal in S .

Let $m := \lambda_{S_P} \left(\left(\frac{K+P^{[q]}}{P^{[q]}} \right)_P \right)$. Then Proposition 3.1 allows us to write

$$s_e(\overline{P}) = \lambda_{S_P} \left(\left(\frac{K + P^{[q]}}{P^{[q]}} \right)_P \right) \cdot \frac{1}{q^{\text{ht}(P/I)}} = m \cdot \frac{1}{q^{\text{ht}(P/I)}}.$$

Consider $P^{[q]} \subseteq K + P^{[q]} \subseteq S$. Applying Lemma 3.2 to $P^{[q]} \subseteq K + P^{[q]}$ and $K + P^{[q]} \subseteq S$, we can find x in $S \setminus P$ and a filtration

$$P^{[q]}_x = M_0 \subseteq M_1 \subseteq \dots \subseteq M_m \subseteq M_{m+1} \subseteq \dots \subseteq M_{m+n} = S_x$$

such that $M_m = (K + P^{[q]})_x$ and $M_i/M_{i-1} \simeq (S/P)_x$ for all $i = 1, \dots, m+n$.

Since R is excellent, the regular locus of S/P is open (and non-empty), so there exists $y \notin P$ such that $R/\overline{P} = S/P$ becomes regular when localizing at y . We can replace x by xy and hence simply assume that $(S/P)_x$ is in fact regular.

Consider any $Q \in D(x) \subseteq \text{Spec}(S)$ such that $P \subseteq Q$. Since $(S/P)_x$ is regular we obtain that $(S/P)_Q$ is regular as well. Let us choose $\mathbf{x} = x_1, \dots, x_d$ in S such that their images in $(S/P)_Q$ form a regular system of parameters (of the regular local ring $(S/P)_Q$). More explicitly, we have $Q_Q = (P + (\mathbf{x}))_Q$ and $d = \dim((R/P)_Q) = \text{ht}(Q/P)$.

Note that Proposition 3.1 also allows us to write

$$s_e(Q/I) = \lambda_{S_Q} \left(\left(\frac{K + Q^{[q]}}{Q^{[q]}} \right)_Q \right) \cdot \frac{1}{q^{\text{ht}(Q/I)}} = \lambda_{S_Q} \left(\left(\frac{K + (P + (\mathbf{x}))^{[q]}}{P^{[q]} + (\mathbf{x})^{[q]}} \right)_Q \right) \cdot \frac{1}{q^{\text{ht}(Q/I)}}.$$

Remember that $m = \lambda_{S_P} \left(\left(\frac{K+P^{[q]}}{P^{[q]}} \right)_P \right)$ and $d = \text{ht}(Q/P)$, which equals $\dim(R_{\overline{Q}}) - \dim(R_{\overline{P}})$ since R is locally equidimensional.

Our plan is to show that

$$(*) \quad \lambda_{S_Q} \left(\left(\frac{K + (P^{[q]} + (\mathbf{x})^{[q]})}{P^{[q]} + (\mathbf{x})^{[q]}} \right)_Q \right) = m \cdot \lambda_{S_Q} \left(\left(\frac{S}{P + (\mathbf{x})^{[q]}} \right)_Q \right) = m \cdot q^d.$$

This will show that $s_e(Q/I) = s_e(P/I)$.

In order to prove our claim let us return to the filtration we considered above

$$P^{[q]}_x = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m \subseteq M_{m+1} \subseteq \cdots \subseteq M_{m+n} = S_x,$$

in which $M_m = (K + P^{[q]})_x$ and $M_i/M_{i-1} \simeq (S/P)_x$ for all $i = 1, \dots, m+n$.

Applying Lemma 3.3 (1) to the filtration $M_0 \subseteq M_m \subseteq \cdots \subseteq M_{m+n}$ with $A = S_x \rightarrow B = S_Q$ and the sequence $\mathbf{x}^{[q]} := x_1^q, \dots, x_d^q$ which is regular on $(S/P)_Q$, we get

$$(\dagger) \quad \left(\frac{(K + P^{[q]}) + (\mathbf{x})^{[q]}}{P^{[q]} + (\mathbf{x})^{[q]}} \right)_Q \simeq \frac{(K + P^{[q]})_x}{P^{[q]}_x} \otimes \frac{S_Q}{(\mathbf{x})^{[q]}S_Q}.$$

Then, applying Lemma 3.3 (2) to the filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$, we see

$$\lambda_{S_Q} \left(\frac{(K + P^{[q]})_x}{P^{[q]}_x} \otimes \frac{S_Q}{(\mathbf{x})^{[q]}S_Q} \right) = \sum_{i=1}^m \lambda_{S_Q} \left(\left(\frac{S}{P} \right)_x \otimes \frac{S_Q}{(\mathbf{x})^{[q]}S_Q} \right) = m \cdot \lambda_{S_Q} \left(\frac{S_Q}{(P + (\mathbf{x})^{[q]})S_Q} \right).$$

Since \mathbf{x} are regular system of parameters on $(S/P)_Q$, we deduce $\lambda_{S_Q} \left(\frac{S_Q}{(P + (\mathbf{x})^{[q]})S_Q} \right) = q^d = q^{\text{ht}(Q/P)}$ and hence

$$(\ddagger) \quad \lambda_{S_Q} \left(\frac{(K + P^{[q]})_x}{P^{[q]}_x} \otimes \frac{S_Q}{(\mathbf{x})^{[q]}S_Q} \right) = m \cdot q^d = m \cdot q^{\text{ht}(Q/P)}.$$

Combining (\dagger) and (\ddagger) , we see our claim $(*)$ that $\lambda_{S_Q} \left(\left(\frac{K + (P^{[q]} + (\mathbf{x})^{[q]})}{P^{[q]} + (\mathbf{x})^{[q]}} \right)_Q \right) = m \cdot q^{\text{ht}(Q/P)}$, which then implies

$$\begin{aligned} s_e(Q/I) &= \lambda_{S_Q} \left(\left(\frac{K + (P + (\mathbf{x}))^{[q]}}{P^{[q]} + (\mathbf{x})^{[q]}} \right)_Q \right) \cdot \frac{1}{q^{\text{ht}(Q/I)}} \\ &= m \cdot q^{\text{ht}(Q/P)} \cdot \frac{1}{q^{\text{ht}(Q/I)}} = \lambda_{S_P} \left(\left(\frac{K + P^{[q]}}{P^{[q]}} \right)_P \right) \cdot \frac{1}{q^{\text{ht}(P/I)}} = s_e(P/I). \end{aligned}$$

In summary, for any given prime ideal P/I of R (i.e., $P \in V(I) \subseteq \text{Spec}(S)$), there is $x \in S \setminus P$ such that $s_e(Q/I) = s_e(P/I)$ for all $Q \in D(x) \cap V(P)$.

Now, to prove the lower semicontinuity of $s_e : \text{Spec}(R) \rightarrow \mathbb{Q}$, let $r \in \mathbb{R}$ be a number and let $U = \{\overline{Q} : s_e(\overline{Q}) > r\} \subseteq \text{Spec}(R)$. To show that U is open it is enough to apply Proposition 2.6.

Part (i) in Proposition 2.6 is satisfied by [14, Proposition 5.2] where it is shown that the normalized Frobenius splitting numbers can only increase by localization.

Part (ii) follows from what we just proved earlier: For any $\overline{P} = P/I \in U$, the work right above shows $D(\overline{x}) \cap V(\overline{P}) \subseteq U$, in which $x \in S \setminus P$ is as obtained above and \overline{x} denotes the image of x in R . This completes the proof. \square

Remark 3.5. The reader should note that under the conditions of Theorem 3.4, the same proof shows that the set $X_{\geq r} = \{Q \in \text{Spec}(R) : s_e(Q) \geq r\}$ is open for all $r \in \mathbb{R}$.

4. GORENSTEIN EXCELLENT RINGS

We state the following lemma about Gorenstein rings.

Lemma 4.1. *Let (R, \mathfrak{m}, k) be a local Gorenstein ring of prime positive characteristic p and dimension d . Let $\mathbf{x} = x_1, \dots, x_d$ be any system of parameters for R .*

Then, for any $u \in R$ such that its image generates the socle of $R/(\mathbf{x})$, we have

$$s_e(R) \cdot q^d = \lambda \left(\frac{Ru^q + (\mathbf{x})^{[q]}}{(\mathbf{x})^{[q]}} \right) \quad \text{for all } e \geq 0.$$

(When $\dim(R) = 0$, we agree that $(\mathbf{x}) = 0$ by convention.)

Proof. Note that the statement is clear when $\dim(R) = 0$. (In fact, the following proof also works for the case of $\dim(R) = 0$ if we agree that $x = 1$ and $(\mathbf{x})^{[n]} = 0$ for all $n \geq 0$.)

Denote $x := \prod_{i=1}^d x_i$ and $(\mathbf{x})^{[n]} := (x_1^n, \dots, x_d^n)$ for all $n \geq 0$. As R is Gorenstein, the injective hull of $k = R/\mathfrak{m}$ can be obtained as

$$E = E_R(k) = \varinjlim \left\{ \frac{R}{(\mathbf{x})} \xrightarrow{\cdot x} \frac{R}{(\mathbf{x})^{[2]}} \xrightarrow{\cdot x} \dots \xrightarrow{\cdot x} \frac{R}{(\mathbf{x})^{[n]}} \xrightarrow{\cdot x} \frac{R}{(\mathbf{x})^{[n+1]}} \xrightarrow{\cdot x} \dots \right\}.$$

Note that all the maps in the direct system above are injective.

Consider the R -linear injective map $f : k = \frac{R}{\mathfrak{m}} \xrightarrow{\cdot u} \frac{R}{(\mathbf{x})}$ defined by $f(r + \mathfrak{m}) = ru + (\mathbf{x})$ for $r \in R$, which induces an injective R -linear map $\psi : k \rightarrow E$ as all the maps in the direct system above are injective.

By Definition 1.1, we see

$$s_e(R) \cdot q^d = \lambda_{R^{(e)}} \left(\text{Im} \left(R^{(e)} \otimes_R k \xrightarrow{1 \otimes \psi} R^{(e)} \otimes_R E \right) \right).$$

Moreover, by the property of tensor product and direct limit, we see

$$R^{(e)} \otimes_R E = \varinjlim \left\{ R^{(e)} \otimes \frac{R}{(\mathbf{x})} \xrightarrow{1 \otimes (\cdot x)} \dots \xrightarrow{1 \otimes (\cdot x)} R^{(e)} \otimes \frac{R}{(\mathbf{x})^{[n]}} \xrightarrow{1 \otimes (\cdot x)} R^{(e)} \otimes \frac{R}{(\mathbf{x})^{[n+1]}} \xrightarrow{1 \otimes (\cdot x)} \dots \right\}.$$

Thus, the image of the homomorphism $R^{(e)} \otimes_R k \xrightarrow{1 \otimes \psi} R^{(e)} \otimes_R E$ is exactly the image of $R^{(e)} \otimes_R \frac{R}{\mathfrak{m}}$ in the direct limit of the following direct system

$$R^{(e)} \otimes \frac{R}{\mathfrak{m}} \xrightarrow{1 \otimes (\cdot u)} R^{(e)} \otimes \frac{R}{(\mathbf{x})} \xrightarrow{1 \otimes (\cdot x)} \dots \xrightarrow{1 \otimes (\cdot x)} R^{(e)} \otimes \frac{R}{(\mathbf{x})^{[n]}} \xrightarrow{1 \otimes (\cdot x)} R^{(e)} \otimes \frac{R}{(\mathbf{x})^{[n+1]}} \xrightarrow{1 \otimes (\cdot x)} \dots.$$

However, by the meaning of $R^{(e)}$, the above direct system can be written as

$$\frac{R}{\mathfrak{m}^{[q]}} \xrightarrow{\cdot u^q} \frac{R}{(\mathbf{x})^{[q]}} \xrightarrow{\cdot x^q} \dots \xrightarrow{\cdot x^q} \frac{R}{(\mathbf{x})^{[nq]}} \xrightarrow{\cdot x^q} \frac{R}{(\mathbf{x})^{[(n+1)q]}} \xrightarrow{\cdot x^q} \dots,$$

in which $\frac{R}{(\mathbf{x})^{[nq]}} \xrightarrow{\cdot x^q} \frac{R}{(\mathbf{x})^{[(n+1)q]}}$ is injective for every n since \mathbf{x} is regular on R .

Putting things together, we see

$$\begin{aligned} s_e(R) \cdot q^d &= \lambda_{R^{(e)}} \left(\operatorname{Im} \left(R^{(e)} \otimes_R k \xrightarrow{1 \otimes \psi} R^{(e)} \otimes_R E \right) \right) \\ &= \lambda_R \left(\operatorname{Im} \left(\frac{R}{\mathfrak{m}^{[q]}} \xrightarrow{\cdot u^q} \frac{R}{(\mathbf{x})^{[q]}} \right) \right) = \lambda_R \left(\frac{Ru^q + (\mathbf{x})^{[q]}}{(\mathbf{x})^{[q]}} \right). \quad \square \end{aligned}$$

Theorem 4.2. *Let R be an excellent Gorenstein ring. Then the normalized Frobenius splitting numbers are lower semicontinuous, i.e., s_e is lower semicontinuous for every $e \geq 0$.*

Proof. Let $r \in \mathbb{R}$ and let P be a prime ideal in R such that $s_e(P) > r$.

By prime avoidance, we can choose $\mathbf{x} = x_1, \dots, x_k$ that is a regular sequence on R such that their images form a system of parameters on R_P , where $k = \operatorname{ht}(P)$. Let $u \in R$ be such that its image in R_P generates the socle modulo (\mathbf{x}) , the ideal generated by \mathbf{x} .

Let $m := \lambda_{R_P} \left(\left(\frac{(u^q) + (\mathbf{x})^{[q]}}{(\mathbf{x})^{[q]}} \right)_P \right)$. By Lemma 4.1 we get that

$$m = \lambda_{R_P} \left(\left(\frac{(u^q) + (\mathbf{x})^{[q]}}{(\mathbf{x})^{[q]}} \right)_P \right) = s_e(P) \cdot q^{\operatorname{ht}(P)}.$$

Now we proceed as in the proof of Theorem 3.4. Applying Lemma 3.2 to $(\mathbf{x})^{[q]} \subseteq (u^q) + (\mathbf{x})^{[q]} \subseteq R$ and $(\mathbf{x}) \subseteq (u) + (\mathbf{x}) \subseteq R$, we therefore obtain $x \in R \setminus P$ and filtrations

$$(4.2.1) \quad (\mathbf{x})^{[q]}_x = M_0 \subseteq M_1 \subseteq \dots \subseteq M_m \subseteq M_{m+1} \subseteq \dots \subseteq M_{m+n} = R_x$$

$$(4.2.2) \quad \text{and } (\mathbf{x})_x = N_0 \subseteq N_1 \subseteq \dots \subseteq N_t = R_x,$$

in which $M_m = ((u^q) + (\mathbf{x})^{[q]})_x$, $N_1 = (Ru + (\mathbf{x}))_x$, and $M_i/M_{i-1} \simeq N_j/N_{j-1} \simeq (R/P)_x$ for all $i = 1, \dots, m+n$ and all $j = 1, \dots, t$.

Similarly, since R is excellent and thus the regular locus of R/P is open, we may just as well further assume that $(R/P)_x$ is regular.

Let Q be any prime ideal in $D(x) \cap V(P)$. As $(R/P)_Q$ is regular, let $\mathbf{y} = y_1, \dots, y_h$ be chosen such that their images in $(R/P)_Q$ form a regular system of parameters. (In particular, we have $Q_Q = (P + (\mathbf{y}))_Q$ and $h = \operatorname{ht}(Q/P)$.) It then follows that \mathbf{y} is a regular sequence on $(R/(\mathbf{x}))_Q$ (because of the filtration (4.2.1)). Thus \mathbf{x}, \mathbf{y} form a system of parameters of R_Q .

As in the proof of Theorem 3.4, we apply Lemma 3.3 to the filtration (4.2.1) and $R_x \rightarrow R_Q$ to deduce that

$$\lambda_{R_Q} \left(\left(\frac{(u^q) + (\mathbf{x})^{[q]} + (\mathbf{y})^{[q]}}{(\mathbf{x})^{[q]} + (\mathbf{y})^{[q]}} \right)_Q \right) = m \cdot q^h = m \cdot q^{\operatorname{ht}(Q/P)}.$$

Next, we show that the image of u generates the socle of $\left(\frac{R}{(\mathbf{x}, \mathbf{y})} \right)_Q$: Note that R_Q is Gorenstein and $QR_Q = (P + (\mathbf{y}))R_Q$. Applying Lemma 3.3 (1) to the filtration (4.2.2), $R_x \rightarrow R_Q$ and the sequence \mathbf{y} that is regular on $(R/P)_Q$, we deduce that the image of u

generates the following submodule in $\left(\frac{R}{(\mathbf{x}, \mathbf{y})}\right)_Q$

$$\begin{aligned} \left(\frac{(u) + (\mathbf{x}) + (\mathbf{y})}{(\mathbf{x}) + (\mathbf{y})}\right)_Q &= \left(\frac{N_1 + (\mathbf{y})N_t}{N_0 + (\mathbf{y})N_t}\right)_Q \\ &\simeq \frac{N_1}{N_0} \otimes \frac{R_Q}{(\mathbf{y})R_Q} \simeq \frac{R}{P} \otimes \frac{R_Q}{(\mathbf{y})R_Q} = \frac{R_Q}{(P + (\mathbf{y}))R_Q} \simeq \frac{R_Q}{QR_Q}. \end{aligned}$$

Thus, as R_Q is Gorenstein, the image of u generates the socle of $\left(\frac{R}{(\mathbf{x}, \mathbf{y})}\right)_Q$.

Applying again Lemma 4.1 to R_Q and \mathbf{x}, \mathbf{y} , we get that

$$s_e(Q) \cdot q^{\text{ht}(Q)} = m \cdot q^{\text{ht}(Q/P)} = s_e(P) \cdot q^{\text{ht}(P) + \text{ht}(Q/P)}.$$

Note that $\text{ht}(P) + \text{ht}(Q/P) = \text{ht}(Q)$ because R is Gorenstein, hence locally equidimensional. So in summary, we see

$$s_e(Q) = s_e(P) \quad \text{for all } Q \in D(x) \cap V(P).$$

Finally, we show the lower semicontinuity on the normalized Frobenius splitting numbers by applying Proposition 2.6. Note that the normalized Frobenius splitting number can only increase under localization by in [14, Proposition 5.2]. Thus we need to check only part (ii) of Proposition 2.6 for $U = \{Q : s_e(Q) > r\}$. But part (ii) is clear by now since, for any $P \in U$, there is $x \in R \setminus P$ such that $D(x) \cap V(P) \subseteq U$ by the work above. It follows that $U = \{Q : s_e(Q) > r\}$ is open. \square

Remark 4.3. The reader should note that under the conditions of Theorem 4.2, the same proof shows that the set $X_{\geq r} = \{Q \in \text{Spec}(R) : s_e(Q) \geq r\}$ is open for all $r \in \mathbb{R}$.

5. ESSENTIALLY OF FINITE TYPE OVER A SEMI-LOCAL EXCELLENT RING

In the following theorem, \widehat{A} denotes the completion of a semi-local ring A with respect to its Jacobson radical, which is isomorphic to $\prod_{\mathfrak{m}} \widehat{A}_{\mathfrak{m}}$ in which \mathfrak{m} runs over all the (finitely many) maximal ideals \mathfrak{m} of A while $\widehat{A}_{\mathfrak{m}}$ is the \mathfrak{m} -adic completion of A (or of $A_{\mathfrak{m}}$).

Theorem 5.1. *Let R be a ring of prime characteristic p that is essentially of finite type ring over an excellent semi-local ring A . Assume that $R \otimes_A \widehat{A}$ is locally equidimensional.*

Then the normalized Frobenius splitting numbers are lower semicontinuous, i.e., s_e is lower semicontinuous for every $e \geq 0$.

Proof. Let us fix a nonnegative integer e and let $r \in \mathbb{R}$ be any number. We plan to show that the set $W = \{Q : Q \in \text{Spec}(P), s_e(Q) > r\}$ is open by applying Proposition 2.6. The behavior under localization (cf. [14, Proposition 5.2]) implies that we need to check only part (ii) of Proposition 2.6 for W . Fix a prime ideal $P \in W$, i.e., $s_e(P) > r$.

Let $S = R \otimes_A \widehat{A}$ and let $C = \{P' \in \text{Spec}(S) : s_e(P') \leq r\}$. Note that Theorem 3.4 applies to S since S is essentially of finite type over \widehat{A} . Therefore, the normalized Frobenius splitting numbers are lower semicontinuous on S . Thus, C is closed in $\text{Spec}(S)$ so there exists an ideal I of S such that $V(I) = C$. Let $J = I + PS$.

We claim that $J \cap (R \setminus P) \neq \emptyset$: By way of contradiction, suppose $J \cap (R \setminus P) = \emptyset$. Then there exists a prime ideal $P' \in \text{Spec}(S)$ such that $P' \in V(J) \subseteq V(I) = C$ and

$P' \cap (R \setminus P) = \emptyset$. This forces $P' \cap R = P$. However, since A is excellent, we see that $R \rightarrow S$ has geometrically regular fibers ([8, 33.E Lemma 4]) and hence the map $R_P \rightarrow S_{P'}$ has regular closed fiber. Now, by [14, Theorem 5.6], we see $s_e(P') = s_e(P)$, contradicting the choice of P' with $P' \in C$.

Now that $J \cap (R \setminus P) \neq \emptyset$, let $x \in J \cap (R \setminus P)$ and consider $Q \in \text{Spec}(R)$ such that $P \subseteq Q$ and $x \notin Q$ (that is, $Q \in D(x) \cap V(P) \subseteq \text{Spec}(R)$).

As S is faithfully flat over R , there exists $Q' \in \text{Spec}(S)$ such that $Q' \cap R = Q$. Hence $x \notin Q'$ and thus $J \not\subseteq Q'$. Note that this further implies that $I \not\subseteq Q'$ (as it automatically holds that $PS \subseteq Q'$). This says that $Q' \notin V(I) = C$, or simply, $s_e(Q') > r$.

We can see that $R_Q \rightarrow S_{Q'}$ is faithfully flat and so $s_e(Q) \geq s_e(Q')$ by [14, Lemma 5.1]. (Or use [14, Theorem 5.6] to deduce that $s_e(Q) = s_e(Q')$.) Thus $s_e(Q) > r$.

Putting everything together, we get $s_e(Q) > r$ for all $Q \in D(x) \cap V(P)$. That is

$$D(x) \cap V(P) \subseteq U.$$

By Proposition 2.6, this shows that the normalized Frobenius splitting numbers are lower semicontinuous on R . \square

By Ratliff's theorem, the completion of an equidimensional excellent local ring remains equidimensional (see [12, Corollary B.4.3 and Theorem B.5.1]). We have the following

Corollary 5.2. *If R is an excellent locally equidimensional semi-local ring (e.g., R is an excellent equidimensional local ring) of prime characteristic p , the function $s_e : \text{Spec}(R) \rightarrow \mathbb{Q}$ is lower semicontinuous.*

Remark 5.3. The reader should note that under the conditions of Theorem 5.1, the proof shows that the set $X_{\geq r} = \{Q \in \text{Spec}(R) : s_e(Q) \geq r\}$ is open for all $r \in \mathbb{R}$. (To do this, one accordingly defines $C = \{P' \in \text{Spec}(S) : s_e(P') < r\} = V(I)$ because of Remark 3.5.)

Remark 5.4. Note that the proof of Theorem 5.1 relies on the lower semicontinuity of s_e on the ring $S = R \otimes_A \widehat{A}$. Although this is covered in Theorem 3.4, we would like to point out that the lower semicontinuity of s_e on S can also be proved via Γ -construction (see [5]). By applying Γ -construction, one may reduce the lower semicontinuity of s_e on S to the F-finite case and therefore Corollary 2.5 applies.

Remark 5.5. In general, one may study the notion of e^{th} normalized Frobenius splitting numbers of any finitely generated module M . Indeed, for any local ring (R, \mathfrak{m}, k) of prime characteristic p and any finitely generated module R -module M , one may define $s_e(M) := \frac{\#(^e M)}{q^{\dim(R)}}$, in which $\#(^e M)$ is the same as defined in [14, Definition 2.2].

Thus, for any fixed $e \geq 0$, there is a function $s_{e,M} : \text{Spec}(R) \rightarrow \mathbb{Q}$ defined by $P \mapsto s_e(M_P)$. Note that $s_{e,R} = s_e$ when $M = R$. Correspondingly, one may conjecture that $s_{e,M}$ is lower semicontinuous if the ring is excellent and locally equidimensional (cf. Conjecture 2.1). In fact, the lower semicontinuity of $s_{e,M}$ can be verified in the cases when R is as in Corollary 2.5, as in Theorem 4.2 and further assume M is (locally) maximal Cohen-Macaulay, or as in Theorem 5.1. Note that when R is as in Theorem 5.1, one may first prove the lower semicontinuity of $s_{e,M \otimes_A \widehat{A}}$ on $R \otimes_A \widehat{A}$ via Γ -construction (cf. Remark 5.4) and then deduce the lower semicontinuity of $s_{e,M}$ over R in a similar fashion as in the proof of Theorem 5.1.

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