Florian Enescu, Fall 2010 Polynomials: Lecture notes Week 7.

1. Complex roots.

Theorem 1.1. *Let f be a polynomial with real coefficients and define*

$$\xi(z) = z - n\frac{f(z)}{f'(z)}.$$

Then all the roots of f are real if and only if $Im(z) \cdot Im(\xi(z)) < 0$ for any $z \in \mathbf{C} \setminus \mathbf{R}$.

Proof. First, let us assume that all zeroes of f are real. Let z be a complex number such that $Im(z) = a > 0$. We will show that $Im(\xi(z)) < 0$. Let $0 < \epsilon < a$.

The half-plane $\Gamma = \{z : Im(z) < \epsilon\}$ contains all the zeroes of f and does not contain z. By Theorem 1.6 from Lecture 6 the curvilinear polygon associated to the roots with respect to z must be in Γ. Since this polygon separates z from $\xi(z)$, and z is not in the polygon (as it is not in Γ), then $\xi(z) \in \Gamma$ and so $Im(\xi(z)) < \epsilon$. By letting ϵ approach zero we get $Im(\xi(z)) \leq 0$.

But $Im(\xi(z)) = 0$ is impossible, since if $\xi(z)$ lies on the real axis then we can find a circle that contains z and $\xi(z)$ in its interior while the roots of f are outside. This contradicts Theorem 1.6 from Lecture 6.

The case $Im(z) < 0$ is similar.

Now let us assume that $Im(z) \cdot Im(\xi(z)) < 0$ for any $z \in \mathbf{C} \setminus \mathbf{R}$. Let z not real approaching z_1, where z_1 is a root of f. Then $Im(z)$ approaches $Im(z_1)$ while $\xi(z) \to z_1$, and so $Im(\xi(z)) \to Im(z_1)$.

Hence $Im(z) \cdot Im(\xi(z)) < 0$ produces $Im(z_1)^2 \leq 0$ and so z_1 must have imaginary part equal to zero, hence it is real. The same can applied to all roots of f, so f has real roots. □

Example 1.2. Let $f(z) = z^4 - 3z + 2$.

The center of mass of the roots of f with respect to $z = 2$ is

$\xi(2) = 2 - 4\frac{f(2)}{f'(2)} = 2 - 4\frac{12}{29} = \frac{10}{29}$.

Let us compute the center of mass with respect to $z_1 = 1$ of the other roots of f.

Note that $f(1) = 0$ and $f'(1) = 1 \neq 0$ so 1 is a simple root.

Hence the center of mass $X(1) = 1 - 2 \cdot 3\frac{f'(1)}{f''(1)} = 1 - 6\frac{1}{12} = .5$

Theorem 1.7 from Lecture 7 can be applied also to this example. If a circle passes through 1 and does not contain .5 in its interior then at least one other root for f must be outside of the interior of the circle.

Indeed, if all other roots are inside the circle, then Theorem 1.7 says that $X(1) = .5$ must be inside the circle as well, which is a contradiction.

We would like to show how Laguerre's work can be applied to an important chapter in the theory of polynomials, the subject of classical orthogonal polynomials. These polynomials are important in applied mathematics and they arise as solutions of differential equations of order two.

A polynomial $y = f(z)$ is called a polynomial solution of the differential equation

$$P(z)y'' + Q(z)y' + R(z)y = 0$$

if and only if

$$P(z)f''(z) + Q(z)f'(z) + R(z)f(z) = 0.$$

Lemma 1.3. *Let $f(z)$ be a polynomial solution for*

$$P(z)y'' + Q(z)y' + R(z)y = 0.$$

Then for each z_0 such that $f(z_0) = 0$ we have

$$P(z_0)f''(z_0) + Q(z_0)f'(z_0) = 0.$$

Proof. Since

$$P(z)f''(z) + Q(z)f'(z) + R(z)f(z) = 0,$$

we get the stament of the Lemma by simplying letting $z = z_0$.

□

Corollary 1.4. *Let $f(z)$ be a polynomial of degree n and define the function:*

$$X(z) = z + 2(n-1)\frac{P(z)}{Q(z)}.$$

Let γ be a circle/line that passes through a simple root z_1 of f such that all other roots belong to one of the domains determined by γ. Then $X(z_1)$ belongs to the same domain (closed or open, that is, with or without its border), as long as $Q(z_1) \neq 0$.

Proof. This is an immediate consequence of Theorem 1.7 from Lecture 6 and the above Lemma, since one can note that

$$z_1 + 2(n-1)\frac{P(z_1)}{Q(z_1)} = z_1 - 2(n-1)\frac{f'(z_1)}{f''(z_1)}$$

for a zero z_1 of f.

Moreover, z_1 is a simple root so $f'(z_1) \neq 0$. Now, $Q(z_1) \neq 0$ together with the above Lemma guarantees that $f''(z_1) \neq 0$ so Theorem 1.6 (Lecture 6) can indeed be applied.

□

We state now a Proposition that will be proven a bit later.

Proposition 1.5. *The polynomial solutions of*

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

have only simple zeroes.

Now let us go back to the subject of classical orthogonal polynomials. We will study a number of classical orthogonal polynomials due to Hermite, Laguerre and Legendre respectively, who have applications in applied mathematics, mathematical physics, theory of random matrices, etc..

For each $n \in \mathbf{N}$, one gets a Hermite Polynomial H_n as a solution of

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

where $P(z) = 1, Q(z) = -2z, R(z) = 2n$.

For each $n \in \mathbf{N}$, one gets a Laguerre Polynomial L_n as a solution of

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

where $P(z) = z, Q(z) = -1 + \alpha - z, R(z) = n$ and $\alpha > -1$ a fixed real number.

For each $n \in \mathbf{N}$, one gets a Legendre Polynomial P_n as a solution of

$$P(z)y'' + Q(z)y' + R(z)y = 0,$$

where $P(z) = 1 - z^2, Q(z) = -2z, R(z) = n(n+1)$.

It is known that H_n, L_n, P_n have degree n (we won't prove this here).

The following is an application of Laguerre's work presented in Lecture 6.

Theorem 1.6. *The Hermite, Laguerre, Legendre polynomials have simple real roots.*

Proof. We already know that the roots are simple.

We will show the proof for the case of Hermite Polynomials H_n.

If $n = 1$, then since H_1 has degree 1, then $H(z) = az + b$, and so $H_1' = a, H_1'' = 0$. By plugging these values in the defining differential equation we get

$$(-2z)a + 2(az + b) = 0$$

and so $b = 0$.

Hence $H_1(z) = az$, and obviously its only root is 0.

Now, assume $n > 1$.

We will prove the statement by contradiction.

Assume that $z_1 = a + ib$ is an complex root for H_n with nonzero imaginary part. We can also assume that z_1 is chosen such that among all roots for H_n it has the largest imaginary part b.

Assume $b > 0$

With this choice, the closed half-plane $\Gamma = \{z \in \mathbf{C} : Im(z) \leq b\}$ contains all other roots of H_n and the Corollary of Lecture 7 shows that $X(z_1)$ must belong to Γ as well.

But $X(z_1) = z_1 + 2(n-1)\frac{P(z_1)}{Q(z_1)} = a + ib + 2(n-1)\frac{1}{-2(a+ib)} = a + ib + (n-1)\frac{1}{a+ib}$.

A quick computation shows that the imaginary part of $X(z_1)$ is

$$b + (n-1)\frac{b}{a^2 + b^2}$$

which is clearly greater than b (if $b > 0$) so $X(z_1)$ cannot belong to Γ, a contradiction. So, $b \leq 0$.

This means that all roots have imaginary parts less or equal to zero.

Choose now a complex root with least nonzero imaginary part, say $z_1 = a + ib$. Repeat the argument with $\Gamma = \{z \in \mathbf{C} : Im(z) \geq b\}$. As above, we obtain a contradiction, since $b < 0$.

This proves that H_n has only real roots.

$\square$

There are situations in mathemarics where it is important to establish whether a certain polynomial has all the roots with real part negative. This is referred to the Routh-Hurwitz problem. We will present a result that can be used to solve this problem.

First of if f is a polynomial with complex coefficients, then let $f(z) = (z - z_1) \cdot (z - z_n)$, where $z_1, ..., z_n$ are all the roots of f.

Then $\overline{f(\overline{z})} = (z - \overline{z_1}) \cdots (z - \overline{z_n})$.

This shows that

$$F(z) = f(z) \cdot \overline{f(\overline{z})} = (z - z_1)(z - \overline{z_1}) \cdots (z - z_n)(z - \overline{z_n}) = (z^2 - 2Re(z_1)z + \mid z_1 \mid^2) \cdots (z^2 - 2Re(z_n)z + \mid z_n \mid^2)$$

is a polynomial with real cofficients.

The roots of F are roots $z_1, \ldots z_n, \overline{z_1}, \ldots, \overline{z_n}$, and these have the real parts as $z_1, \ldots z_n$.

Hence, if we want o find out whether f has roots with only negative real parts it is enough to consider this question for the polynomial F. In conclusion, this problem can be reduced to polynomials with real cofficients.

Now let $f(z)$ be a polynomial with real coefficients. Let g be the polynomial that has roots all $z_i + z_j$ for all subsets $\{i, j\}$ of $\{1, \ldots, n\}$, where $z_1, ..., z_n$ stand for the roots of f. The polynomial g has degree $\binom{n}{2}$ that is $n(n-1)/2$.

One should note that the coefficients of g are symmetric in the roots $z_1, ..., z_n$.

Theorem 1.7. *With the notations just introduced, if f has real coefficients then f is stable (i.e., all roots have negative real part) if and only if f and g have positive coefficients.*

Proof. Suppose that f is stable. Then it can easily be shown that f, g have real coefficients.

□