Are there still open problems about polynomials? Old and new questions with an elementary flavor

Florian Enescu, Georgia State University

Atlanta, GA

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Let $\mathbb{C}$ the set of complex numbers. Recall that $\mathbb{C}$ can be naturally identified to the real plane $\mathbb{R}^2$.

Definition

Let $f = f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$, where $a_0, \ldots, a_n$ are complex numbers, $a_n \neq 0$. Then $\alpha \in \mathbb{C}$ is called a *root* for f if $f(\alpha) = 0$.

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Proposition

Given a polynomial $f \in \mathbb{C}[X]$ and $\alpha \in \mathbb{C}$, there exists a unique $q(x) \in \mathbb{C}[X]$ such that $f(x) = (X - \alpha)g(X) + f(a)$. In fact $X - \alpha$ divides $f(X)$ if and only if $f(\alpha) = 0$.

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Let $f(x) \in \mathbb{C}[X]$ and $x \in \mathbb{C}$, a root of f. We say that the root $\alpha \in \mathbb{C}$ for f has multiplicity k if $(X - \alpha)^k$ divides f but $(X - \alpha)^{k+1}$ does not. This is equivalent to saying that $f = (X - \alpha)^k g$, $g \in \mathbb{C}[X]$ and $g(x) \neq 0$.

Theorem (Fundamental Theorem of Algebra)

Let $f = f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$, where $a_0, \ldots, a_n$ are complex numbers, $a_n \neq 0$ and $n \geqslant 1$. Then there exists $\alpha \in \mathbb{C}$ root for f.

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Definition

The roots of the derivative of f are called the *critical points* of f.

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A set S in $\mathbb{R}^2$ is convex if for any two points $a, b \in S$ the segment joining a and b is also in S. The convex hull of an arbitrary set A is the smallest convex hull containing A, i.e. the intersection of all convex sets in the plane containing the given set A.

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Theorem (Gauss-Lucas)

The critical points of a polynomial f lie inside the convex hull made by its roots.

Let $f(x) = a \cdot (x - \alpha)^n$, with $n \geqslant 2$. Then
$f^{(i)}(x) = a \cdot n \cdots (n - i + 1)(x - \alpha)^{n-i}$.

Clearly f and $f^{(i)}$ and share a root for all $1 \geqslant i \leqslant n$.

In 2001, E. Casas-Alvero formulated the following conjecture

Conjecture (Casas-Alvero)

For a polynomial f of degree $n \geqslant 2$, if f and $f^{(i)}$ share a root for every $1 \leqslant i \leqslant n$, then $f(X) = a(x - \alpha)^n$, for constants a, α.

One of the basic approaches to this conjecture is baed upon the concept of *resultant*. The resultant of two polynomials is a tool that allows us to check whether two polynomials share a root, without finding the respective root.

Let $f(x) = a_0x^n + \cdots + a_{n-1}x + a_n$, where $a_0 \neq 0$ and
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The resultant is the determinant of the so called the Sylvester matrix of
f, g (size $n + m$ by $n + m$):

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & a_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{m-1} & b_m & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

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Note that n, m are different that in fact one should not infer from the
above presentation that a_n and b_m lie on same column. Also, the first m
rows contain only a's and the remaining n rows contain only b's.

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Among techniques: commutative algebra and homological algebra such as regular sequences, Koszul cohomology, and homology computations.

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In 1998 at an Invited Address in Baltimore at the Joint Meetings of AMS and MAA, Hochster showed the following statement:

Theorem

Let $f, g, h \in R = \mathbb{C}[x, y]$. Then

$$f^2 g^2 h^2 \in (f^3, g^3, h^3).$$

Hochster has asked for an elementary proof of this statement.

Example (Polstra)

We have

$$(x+y)^2(x-y)^2x^2 = \frac{x^2y}{2}(x+y)^3 + (-\frac{x^2y}{2})(x-y)^3 + (x^3 - 2xy^2 - 3y^2)x^3.$$

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Proposition (Polstra)

Let $f, g, h \in R$ and further assume $= x + h_1(y)$, where $h_1 \in \mathbb{C}[y]$. Then

$$f^2g^2h^2 \in (f^3, g^3, h^3).$$

The techniques behind the proof offered by Hochster go beyond a standard course in commutative algebra:
Integral closure of ideals, reductions of ideals and minimal reductions, analytic spread and Briancøn-Skoda Theorem.

The approach offers a generalization as well to *n* variables.

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Conjecture (Sendov)

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geqslant 1$. Assume that the roots are inside the closed unit disk, ie inside

$$D = \{z \in \mathbb{C} : |z| \leqslant 1\}.$$

Then any root of f is within distance less than or equal to one from a critical point of f.

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The conjecture has been proven for $n < 9$ by Brown-Xiang (1999) and for n sufficiently large by Tao (2020).