Florian Enescu, Polynomials, Fall 2010: Lecture notes Week 1.

1. Introduction. Complex numbers.

What are the solutions of $x^2 + 3x + 7 = 0$? How about $x^3 - 3x + 1 = 0$? More generally, can one solve any equation of the form $f(x) = 0$ where f is a polynomial with real coefficients?

As soon as we inspect the first equation we see that it does not have any real solutions. It admits two complex solutions and this suggests that if we want to answer general questions about polynomials with real coefficients then we have to look at a larger class of numbers than real numbers. Therefore we will spend some time discussing *complex numbers*.

We will denote the set of real number by $\mathbf{R}$. A complex number z is a pair of real numbers $z = (x, y)$. As such the definition is not useful since we need to explain how we add and multiply the complex numbers as well as relate real numbers to complex ones.

For $z = (x, y), w = (u, v)$ we define $z + w := (x + u, y + v)$ while the multiplication is defined by $(x, y)(u, v) := (xu - yv, xv + uy)$. One can check the set of all complex numbers together with the addition and multiplication defined as above form a field. In intuitive terms, this means that addition and multiplication obey the same natural rules that they have for real numbers: commutativity, associativity, distributivity. These are somewhat lengthy verifications that we will not perform here.

We can see that $(x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0)$, as $(0, 1)(y, 0) = (0, y)$. Moreover, $(0, 1)^2 = (1, 0)$. With this in mind, it makes sense to introduce the following notations: for any real number x, we denote $x = (x, 0)$ and we also set $i := (0, 1)$. Therefore we can now write a complex number $z = (x, y)$ as $z = x + iy$. We also denote $Re(z) = x, Im(z) = y$ called the *real*, respectively *imaginary* part of z. the set of complex numbers will be denoted by $\mathbf{C}$.

The *absolute value* of $z = x+iy$ is denoted by $\mid z \mid$ and is by definition equal to $\sqrt{x^2 + y^2}$. One can check that for any two complex numbers z, w, we have $\mid zw \mid = \mid z \mid\mid w \mid$. Also, $z \neq 0$ if and only if $\mid z \mid \neq 0$.

It can be easily checked that $0z = z0 = 0$ and $1z = z1 = z$ for all complex numbers z. Moreover for every nonzero complex number z we can check that there exist an unique multiplicative inverse of z, z^{-1}. It can be easily checked that

$$z^{-1} = \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} = \frac{z}{\mid z \mid^2}.$$

Division by a complex number w is defined as:

$$\frac{z}{w} := z \cdot w^{-1}.$$

Example 1.1. Compute

$$\frac{1}{2 + 3i} \cdot \frac{1 + 2i}{1 - i}.$$

Another important property of complex numbers is that, similarly to the case of real numbers, whenever $zw = 0$ then $z = 0$ or $w = 0$.

It is useful to introduce another concept, the *complex conjugate* of a complex number $z = x+iy$. It is defined as $\overline{z} = x - iy$. Clearly, $\overline{z + w} = \overline{zw}$, and $\overline{zw} = \overline{zw}$. Also, $\mid \overline{z} \mid = \mid z \mid$, and $z\overline{z} = \mid z \mid^2$.

Proposition 1.2 (Triangle Inequality)**.** *Let z_1, z_2 be two complex numbers. Then*

$$| z_1 + z_2 | \leq | z_1 | + | z_2 |, | z_1 - z_2 | \geq | | z_1 | - | z_2 | |.$$

Proof. Let $z_1 = x + iy, z_2 = u + iv$.

The first inequality translates into:

$\sqrt{(x+u)^2 + (y+v)^2} \leq \sqrt{x^2+y^2} + \sqrt{u^2+v^2}$. Squaring both sides, we need to show that $(x+u)^2 + (y+v)^2 \leq x^2 + y^2 + u^2 + v^2 + 2\sqrt{(x^2+y^2)(u^2+v^2)}$, or after simplifications $xu + yv \leq \sqrt{(x^2+y^2)(u^2+v^2)}$.

Squaring again, we need to show that

$x^2u^2 + 2xuyv + y^2v^2 \leq x^2u^2 + x^2v^2 + y^2v^2 + y^2u^2$, or $2xuyv \leq x^2v^2 + y^2u^2$ which is equivalent to $(xv - yu)^2 \geq 0$.

The second inequality follows from the first since by the triangle inequality

$| z_1 - z_2 | \geq | z_1 | - | z_2 |, | z_2 - z_1 | \geq | z_2 | - | z_1 |$ and $| z_1 - z_2 | = | z_2 - z_1 |$.

□

We will discuss now the geometric interpretation of complex numbers.

Since $z = x + iy$ represents a pair (x, y) we can simply plot this point into the xy-plane. Then $| z | = \sqrt{x^2 + y^2}$ is in fact the distance from z to the origin of the plane.

The distance between two complex numbers z_1, z_2 is defined as $| z_1 - z_2 |$.

Example 1.3. Identify the collection of points z such that $| z - 1 - 2i |$.

Every complex number $z = (x, y) = x + yi$ has polar coordinates r, θ such that $z = r(\cos\theta + i\sin\theta)$ with $r \geq 0$. Clearly, $r = | z |$.

Also $z = r(\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n))$, for all integers n.

We denote $\theta = Arg(z)$ and as we can see from the identity above there are many possible choices for θ all different from each other by a multiple of 2π.

The principal value of $Arg(z)$ is defined as the unique value of $Arg(z)$ between 0 and 2π (some authors use a value between $-\pi$ and π). The principal value is denote by $arg(z)$. So, $0 \leq arg(z) < 2\pi$ and formally $Arg(z) = \{arg(z) + 2\pi n : n \in \mathbb{Z}\}$.

Example 1.4. Let $z = i$. To find the argument of z we need to first note that $r = | z | = 1$ and consider the system of equations $0 = \cos\theta$, $1 = \cos\theta$, which gives $\theta = \pi/2 + 2\pi n$, $n \in \mathbb{Z}$.

So, $arg(z) = \pi/2$ while $Arg(z) = \{\pi/2 + \pi n, n \in \mathbb{Z}\}$.

An important identity involving arguments is

$$Arg(z_1 z_2) = Arg(z_1) + Arg(z_2),$$

where one should note that this is an equality of sets.

To verify this write $z = r_1(\cos\theta_1 + i\sin\theta_1)$, $z = r_2(\cos\theta_2 + i\sin\theta_2)$. Multiplying $z_1 z_2$, we get

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)],$$

which can be rewritten as

$$z = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Example 1.5. Verify that $z^{-1} = \frac{1}{r}(\cos -\theta + i\sin -\theta)$.

The equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is called *Euler's formula*.

Hence, we can write $z = re^{i\theta}$. Due to the formula on the additive property of the argument, we have also that

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

The exponential notation obeys the rules of the exponential function (although we have not explained the meaning of e^z strictly speaking).

In particular if $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$, for all integers n.

If we take $r = 1$ and rewrite this in trigonometric form we get

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta,$$

called *de Moivre's formula*.

We will discuss now the nth roots of unity, that is the solutions of the equation $z^n = 1$ in the set of complex numbers.

Definition 1.6. Fix n a natural number. A complex number z is called a nth root of unity if $z^n = 1$.

To solve the equation $z^n = 1$, write $z = re^{i\theta}$, so $z^n = 1$ implies $r^n(\cos n\theta + i\sin n\theta = 1$.

Therefore $r = 1$ and $n\theta = 2k\pi$, where k is an integer which gives $\theta = \frac{2k\pi}{n}$, $k \in \mathbf{Z}$.

So the solutions are $z = \exp(i\frac{2k\pi}{n}) = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$, where k integer. Although there are infinitely many choices possible for k, there are only n distinct roots, and they can be obtained for $k = 0, \ldots, n-1$.

If we denote $\xi_n = \exp(2\pi/n)$, we can see that $\xi_n^k = \exp(i\frac{2k\pi}{n})$, so all nth roots of unity are of the form ξ_n^k, for $k = 0, 1, ..., n-1$.

Let us denote $U_n = \{z \in \mathbf{C} : z^n = 1\}$. We have remarked that $\mid U_n \mid= n$. We can also see that $1 \in U_n$ and that whenever $z, w \in U_n$, $zw^{-1} \in U_n$. This makes U_n a group under multiplication. Moreover this group is generated by ξ_n, since we have seen that for any $z \in U_n$ there exists k such that $z = \xi_n^k$.

What other generators can one find for U_n? In other words, for what roots $\exp(2\pi l/n)$, is it true that for any $z \in U_n$, there exists k such that $z = \exp(2\pi l/n)^k$.

To answer this question it is enough to consider the case of $z = \xi_n$, since every $z \in U_n$ is a power of ξ_n.

But $\xi_n = \exp(2\pi/n) = \exp(2\pi l/n)^k = \exp(2\pi kl/n)$, which implies that

$$2\pi/n = 2\pi kl/n + 2\pi m,$$

or in other words, $kl \equiv 1$ modulo n. We know that an integer l is invertible modulo n if and only if $(l, n) = 1$.

This shows that $\exp(2\pi l/n) = \xi_n^l$ is a generator for U_n if and only if $(l, n) = 1$.

We can use the above method to solve any equation of the form $z^n = z_0$.

If we write $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$, we can similarly obtain that the roots of $z^n = z_0$ are of the form

$$\sqrt[n]{r_0}\exp[i(\frac{\theta_0}{n} + \frac{2k\pi}{n})],$$

$k = 0, 1, \ldots, n-1$.

In fact, we can rewrite these solutions as

$$\sqrt[n]{r_0}\exp i(\frac{\theta_0}{n}) \cdot \omega_n^k,$$

$k = 0, 1, \ldots, n-1$.

Indeed, if we have a solution of $z^n = z_0$ say η, then the rest of the solutions are $\eta\omega_n^k$ for some integer k. This observation comes from the fact that if η_1, η_2 are solutions of $z^n = z_0$, then $\frac{\eta_1}{\eta_2}$ is a solution of $z^n = 1$, since

$$(\frac{\eta_1}{\eta_2})^n = \frac{z_0}{z_0} = 1.$$

Solving a quadratic equation:

If we consider a quadratic equation such as $x^2 + 3x + 7 = 0$, then the quadratic formula leads to solutions of the form

$$x = \frac{-3 \pm \sqrt{-19}}{2} = \frac{-3 \pm i\sqrt{19}}{2}.$$

In fact, we can use the quadratic formula to solve any quadratic equation even if it has complex coefficients. This is so because the algebraic manipulation that one makes to get the quadratic formula are well-defined in the case of complex numbers and hence can be reproduced without modification.

Therefore, given $ax^2 + bx + c = 0$, with $a, b, c \in \mathbf{C}$, the solutions are obtained with

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

One should exercise caution since $b^2 - 4ac$ is now a complex number and so taking sqaure roots usually requires the use of trigonometric form of complex numbers.

Example 1.7. Solve $x^2 - (1+3i)x + (i-2) = 0$.