

Florian Enescu, Fall 2010 Polynomials: Lecture notes Week 9.

1. GAUSS-LUCAS THEOREM; JENSEN DISKS, JENSEN THEOREM

The following theorem by Gauss and rediscovered by Lucas describes a beautiful relationship between the roots of a polynomial and the roots of its derivative.

For a polynomial P , we call the roots of its derivative critical points.

First we need to state a result that is interesting in its own right and will be used in the proof of the Gauss-Lucas theorem. We will refer to a *half-plane* as any of the two regions determined by a line in the plane.

Lemma 1.1. *The sum of a finite number of complex numbers that are situated in a open half-plane is situated in that open half-plane.*

Proof. We can translate the line and the number such that the line that determines the half-plane passes through the origin.

It is enough to show the statement for two complex numbers, since we can then iterate our sum.

We can rotate our line so that the line is the x -axis and the complex numbers are above it. It is clear then that the principal arguments of the two complex numbers are θ_1 and θ_2 , then their sum will have the principal argument between θ_1 and θ_2 , since the sum of two complex number z_1, z_2 is the vertex of the parallelogram with the other vertices equal to $z_1, z_2, 0$.

In conclusion θ will be anywhere between 0 and π and hence in the upper half-plane. □

We can now state the Gauss-Lucas Theorem.

Theorem 1.2. *Let P be a complex polynomial. Then the roots of P' belong to the convex hull determined by the roots of P .*

Proof. Write $P(z) = (z - z_1) \cdots (z - z_n)$, where $n = \deg(P)$.

Then $P'/P = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_n}$.

Suppose that $P'(w) = 0$ and $P(w) \neq 0$ and that w does not belong to the convex hull determined by z_1, \dots, z_n .

We can draw a line through w that does not intersect the convex hull of z_1, \dots, z_n .

Then all numbers $w - z_i$ will all be situated in one of the open half-planes determined by the translation of the line through the origin.

The inverses of $w - z_i$ will have the principal arguments given by subtracting the original principal arguments from 2π respectively. □

We will now move to another statement about the position of the critical points of a polynomial with respect to the roots of the polynomial itself.

We have noted before that for a real polynomial P , if $P(z) = 0$, then $P(\bar{z}) = 0$.

Let us consider z_0 a root of P such that $z_0 \in \mathbf{C} \setminus \mathbf{R}$.

Then \bar{z}_0 is also a root. The circle with center at $Re(z_0)$ and radius $Im(z_0)$ is called a *Jensen disk* of P .

Theorem 1.3 (Jensen). *Let P be a polynomial with real coefficients. Then any non-real critical point of P lie inside or on the boundary of a Jensen disk of P .*

Proof. Let $n = \deg(P)$ and let z_1, \dots, z_n be its complex roots, possible non distinct.

Then as in proof of the Gauss-Lucas Theorem,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - z_i}.$$

Assume that w is non-real critical point of P , so $P'(w) = 0$. If $P(w) = 0$, then $w = z_i$ for some i and w lies on the boundary of the Jensen disk determined by $w = z_i$.

So, let us assume that $P(w) \neq 0$. Let $w = u + iv, v \neq 0$.

For a nonreal root $z_i = a + bi, a, b \in \mathbf{R}$, its conjugate is also a root and equals $a - bi$.

Then we have

$$\frac{1}{w - a - bi} + \frac{1}{w - a + bi} = \frac{2w - 2a}{(w - a)^2 + b^2}.$$

But $\overline{(w - a)^2 + b^2} = (\overline{w} - a)^2 + b^2$.

So,

$$\frac{2w - 2a}{(w - a)^2 + b^2} = \frac{(2w - 2a)(\overline{w} - a)^2 + b^2}{|(w - a)^2 + b^2|^2}.$$

We would like to compute the sign of the imaginary part of this expression. Since the denominator is real and positive then it is enough to look at the sign of the imaginary part of the numerator.

But

$(2w - 2a)(\overline{w} - a)^2 + b^2 = 2(|w - a|^2 (u - iv - a) + b^2(u + iv - a))$, and the imaginary part of this number is

$$2(-v |w - a|^2 + b^2v) = 2v(-|w - a|^2 + b^2).$$

Note that this expression has the sign of $-v$, because w outside the Jensen disk of $z = a + ib$ means exactly that

$$b^2 < |w - a|^2.$$

Let us look at the sign of the imaginary part of the expression

$$\frac{1}{w - z_i},$$

in the case $z_i = a$ real number.

Then

$$\frac{1}{w - a} \cdot \frac{\overline{w} - a}{\overline{w} - a} = \frac{u - a - iv}{|w - a|^2},$$

and the imaginary of this expression is exactly the sign of $-v$.

Now in the sum

$$0 = \frac{P'(w)}{P(w)} = \sum_{i=1}^n \frac{1}{w - z_i},$$

appear two types of roots: complex non-real roots which can be paired together and the sum corresponding to them has the sign of $-v$, and terms corresponding to real roots which have also sign equal to the sign of $-v$. In conclusion, the above sum has sign exactly the sign of $-v$, and we know that $-v \neq 0$, so the sum cannot be zero. Contradiction!

Therefore our w must belong inside or on the boundary of one of the Jensen disks. □

In what follows we show some more applications of Laguerre's theory developed in lecture 6.

Let f be a polynomial of degree n , and ξ a complex number different from ∞ .

The (apolar) derivative of f with respect to ξ is defined as

$$A_\xi f(z) = (\xi - z)f'(z) + nf(z), \text{ for } \xi \neq \infty,$$

or

$$A_\xi f(z) = f'(z), \text{ for } \xi = \infty.$$

One should note that this definition is such that for a polynomial of degree n , the center of mass of z with respect to the roots of f equals ξ if and only if $A_\xi f(z) = 0$.

To facilitate the computations we will write our polynomials in a different form. Namely, every polynomial $f(z) = \sum_{k=0}^n d_k z^k$, can be rewritten as $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$.

Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

So for example $\binom{3}{1} = 3$, $\binom{3}{3} = 1$, and

$$f(z) = z^3 - z = \binom{3}{3} z^3 - \binom{3}{1} \frac{1}{3} z.$$

The benefits of this writing will be clear later.

Until then, let us note that

$$f'(z) = \sum_{k=1}^n \binom{n}{k} k a_k z^{k-1},$$

and so

$$\frac{f'(z)}{n} = \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} a_k z^{k-1} = \sum_{k=1}^n \binom{n-1}{k-1} a_k z^{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{k+1} z^k.$$

With this computation now one can see that

$$\frac{A_\xi f(z)}{n} = (\xi - z) \sum_{k=0}^{n-1} \binom{n-1}{k} a_{k+1} z^k + \sum_{k=0}^n \binom{n}{k} a_k z^k.$$

But,

$$\xi \binom{n-1}{k} a_{k+1} z^k - \binom{n-1}{k-1} a_k z^k + \binom{n}{k} a_k z^k = \binom{n-1}{k} (a_k + a_{k+1} \xi) z^k,$$

since

$$\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}.$$

Therefore

$$\frac{A_\xi f(z)}{n} = \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + a_{k+1} \xi) z^k.$$

This way we obtained a new polynomial of degree $n-1$.

If we repeat our computation and compute

$$\frac{A_\eta(A_\xi)f(z)}{(n-1)n}$$

we will get

$$\sum_{k=0}^{n-2} \binom{n-2}{k} ((a_k + a_{k+1} \xi) + (a_{k+1} + a_{k+2} \xi) \eta) z^k.$$

Note that $a_k + a_{k+1} \xi + (a_{k+1} + a_{k+2} \xi) \eta = a_k + a_{k+1} (\xi + \eta) + a_{k+2} \xi \eta$.

So, if we iterate the procedure we can see that

$$\frac{1}{n!} A_{\xi_1} A_{\xi_2} \dots A_{\xi_n} f(z) = a_0 + a_1 \sigma_1 + \dots + a_n \sigma_n,$$

where

$$\sigma_1 = \xi_1 + \dots + \xi_n$$

$$\sigma_2 = \xi_1 \xi_2 + \xi_1 \xi_3 + \dots + \xi_{n-1} \xi_n$$

...

$$\sigma_n = \xi_1 \dots \xi_n.$$

Let consider another polynomial $g(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k$.

Denote z_1, \dots, z_n the roots of $f(z)$ and ξ_1, \dots, ξ_n be the roots of g .

Therefore Viète relations for the roots of g tell us that

$$\sigma_1 = -\binom{n}{1} \frac{b_{n-1}}{b_n},$$

$$\sigma_2 = \binom{n}{2} \frac{b_{n-2}}{b_n},$$

...

$$\sigma_n = (-1)^n \binom{n}{0} \frac{b_0}{b_n}.$$

Therefore we can notice now that

$$A_{\xi_1} A_{\xi_2} \dots A_{\xi_n} f(z) = 0$$

is in fact equivalent to

$$a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \dots + (-1)^n a_n b_0 = 0.$$

Definition 1.4. Two polynomials $f(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k$ and $g(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k$, are called *apolar* if and only if their coefficients satisfy

$$a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \dots + (-1)^n a_n b_0 = 0.$$

Let us construct an apolar polynomial for $f(z) = 2z^2 + 4z + 1 = 1 + 2\binom{2}{1}z + 2\binom{2}{2}z^2$.

So, $a_0 = 1, a_1 = 2, a_2 = 2$.

Then we are looking for a polynomial g of degree 2 such that

$g(z) = b_0 + 2b_1z + b_2z^2$, and

$$1b_2 - \binom{2}{1}2b_1 + 2b_0 = 0.$$

There are infinitely many possibilities for b_0, b_1, b_2 therefore and we can choose for example

$$b_0 = 1, b_1 = 1, b_2 = 2,$$

or in other words

$$g(z) = 1 + 2z + 4z^2.$$

The following result is a consequence of Laguerre's work and it is what is needed in our applications.

We remind you that a circular domain (open or closed) is the interior or the exterior of a circle or a half-plane.

Theorem 1.5 (J. H. Grace, 1902). *Let f, g be two apolar polynomials. If all the roots of g belong to a circular domain K , then at least one of the roots of f also belongs to K .*

We postpone the proof of this theorem so that we can illustrate how it can be applied.

Let $f(z) = 1 - z + cz^n$, with $c \in \mathbf{C}, n \geq 2$. We will show that f has at least one root inside the disk $|z - 1| \leq 1$.

To this purpose, we look for a polynomial $g(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k$, such that

$$b_n - n\left(-\frac{1}{n}\right)b_{n-1} + cb_0 = 0.$$

Let us take $b_n = 1$, so we need to choose b_{n-1}, b_0 such that $1 + b_{n-1} + cb_0 = 0$. Note that there are NO conditions imposed on b_1, \dots, b_{n-2} .

A possibility would hence be any polynomial with $b_0 = 0, b_{n-1} = -1$.

Let $\xi_k = 1 - \epsilon_k$, where ϵ_k are n th roots of unity, $k = 1, \dots, n$.

Note that ξ_k are on the circle $|z - 1| = 1$.

Construct the polynomial

$$g(z) = \prod_{k=1}^n (z - \xi_k).$$

The sum of the roots of g is n and the product of the roots is 0 (since one of the roots is actually 0).

Hence

$$g(z) = z^n - nz^{n-1} + \dots + 0,$$

which is a polynomial with $b_{n-1} = -1, b_0 = 0$. This means that $g(z)$ is apolar with $f(z)$.

Moreover all its roots belong to the closed circular domain $|z - 1| \leq 1$, and hence by Grace's Theorem we know that at least a root of f must be inside K .

One can agree that this is quite a beautiful application of Laguerre's work.

Our next lecture will deal with a proof of Grace's Theorem.