Florian Enescu, Polynomials: Lecture notes Week 5.

1. Estimates for roots of polynomials, continuation.

Theorem 1.1. *a)[Eneström-Kakeya]*

If all the coefficients of the polynomial $g(x) = a_0x^{n-1} + \cdots + a_{n-1}$ are positive then for any roots ξ of g we have

$$\min\{\frac{a_i}{a_{i-1}} : i = 1, ..., n-1\} \leq \mid \xi \mid \leq \gamma = \max\{\frac{a_i}{a_{i-1}} : i = 1, ..., n-1\}.$$

b) [Ostrovsky]

Let $\frac{a_k}{a_{k-1}} < \gamma$ for all $k_1, ..., k_m$ with notations exactly as in the preceding proof. If the greatest divisor of the numbers $n, k_1, \ldots, k_m$ is equal to 1, then $\mid \xi \mid < \gamma$.

Proof. Let $f(x) = (x - \gamma)g(x)$.

First note that $f(\gamma) = 0$, so γ is a positive root for f.

Expand $(x-\gamma)g(x) = a_0x^n + (a_1 - \gamma a_0)x^{n-1} + (a_2 - \gamma a_1)x^{n-2} + \cdots + (a_{n-1} - \gamma a_{n-2})x + \gamma a_{n-1}$. So,

$$f(x) = (a_0)(x^n + \frac{a_1 - \gamma a_0}{a_0}x^{n-1} + \frac{a_2 - \gamma a_1}{a_0}x^{n-2} + \cdots + \frac{a_{n-1} - \gamma a_{n-2}}{a_0}x + \frac{\gamma a_{n-1}}{a_0}).$$

We will apply Cauchy's Theorem to

$$x^n + \frac{a_1 - \gamma a_0}{a_0}x^{n-1} + \frac{a_2 - \gamma a_1}{a_0}x^{n-2} + \cdots + \frac{a_{n-1} - \gamma a_{n-2}}{a_0}x + \frac{\gamma a_{n-1}}{a_0}.$$

which has γ as positive root since $f(\gamma) = 0$.

This polynomial can be rewritten as

$$x^n - (-\frac{a_1 - \gamma a_0}{a_0})x^{n-1} - (-\frac{a_2 - \gamma a_1}{a_0})x^{n-2} - \cdots - (-\frac{a_{n-1} - \gamma a_{n-2}}{a_0})x - (-\frac{\gamma a_{n-1}}{a_0}).$$

The coefficientsof the above polynomial are of the form

$$-\frac{a_i - \gamma a_{i-1}}{a_0} = \frac{-a_i + \gamma a_{i-1}}{a_0} \geq 0,$$

since a_0 is positive and due to the special way γ was chosen.

Applying Cauchy's theorem to this polynomial we get that all roots of it will have absolute value less than γ.

However note that a root for this polynomial other than γ is a root for f other than γ. But, the roots of f different than γ are the roots of g.

Hence we obtain the first half of a). It is clear that b) is obtained from Ostrovsky's theorem in similar fashion. Let us comment on the condition in part b) that $[k_1, \ldots, k_m, n] = 1$.

The polynomial for which we apply Ostrovky's Theorem is:

$$h = x^n - (-\frac{a_1 - \gamma a_0}{a_0})x^{n-1} - (-\frac{a_2 - \gamma a_1}{a_0})x^{n-2} - \cdots - (-\frac{a_{n-1} - \gamma a_{n-2}}{a_0})x - (-\frac{\gamma a_{n-1}}{a_0}).$$

To match the notations in Ostrovky's Theorem we let $b_i = -\frac{a_i - \gamma a_{i-1}}{a_{i-1}}, i = 1, .., n$. Moreover $b_n = -\frac{\gamma a_{n-1}}{a_0}$. What b_i are nonzero? First of all $b_n \neq 0$. Then $b_i \neq 0$ if and only if $\gamma \neq \frac{a_i}{a_{i-1}}$, that is $\gamma < \frac{a_i}{a_{i-1}}$. Hence the indices $k_1, ..., k_m$ are exactly those indices i such that $\gamma < \frac{a_i}{a_{i-1}}$.

For the remaining part of a):

It remains to show that if ξ is a root for g then

$$\min\{\frac{a_i}{a_{i-1}} : i = 1, ..., n-1\} \leq \mid \xi \mid .$$

Note that $g(1/z) = \frac{a_{n-1}z^n + \cdots + a_0}{z^{n-1}}$, which means that any nonzero u root for $h(x) = a_{n-1}x^n + \cdots + a_0$ gives $g(1/u) = 0$. In other words, if ξ is a root for g, then $\eta = \frac{1}{\xi}$ is a root for h.

Write $h(x) = a_{n-1}x^n + \cdots + a_0 = b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1}$, that is, we denote $b_i = a_{n-i-1}$.

Applying the part of a) that we already proved we get

$$\mid \eta \mid \leq \max\{\frac{b_i}{b_{i-1}} : i = 1, \ldots, n-1\},$$

for any η root for f.

But $\eta = \frac{1}{\xi}$ with $\xi = \frac{1}{\eta}$ root for g and

$$\max\{\frac{b_i}{b_{i-1}} : i = 1, \ldots, n-1\} = \max\{\frac{a_{n-i-1}}{a_{n-i}} : i = 1, \ldots, n-1\} = \frac{1}{\min\{\frac{a_{n-i}}{a_{n-i-1}} : i = 1, \ldots, n-1\}}.$$

But

$$\frac{1}{\min\{\frac{a_{n-i}}{a_{n-i-1}} : i = 1, \ldots, n-1\}} = \frac{1}{\min\{\frac{a_i}{a_{i-1}} : i = 1, \ldots, n-1\}}.$$

So, we got

$$\frac{1}{\mid \xi \mid} \leq \frac{1}{\min\{\frac{a_i}{a_{i-1}} : i = 1, \ldots, n-1\}},$$

which proves that

$$\min\{\frac{a_i}{a_{i-1}} : i = 1, \ldots, n-1\} \leq \mid \xi \mid .$$

□

In what follows we will present a proof of the Fundamental theorem of Algebra following Rudin and Terkelsen.

Theorem 1.2. *Every polynomial with complex coefficients and positive degree has a complex root.*

Proof. Let $P(z)$ be a polynomial with complex coefficients.

As $z \to \infty$, we have $\mid P(z) \mid \to \infty$. So, for any $N > 0$, there exists $M > 0$ such that $\mid z \mid > M$ implies $\mid P(z) \mid > N$. Fix an arbitray $N > 0$ such that $N > \mid P(0) \mid$. Then whenever z has the property $\mid z \mid \leq M$, then $\mid P(z) \mid \leq N$.

Let $D = \{z \in \mathbf{C} : \mid z \mid \leq M\}$. D is a closed disk of radius M. The function $\mid P(z) \mid$ is continuous so it has a minimum on D. That is, there exists $z_0 \in D$ such that $\mid P(z_0) \mid \leq \mid P(z) \mid$ for all $z \in D$. But 0 is in D so $\mid P(z_0) \mid \leq \mid P(0) \mid < N$. For $z \notin D$ we know that $\mid P(z) \mid > N > \mid P(z_0) \mid$.

In conclusion, $\mid P(z) \mid \geq \mid P(z_0) \mid$, for all $z \in \mathbf{C}$.

We will show that $P(z_0) = 0$.

First of all, denote $P(z + z_0) = T(z)$. We see that $\mid T(z) \mid \geq \mid T(0) \mid$ for all complex numbers z.

Clearly T is a polynomial as well. It can be written as $T(z) = a + bz^n + z^{n+1}Q(z)$, such that $b \neq 0$ and $Q(z)$ is a polynomial.

Suppose that $T(0) = a \neq 0$. We will derive a contradiction.

Chose an nth root w of $-a/b$, that is $w^n = -a/b$. There exist t with $0 < t < 1$ such that $t \cdot \mid w^{n+1}Q(tw) \mid < \mid a \mid$.

But

$T(tw) = a + b(tw^n) + (tw)^{n+1}Q(tw) = (1 - t^n)a + (tw)^{n+1}Q(tw)$, since $w^n = -a/b$.

Hence

$$T(tw) \leq (1 - t^n) \mid a \mid + t^{n+1} \mid w^{n+1}Q(tw) \mid < (1 - t^n) \mid a \mid + t^n \mid a \mid = \mid a \mid = T(0).$$

This contradicts the fact that T has a minimum at 0.

So, $T(0) = 0$ and hence $P(z_0) = 0$.

$\square$