Florian Enescu, Polynomials: Lecture notes Week 11.

1. Discriminants; equations of degree $3, 4$ revisited

Several times in this course we have used properties of the symmetric polynomial and in fact we needed a fundamental result on them that says that any symmetric polynomial in n variables is a polynomial in the nth fundamental symmetric polynomials in those variables.

Let $f \in K[X_1, ..., X_n]$ be polynomial where K denotes the real numbers, the complex numbers or the rational numbers.

A permutation σ is a one-to-ne coresponde on the set $\{1, 2, \ldots, n\}$. For example $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ is a permutation of $\{1, 2, 3\}$. For the set $\{1, 2, \ldots, n\}$ there exist $n!$ distinct possible permutations. Their set will be denoted by S_n.

For a permutation σ and a polynomial f, we let $\sigma(f) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. For example, let $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ and $f(x_1, x_2, x_3) = x_1x_2$. Then $\sigma(f) = x_2x_3$.

A polynomial f is called *symmetric* if for all permutation $\sigma \in S_n$ we have $\sigma(f) = f$. This means that for every monomial of f, say $x_{i_1}^{a_1} \cdots x_{i_n}^{a_n}$, the monomial $x_{\sigma(i_1)}^{a_1} \cdots x_{\sigma(i_n)}^{a_n}$ is also present in f, for all permutations σ.

We remind the reader the nth fundamental symmetric polynomials:

$$s_1 = x_1 + x_2 + \cdots + x_n$$

$$s_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$

$$\cdots$$

$$s_n = x_1x_2 \cdots x_n.$$

For the next Theorem we need to introduce the lexicographic order on monomials. We say that

$$x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$$

if and only if the first nonzero entry in $(a_1 - b_1, \ldots, a_n - b_n)$ is positive.

As an example, $x_1^2x_2 > x_2^5$.

Also, a homogeneous polynomial $f(x_1, ..., x_n$ is a polynomial such that all its monomials have the same total degree.

Let us state the major result on symmetric polynomials

Theorem 1.1. *Let $f(x_1, ..., x_n)$ be a symmetric polynomial. Then there exist a polynomial $F(T_1, ..., T_n)$ such that*

$$f(x_1, ..., x_n) = F(s_1, ..., s_n).$$

Proof. First let us note that a polynomial is a sum of homogeneous polynomials. This can be seen by collecting the terms of the polynomial according to their total degree.

So, it is enough to write every homogeneous part of f as polynomial in $s_1, ..., s_n$. So, let us assume that f is homogeneous of degree d.

Let $ax_1^{i_1}\cdots x_n^{i_n}$ be the term containing the largest monomial of f with regard to the lexicographic order. We claim that $i_1 \geq i_2 \geq \ldots \geq i_n$.

Assume that there exist j such that $i_j < i_{j+1}$. We will show that this leads to a contradiction.

Consider the permutation σ such that $\sigma(j) = j+1, \sigma(j+1) = j$, and $\sigma(k) = k$ for all other k. Then $\sigma(f) = f$ must contain the following monomial $x_1^{i_1}\cdots x_j^{i_{j+1}} x_{j+1}^{i_j}\cdots x_n^{i_n}$ which is a monomial that is greater than $x_1^{i_1}\cdots x_n^{i_n}$. This is a contradiction, so $i_1 \geq i_2 \geq \ldots \geq i_n$.

Set $g = f - as_1^{i_1-i_2}s_2^{i_2-i_3}\cdots s_n^{i_n}$. One can see that, since the largest term in f is cancelled by the largest term in $as_1^{i_1-i_2}s_2^{i_2-i_3}\cdots s_n^{i_n}$, the largest monomial in the polynomial g is smaller than that of f. Also, note that g is till homogeneous of same degree as f. Continuing in similar fashion we can knock each monomial off and replace it by an expression in the symmetric polynomials until we get the zero polynomial. This means that f itself is a polynomial in $s_1, ..., s_n$ therefore ending the proof of our claim.

□

Example 1.2. Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$.

The leading monomial is x_1^2. Compute $g = f - s_1^2 = x_1^2 + x_2^2 + x_3^2 - (x_1 + x_2 + x_3)^2 = -2x_1x_2 - 2x_2x_3 - 2x_3x_1$.

The leading monomial in g is $-2x_1x_2$. Compute $h = g + 2s_2$ and note that $h = 0$.

So $f = s_1^2 + 2s_2 = F(s_1, s_2)$, where $F(T_1, T_2, T_3) = T_1^2 + 2T_2$.

We will now discuss the equation of degree 3 using *Lagrange*'s method.

As before we know that it is enough to show how to solve equations of the form

$$x^3 + px + q = 0, p, q \in \mathbf{C}$$

since the general equation of degree 3 can be reduced to this one after a change of variables.

Let us call x_1, x_2, x_3 the three solutions.

We need to find relations among them that will determine them. We will do that by considering the following polynomial

$$B = (X_1 + \epsilon X_2 + \epsilon^2 X_3)^3,$$

where ϵ is a root of order three of 1 different from 1 itself. Note that this implies that $\epsilon^2 + \epsilon + 1 = 0$.

If we permute the indices in B in all possible ways ($6 = 3!$ possibilities), one can note that we obtain only one other expression that happens to be equal to

$$C = (\epsilon X_1 + X_2 + \epsilon^2 X_3)^3.$$

So, $B + C, BC$ will be left invariant under any permutation of $\{X_1, X_2, X_3\}$ so these expression are symmetric, hence they can be written as polynomials in the fundamental symmetric polynomials s_1, s_2, s_3.

A little bit of computation leads us to precise formuale:

$$B + C = 2s_1^3 - 9s_1s_2 + 27s_3$$
$$BC = (s_1^2 - 3s_2)^3.$$

Note that B, C are hence the roots of the equation

$$z^2-(B+C)z+BC=0.$$

Now plug in x_1, x_2, x_3.

When we evaluate we get $s_1=0, s_2=p, s_3=-q$ from Viete's relations, so $B+C=27q, BC=-27p^3$.

So, B, C are roots of

$$z^2+27qz-27p^3=0,$$

called the *resolvent* of the initial equation.

The solution of these equation are

$$B,C=-\frac{27q}{2}\pm 27\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}=-\frac{27q}{2}\pm\frac{1}{2}\sqrt{-27D},$$

where $D=-27q^2-4p^3$ is the discriminant of the original equation of degree 3.

Note that while $D<0$ suggests that B_1 will be real when p, q are real, the equation won't necessarily have omly real roots as we need to perform more computations involving complex numbers, see what follows below). In fact, one can show that the equation has three real roots if and only if $D\geq 0$.

Denote by b, c the cube roots of B, C respectively. Since $BC=-27p^3$, this means that $bc=-3p$ so $c=\frac{-3p}{b}$.

We have a system of equations now:

$$x_1+x_2+x_3=0$$
$$x_1+\epsilon x_2+\epsilon^2 x_3=b$$
$$x_1+\epsilon^2 x_2+\epsilon x_3=c,$$

system that has the determinant equal to $3(\epsilon^2-\epsilon)\neq 0$ so one can compute the solutions:

$$x_1=(b+c)/3, x_2=(\epsilon^2 b+\epsilon c)/3, x_3=(\epsilon b+\epsilon^2 c)/3.$$

In fact if one determines x_1, then by dividing to $x-x_1$ one obtains a degree two equation that can be solved for x_2, x_3.

Example 1.3. Solve $x^3-15x-126=0$.

Note that $p=-15, q=-126$.

Then $q/2=-63, p/3=-5$ and

$$B,C=-\frac{27q}{2}\pm 27\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}=-27\cdot(-63)\pm\sqrt{63^2-125}.$$

But $\sqrt{63^2-125}=27\cdot 62$.

This shows that $B,C=27\cdot 63\pm 27\cdot 62$.

So, we can take $B=27$ $(C=27\cdot 125.)$

Hence $b=3$ and then $c=15$.

So, $x_1=(3+15)/3=6$. So we know that $x=6$ is a solution and by dividing $x^3-15x-126$ to $x-6$ we get

$$x^3 - 15x - 126 = (x-6)(x^2+6x+21).$$

hence the other two solution are $-3 \pm i\sqrt{2}$.

Now we will discuss Viete's approach to the degree three equation.
First note that $\cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta)$, hence
$u = \cos(\theta)$ is the solution of the equation:

$$x^3 - \frac{3}{4}x - \frac{1}{4}\cos(3\theta) = 0.$$

One can check that the other solutions are:

$$\cos(\theta + 2\pi/3), \cos(\theta + 4\pi/3).$$

Now let us consider a general equation $x^3 + qx + r = 0$.

We will compute a solution for this equation of the form $v = tu$, where t, u are to be determined as follows:

First of all, we need to know that our equation has three real roots. This is equivalent to the discriminant being greater or equal to 0.

Since $(tu)^3 + q(tu) + r = 0$, we get $u^3 + \frac{q}{t^2}u + \frac{r}{t^3} = 0$.

One choose t such that $\frac{q}{t^2} = -\frac{3}{4}$, $\frac{r}{t^3} = -\frac{\cos(3\theta)}{4}$.

The choices are possible since $D = -4q^3 - 27r^2 \geq 0$:

Indeed, the existence of t is guaranteed since q must be a negative number due to the fact that $4q^3 \leq -27r^2$.

Let us check that $\frac{-4r}{t^3}$ is between -1 and 1 (this is necessary and sufficient for the existence of θ).

Since $D \geq 0$ we get that $\frac{9r^2}{q^2} \leq \frac{-4q}{3}$.

So, $\mid \frac{3r}{q} \mid \leq \sqrt{\frac{-4q}{3}} = t$.

So,

$$\mid \frac{-4r}{t^3} \mid = \mid \frac{3r}{q} \mid\mid \frac{1}{t} \mid \leq \frac{t}{t} = 1.$$

Hence $t = \sqrt{\frac{-4q}{3}}$, $\cos(3\theta) = -\frac{4r}{t^3}$.

To determine θ one needs to first compute t, and then use a calculator.

With these choices, there will three possibilities for u:
$\cos(\theta), \cos(\theta + 2\frac{\pi}{3}), \cos(\theta + 4\frac{\pi}{3})$.

Then one gets three values for v after multiplying with t.

Example 1.4. Let us solve $x^3 - 7x + 6 = 0$. Note that the solutions that we expect are $1, 2, -3$ (those can be checked by direct computations).

Note that $q = -7, r = 6$.

First of all, $D = -(27r^2 + 4q^3) = 400 \geq 0$ which says that Viete's method can be applied.

Compute first t: $t = \sqrt{\frac{(-4)(-7)}{3}} \simeq 3.055$.

Then $\cos(3\theta) = -\frac{4r}{t^3} = \frac{-24}{(3.055)^3} \simeq -.842$.

Using a calculator we get that $3\theta \simeq 148$ degrees.

Sp, $\theta \simeq 49$ degrees.

The three possibilities for u are now:

$\cos(49\ degrees), \cos(169\ degrees), \cos(289\ degrees)$.

After computing those we multiply by $t = 3.055$ and get three solutions to our equation:

$2.004, -3, .996$ which are very good approximations to the precise values that we listed at the beginning.