Florian Enescu, Polynomials Fall 2010: Lecture notes Week 6.

1. Complex roots.

Let $z_1, \ldots, z_n$ be complex numbers. The center of mass of these points (with respect to ∞) is by definition

$$\xi = \frac{z_1 + \cdots + z_n}{n}.$$

Example 1.1. Let $z_1 = 1 + i, z_2 = 2 + 6i, z_3 = -i$. Then $\xi = \frac{3+6i}{3} = 1 + 2i$.

Note that whenever $a \leq Re(z_i) \leq b$, one has $a \leq Re(z) \leq b$.

Now we would like to define the center of mass of a set of points $z_1, ..., z_n$ with respect to another point z.

For this we need to discuss the notion of linear fractional transformation.

A function $w = T(z) = \frac{az+b}{cz+d}$ such that $ad - bc \neq 0$ is called a linear fractional transformation or Möbius transformation.

Such a funtion is the composition of the function $1/z$ and suitable linear functions. One can see that $T(\infty) = \frac{a}{c}$, while $T(-\frac{d}{c}) = \infty$ if $c \neq 0$. Also, the inverse function T^{-1} is also linear fractional transformation.

The importance of such transformations comes from the fact that they transform circles and lines into circles and lines.

As an exercise, show that that the image of $x = 2$ under $T(z) = \frac{1}{z}$ is a circle.

Also, note that a linear fractional transformation with $c \neq 0$ can be written as $T(z) = \frac{a}{z-z_0} + b$, $r, s \in \mathbf{C}$ for suitable r, s.

Example 1.2. Let $T(z) = \frac{6z+1}{3z+2}$. Then $T(z) = \frac{6z+4-3}{3z+2} = 2 - \frac{3}{3z+2} = \frac{-1}{z+\frac{2}{3}} + 2$, so we can take $a = -1, b = 2, z_0 = -\frac{2}{3}$.

Given the point $z_1, ..., z_n$, then let C the smallest convex polygon that containing then. This poligon is obtained by joining any two points z_i, z_j and considering the half-planes separated by this line; then C is the intersection of all half-planes obtained in this manner that contain the points $z_1, ..., z_n$. the center of mass ξ will belong to C.

To define the center of mass of $z_1, ..., z_n$ with respect to a point z_0 called pole, we use a linear fractional transformation T that maps z_0 to ∞, compute the images $z'_1, ..., z'_n$ of $z_1, ..., z_n$ under the transformation and denote ξ' to be the center of mass of $z'_1, ..., z'_n$. Then we let the center of mass of $z_1, ..., z_n$ with respect to z_0 to equal the image of ξ' under the inverse function T^{-1}.

Let $T(z) = \frac{a}{z-z_0} + b$. Then $z'_i = T(z_i) = \frac{a}{z_i - z_0} + b$, for all $i = 1, ..., n$.

hence

$$\xi' = \frac{\sum_{i=1}^n z'_i}{n} = \frac{1}{n}\sum_{j=1}^n (\frac{a}{z_i - z_0} + b) = b + \frac{a}{n}\sum_{i=1}^n \frac{1}{z_i - z_0}.$$

But $\xi' = T(\xi) = \frac{a}{\xi - z_0} + b$, so $\frac{1}{\xi - z_0} = \frac{1}{n}\sum_{i=1}^n \frac{1}{z - z_0}$, which gives

$$\xi = z_0 + n\frac{1}{\sum_{i=1}^n \frac{1}{z_i - z_0}}.$$

The formula produced here does show that everything is independent of a and b.

One should note that the linear transformation T^{-1} sends the lines $z_i'z_j'$ to circles that pass through z_0 and z_i, z_j.

A set $S \subset \mathbf{C}$ is called *convex* if for any two points $a, b \in S$, the segment that joins a with b is a subset of S.

For a give set S, the convex hull of S is the smallest convex set containing S. It is obtained by taking the intersection of all convex subsets of $\mathbf{C}$ that contain S. The important part to remember here is that an intersection of convex sets is still convex.

The smallest convex polygon C' that contains $z_1', ..., z_n'$ (basically the convex hull of $z_1', ..., z_n'$) will map into a curvilinear polygon C that will contain $z_1, .., z_n$.

Theorem 1.3. *Let C be curvilinear polygon defined above with respect to $z_1, ..., z_n$ and z_0. Then C separates z_0 from the center of mass ξ.*

Theorem 1.4 (Laguerre)**.** *Let $f(x)$ be a polynomial with complex coefficients, and let $z_1, ..., z_n$ be its roots (counted with multiplicities). Then the center of mass of $z_1, ..., z_n$ with respect to an arbitrary point z_0 is given by*

$$\xi = z_0 - n\frac{f(z_0)}{f'(z_0)},$$

whenever $f'(z_0) \neq 0,\ f(z_0) \neq 0$.

Proof. Let $f(z) = \prod_{i=1}^{n}(z - z_i)$.

Indeed, $f'(z) = \sum_{i=1}^{n} \prod_{j \neq i}(z - z_j)$.

So, $\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j}$.

Now one can easily check that

$$\xi = z_0 + n\frac{1}{\sum_{i=1}^{n} \frac{1}{z_i - z_0}} = z_0 - n\frac{f(z_0)}{f'(z_0)}.$$

□

Theorem 1.5. *Let $f(z)$ be a polynomial of degree n with complex coefficients, and z_1 be a simple root of f, such that $f"(z_1) \neq 0$. Then the center of mass of the remaining zeroes of $f(z)$ with respect to z_1 is*

$$X(z_1) = z_1 - 2(n-1)\frac{f'(z_1)}{f"(z_1)}.$$

Proof. Write $f(z) = (z - z_1)F(z)$. Since z_1 is a simple root for $f(z)$, then $F(z_1) \neq 0$.

Also, $f'(z) = F(z) + (z - z_1)F'(z)$, so $f'(z_1) = F(z_1)$. Moreover, $f"(z) = F'(z) + F'(z) + (z - z_1)F"(z)$, hence $f"(z_1) = 2F'(z_1)$. According to Theorem 1.4 applied to $F(z)$ and z_1, we get that

$$X(z_1) = z_1 - (n-1)\frac{F(z_1)}{F'(z_1)} = z_1 - 2(n-1)\frac{f'(z_1)}{f"(z_1)}.$$

□

Theorem 1.6. *If $z_1, ..., z_n$ are zeroes of a polynomial f and they belong to any circular/linear domain D (i.e either all are outside or inside a circle/line γ) and if z_0 is outside D, then the curvilinear polygon C associated to $z_1, \ldots, z_n$ is in D. Note that that D can either be open or closed.*

Proof. Map z_0 to ∞ and z_i to z'_i $i = 1, \ldots, n$ via a linear fractional transformation T. Then $C = T^{-1}(C')$ where C' is the smallest convex polygon that contains $z'_1, \ldots, z'_n$. Let $D' = T(D)$. Let γ' be the image of γ under T: it is a circle/line and all the $z'_1, \ldots, z'_n$ live inside D'. Moreover since $z_0 \notin D$ and z_0 is mapped under T to ∞, then D' is either the interior of the circle γ' or one of the half-planes determined by γ' in case γ' is a line. Now, D' will contain C' since C' is the smallest convex set that contains all the points $z'_1, \ldots, z'_n$ and D' is convex and contains $z'_1, ..., z'_n$. Since $C' \subset D'$, then $C \subset D$.

□

A direct application to the above theorem is the following result

Theorem 1.7. *Let $f(z)$ be a polynomial of degree n and define the function:*

$$X(z) = z - 2(n-1)\frac{f'(z)}{f"(z)}.$$

Let γ be a circle/line that passes through a simple root z_1 of f such that all other roots belong to one of the domains determined by γ. Then $X(z_1)$ belongs to the same domain (closed or open, that is, with or without its border), as long as $f"(z_1) \neq 0$.

Proof. This is an immediate consequence of Theorems 1.5, 1.6, since $X(z_1)$ is the center of mass of the other roots with respect to z_1 and the center of mass does not stay in the same domain as z_1 by Theorem 1.6.

□

Theorem 1.8. *Let $f(z)$ be a polynomial of degree n and define the function:*

$$X(z) = z - 2(n-1)\frac{f'(z)}{f"(z)}.$$

If x is a simple root of f with maximal absolute value then

$$\mid X(x) \mid \leq \mid x \mid .$$

Proof. All the roots lie in the circular domain $\{z \in \mathbf{C} :\mid z \mid \leq \mid x \mid\}$, hence $X(x)$ also belongs to this disk. □

Theorem 1.9 (Laguerre's Criteria)**.** *Let f be a polynomial with real coefficients and define*

$$\xi(z) = z - n\frac{f(z)}{f'(z)}.$$

Then all the roots of f are real if and only if $Im(z) \cdot Im(\xi(z)) < 0$ for any $z \in \mathbf{C} \setminus \mathbf{R}$.