Florian Enescu, Fall 2010 Polynomials: Lecture notes Week 8, revised version.

1. Stable polynomials. Gauss-Lucas Theorem

We continue by proving in detail the theorem that closed Lecture 7.

Theorem 1.1. *With the notations just introduced, if f has real coefficients then f is stable (i.e., all roots have negative real part) if and only if f and g have positive coefficients.*

Proof. Suppose that f is stable. Then it can easily be shown that f, g have real coefficients:

Since $f(z) = (z - z_1) \cdots (z - z_n)$, then if z_i is real and negative that the factor $z - z_i$ has positive coefficients. If z_i is complex, not real, then its conjugate $\overline{z_i}$ is a root as well (check this), so f has $(z - z_i)(z - \overline{z_i})$ as a factor. But $(z - z_i)(z - \overline{z_i}) = z^2 - 2Re(z_i) + \mid z_i \mid^2$ has only positive coefficients, as $Re(z_i) < 0$.

In conclusion, f is the product of polynomials with positive coefficients, so it has positive coefficients as well.

We can repeat the argument for g, since g is also a stable polynomial as its roots are sums of roots of f, which means that they will have negative real parts as well. We need to make sure that g has real coefficients as well. For every root of f, its conjugate is also a root. Hence for every root of g, say $z_i + z_j$, its conjugate $\overline{z_i + z_j} = \overline{z_i} + \overline{z_j}$ is also a root of g. Hence we can pair up a complex nonreal root w of g with its conjugate and note that g must be a product of terms of the form $(z - w)(z - \overline{w}) = z^2 - 2Re(w)z + \mid w \mid^2$, which have real coefficients, and terms of the form $z - \alpha$, α real, which also have real positive coefficients.

For the converse, if a polynomial has positive coefficients then it is clear that its real roots must be negative. This shows that f has negative real roots. For a complex root of f, $z = a + ib$, we see that $\overline{z} = a - ib$ is a real root as well, so $z + \overline{z} = 2a$ must be a REAL root of g. But g has positive coefficients so $2a$ must be negative, so a is negative, hence the real part of z is negative. This shows that f is stable.

□

Example 1.2. Let $f = z^2 + z + 2$. Let us compute g. It has degree 1 and root $z_1 + z_2$ where z_1, z_2 are roots of f.

Note that Viète's relations tell us that $z_1 + z_2 = -1$, so $g(z) = z - (-1) = z + 1$.

As we can see the above Theorem applies and f is stable.

In fact, one should notice that the coefficients of g are symmetric polynomials in $z_1, ..., z_n$. Therefore the coefficients of g become polynomials in the coefficients of f, after using the Viète's relations.

Now, let us go back to the equation

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

where P, Q, R are polynomials.

The following result was stated in lecture 7 without a proof, so we will provide a proof now.

But first, let us revisit the concept of multiplicity of a root for a polynomial.

Proposition 1.3. *Let $f(z)$ be a polynomial. Then z_0 is a root of multiplicity k if and only if $f^{(i)}(z_0) = 0$ for $i \leq k - 1$ and $f^{(k)}(z_0) \neq 0$, where $f^{(i)}(z)$ stands for the kth order derivative of f.*

Proof. Let $g(z) = f(z+z_0)$. Note that $g(0) = 0$.

Also, $g^{(i)}(z) = f^{(i)}(z+z_0)$.

Moreover $f(z) = (z-z_0)^k h(z)$ is equivalent to $g(z) = z^k h(z+z_0)$ and of course $h(z_0) \neq 0$ is equivalent to $h(0+z_0) \neq 0$. This says that 0 is a root of multiplicity k for g if and only if z_0 is root of multiplicity k for f.

First let us assume that z_0 is root of multiplicity k for f. Hence as we have see above, 0 is root of multiplicity k for g and $g(z) = z^k p(z)$ where p is such that $p(0) \neq 0$.

So, $g(z) = az^k + \ldots, a \neq 0$ and it can be easily checked that $g^{(i)}(0) = 0$ for $i \leq k$, and $g^{(k+1)}(0) \neq 0$. As remarked before, this is equivalent to $f^{(i)}(z_0) = 0$ for $i \leq k-1$ and $f^{(k)}(z_0) \neq 0$.

Now, let us assume that $f^{(i)}(z_0) = 0$ for $i \leq k-1$ and $f^{(k)}(z_0) \neq 0$, that is $g^{(i)}(0) = 0$ for $i \leq k-1$, and $g^{(k)}(0) \neq 0$.

Let $g(z) = a_0 + a_1 z + \ldots$.

But $g(0) = 0$ implies $a_0 = 0$. $g'(0) = a_1$ so this means that $a_1 = 0$. Similarly, $g''(0) = 2a_2$ and hence $a_2 = 0$.

Note that $g^{(k)}(0) = k! a_k$, so $a_k \neq 0$.

So, we can write $g(z) = a_k z^k + \ldots = z^k p(z)$, where p is a polynomial such that $p(0) \neq 0$. Hence g has 0 as a root of multiplicity k, and therefore z_0 is root of multiplicity k for f.

□

Proposition 1.4. *The polynomial solutions of*

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

have only simple zeroes.

Proof. Assume that z_0 is a multiple zeroes for y. Let us say that its multiplicity is $k > 1$.

Case 1: $P(z_0) \neq 0$.

Then by taking the derivative of

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

$k-2$ times we get $P(z)y^{(k)}(z) + F(z) = 0$ where F is an expression in the derivatives of y of order less or equal to $k-1$. (If $k = 1$, there not need to take derivatives). When we plug in $z = z_0$ we get $P(z_0)y^k(z_0) = 0$ so $y_k(z_0) = 0$ which contradicts the fact z_0 has multiplicity exactly k.

Case 2 : $P(z_0) = 0$.

Take the derivative of

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

and get

$$P'(z)y'' + Py'''(z) + Q(z)y'' + Q'(z)y' + R'(z)y + R(z)y' = 0.$$

Now, remark that $R' \equiv 0$ (since R is a constant) and P has only simple roots, for P, R polynomials defining the Hermite, Laguerre, Legendre polynomials.

We either have $P'(z_0) + Q(z_0) \neq 0$, or $Q'(z_0) + R' \neq 0$. As in case 1, take the $k-2$ or $k-1$ derivatives of the newly found expression and note that one gets

$$H(z)y^{(k)}(z) + F = 0$$

where F is an expression depending upon P and the derivatives of y of order less or equal to $k-1$ such that $y^{(i)} = 0, \forall i \leq k-1$ implies that $F = 0$. Here $H(z_0) \neq 0$. (Again, as before, there is no need to take derivatives if $k = 1$.)

When we plug in $z = z_0$ we see that $y^{(i)}(z_0) = 0$, for all $i \leq k-1$, so we get $H(z_0)y^{(k)}(z_0) = 0$, as $F(z_0)$ vanishes. So, $y^{(k)}(z_0) = 0$, contradicting again the fact that the multiplicity of y is k.

Hence $k = 1$ is the only possibility so z_0 is a simple root for y.

□