Florian Enescu, Polynomials, Fall 2010: Lecture notes Week 3, revised.

1. Solving the degree three and four polynomial equations.

We have closed our previous lecture with

Theorem 1.1 (Fundamental theorem of algebra). *Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a complex polynomial. Then inside the circle $\mid z \mid= 1 + \max_i(a_i)$, there are exactly n roots of f, multiplicities counted.*

This theorem can be applied to all polynomials: given $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$, with $a_n \neq 0$ and $a_0, \ldots, a_n$ complex numbers, we see that

$$f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 = a_n(X^n + \frac{a_{n-1}}{a_n}X^{n-1} + \ldots + \frac{a_1}{a_n}X + \frac{a_0}{a_n}),$$

so $f(z) = 0$ if and only if $z^n + \frac{a_{n-1}}{a_n}z^{n-1} + \ldots + \frac{a_1}{a_n}z + \frac{a_0}{a_n} = 0$.

So by applying Theorem 1.1 to $X^n + \frac{a_{n-1}}{a_n}X^{n-1} + \ldots + \frac{a_1}{a_n}X + \frac{a_0}{a_n}$, we get

Theorem 1.2. *Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a complex polynomial. Then f has n roots counted with multiplicities inside the circle $\mid z \mid= 1 + \max_i(\frac{a_i}{a_n})$.*

Another proof, algebraic in nature, to this theorem will be given later.

For now, let us examine how one can solve polynomial equations of degree 3 and 4 in one variable. These equations have an interesting history and their solutions have been found in the first part of the sixteenth century. The equation of degree three has first solved by del Ferro; his solution has been rediscovered by Tartiglia. Cardano published a book describing it, and his student, Ferrari, is the one who obtained the solution for the polynomial of degree four. One should keep in mind that their work faced difficulties such as the lack of understanding of the complex numbers. These numbers have been understood by the mathematicians only later.

The equation of degree three:

Consider $x^3 + a_1x^2 + a_2x + a_3 = 0$ where a_1, a_2, a_3 are complex numbers.

First we perform the substitution $y = x+\frac{a_1}{3}$. It can be easily seen that if one plugs in $x = y-\frac{a_1}{3}$ into the above equation then it transforms into

$$y^3 + py + q = 0,$$

with p, q complex numbers that can be expressed in terms of a_1, a_2, a_3.

By Theorem 1.1 we expect three solutions for our equation, counting possible multiplicities.

To find y, the solution to our equation, we will look for α, β such as $\alpha+\beta = y$ and $\alpha\beta = -p/3$.

So, $(\alpha + \beta)^3 + p(\alpha + \beta) + q = 0$, which gives us after expanding

$$\alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) + p(\alpha + \beta) + q = 0.$$

But $3\alpha\beta + p = 0$, so it follows that

$$\alpha^3 + \beta^3 = -q$$

.

Also, $\alpha\beta = -p/3$ implies that
$$\alpha^3\beta^3 = -p^3/27.$$
In conclusion α^3, β^3 are the roots of the equation
$$x^2 + qx - \frac{p^3}{27} = 0,$$
which allows us to find α^3, β^3 through the formula
$$\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}.$$
Since $y = \alpha + \beta$ and for either α or β there are three possible choices, then we must carefully choose them. One needs to remember that $\alpha \cdot \beta = -\frac{p}{3}$.

Let α be a fixed cube root of $\frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}$.

Let $\epsilon = e^{i\frac{2\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$. We have $\epsilon^3 = 1$ and the other two 3rd roots of unity are $1, \epsilon^2$.

So, $\alpha, \alpha_2 := \epsilon \cdot \alpha, \alpha_3 := \epsilon^2 \cdot \alpha$ are all the roots of
$$\alpha^3 = \frac{-q + \sqrt{q^2 + \frac{4p^3}{27}}}{2}.$$
To find the corresponding β's we will take $\beta = \frac{-p}{3\alpha}$, and then one can check that $\alpha_2\epsilon^2\beta = \epsilon\alpha\epsilon^2\beta = \alpha\beta = \frac{-p}{3}$, and also $\alpha_3\epsilon\beta = \epsilon^2\alpha\epsilon\beta = \epsilon^3\alpha\beta = \frac{-p}{3}$.

So, the three solutions of the degree three equation are:
$$\alpha + \beta$$
$$\epsilon \cdot \alpha + \epsilon^2 \cdot \beta$$
$$\epsilon^2 \cdot \alpha + \epsilon \cdot \beta,$$
with α a solution of $\alpha^3 = \frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}$, and $\beta = \frac{-p}{3\alpha}$. Of course, the case $\alpha = 0$ occurs only when $p = 0$ and in that case the equation is $y^3 = -q$ which can be solved by taking cube roots.

Example 1.3. Let us solve $x^3 - 3x + 1 = 0$.

Here, $p = -3, q = 1$.

So we need first to consider $\alpha^3 = \frac{-1+\sqrt{1-4}}{2}\epsilon = e^{i2\pi/3}$.

We will then take $\alpha = [e^{i2\pi/3}]^{1/3} = e^{i2\pi/9} = \cos(2\pi/9) + \sin(2\pi/9)$.

Therefore, $\beta = \frac{1}{\alpha} = e^{-i2\pi/9} = \cos(2\pi/9) - i\sin(2\pi/9)$.

So, one solution of our equation is $\alpha + \beta = 2\cos(2\pi/9)$.

To compute the rest of the solution we will use the above formulae:

$\epsilon \cdot \alpha + \epsilon^2 \cdot \beta = e^{i2\pi/3+i2\pi/9} + e^{i4\pi/3-i2\pi/9} = e^{i8\pi/9} + e^{i10\pi/9} = e^{i8\pi/9} + e^{-i8\pi/9} = 2\cos(8\pi/9)$,

and

$\epsilon^2 \cdot \alpha + \epsilon \cdot \beta = e^{i4\pi/3+i2\pi/9} + e^{i2\pi/3-i2\pi/9} = e^{i14\pi/9} + e^{-i14\pi/9} = 2\cos(14\pi/9)$.

Now, let us move to the equation of degree four.

The equation of degree four

Let $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, with $a_1, \ldots, a_4$ complex numbers.
By substituting $y = x + \frac{a_1}{4}$ we obtain an equation of the form

$$y^4 + py^2 + qy + r = 0,$$

where p, q, r are complex numbers (and depending on $a_1, \ldots, a_4$).
Let m be a complex number that will be determined later.
The expression $y^4 + py^2 + qy + r$ can be rewritten as

$$y^4 + py^2 + qy + r = (y^2 + \frac{p}{2} + m)^2 - [2my^2 - qy + (m^2 + pm - r + \frac{p^2}{4})],$$

and solving $y^4 + py^2 + qy + r = 0$ amounts to solving

$$(y^2 + \frac{p}{2} + m)^2 = [2my^2 - qy + (m^2 + pm - r + \frac{p^2}{4})] \ (*)$$

Whenever the right hand side is a square, then we can take square roots in both sides and eventually solve for y.

Since relation (*) holds for all m, we will choose m such that

$$2my^2 - qy + (m^2 + pm - r + \frac{p^2}{4}) = 2m(y - a)^2,$$

for some convenient a.

The expression $2my^2 - qy + (m^2 + pm - r + \frac{p^2}{4})$ is quadratic in y and a quadratic function has a double root when its discrimant is zero. Moreover, the double root is the vertex of the parabola defined by the quadratic function.

So, whenever the discriminant $8m^3 + 8pm^2 - 8(r - \frac{p^2}{4})m - q^2 = 0$ our quadratic will be of the form $2m(y - \frac{q}{4m})^2$ (since the vertex is $\frac{q}{4m}$).

In conclusion we will choose m such that

$$8m^3 + 8pm^2 - 8(r - \frac{p^2}{4})m - q^2 = 0.$$

This is a degree three equation in m, and hence the formulae presented earlier can be applied to get a value for m.

After we obtain this value of m, our equation becomes

$$(y^2 + \frac{p}{2} + m)^2 = 2m(y - \frac{q}{4m})^2.$$

By taking square roots in both sides, we obtain two possibilities:

$$y^2 + \frac{p}{2} + m = \sqrt{2m}(y - \frac{q}{4m}),$$

or

$$y^2 + \frac{p}{2} + m = -\sqrt{2m}(y - \frac{q}{4m}).$$

Both of these equations are of degree two in y and hence by solving each of them we will obtain a total of four roots of the initial equation.

As a final note, we need to remark that we need a nonzero solution of the discriminant, that is $m \neq 0$, since only in this case our considerents regarding the quadratic function are valid. However, the only case when the discrimant has triple root equal to zero, is when $p = 0$, $r - p^2/4 = 0$, $q^2 = 0$ which gives $p = q = r = 0$. In this case, $p = q = r = 0$ and the original equation has $y = 0$ as a root of multiplicity 4.

It can happen that the discriminant equation in m has one root equal to zero, fact equivalent to $q = 0$. Assume that $m = 0$ is a root but not of order three. In this case, one has two options: either one takes another value for m that works and is nonzero, or simply remarks that $q = 0$ implies that the original equation is

$$y^4 + py^2 + r = 0$$

and can be solved regarding it as quadratic in y^2:

$y^2 = \frac{-p \pm \sqrt{p^2 - 4r}}{2}$, which leads easily to four values for y by taking square roots (of possible complex numbers).

In conclusion, the degree four equation reduces to one equation of degree three and two equations of degree two. This makes the computation rather lengthy, however it does provide a complete solution to our equation.

Example 1.4. Solve $y^4 + 2\sqrt{6}y - 2 = 0$.

First let us find m. After substituting, the equation that gives m is:

$$8m^3 + 16m - 24 = 0,$$

so we can take $m = 1$.

Now the equations that we need to solve are:

$$y^2 + 1 = \sqrt{2}(y - \frac{2\sqrt{6}}{4}),$$

or

$$y^2 + 1 = -\sqrt{2}(y - \frac{2\sqrt{6}}{4}),$$

and both of them are easy to solve.