

# LOCAL COHOMOLOGY AND F-STABILITY

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ABSTRACT. We study the relationship between the Frobenius stability of an Artinian module over an F-injective ring and its stable part.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m}, k)$  be a local Noetherian ring of positive characteristic  $p$ , where  $p$  is prime. Let  $F : R \rightarrow R$  be the Frobenius homomorphism on  $R$ , that  $F(r) = r^p$ , for all  $r \in R$ . This homomorphism induces a natural Frobenius action on the local cohomology modules  $H_{\mathfrak{m}}^i(R)$ ,  $i = 0, \dots, d = \dim(R)$ . This action is an effective tool in the study of local cohomology modules as it was shown over the years by many authors. Our paper deals with the concept of Frobenius stability which has its roots in an influential paper by Hartshorne and Speiser [6] via the stable part of a module  $M$  endowed with a Frobenius action. Since then, papers by Lyubeznik [7, 8], Fedder and Watanabe [5], Enescu [3], Singh and Walther [10] and Sharp [9] have explored various properties of local cohomology where Frobenius stability played a role in some fashion. The work of Hartshorne and Speiser, Lyubeznik, Singh and Walther dealt with the concept of Frobenius depth and applications to the Grothendieck vanishing problem (see [8], page 1), while Fedder and Watanabe, Enescu and Sharp have explored connections to tight closure theory from a different perspective.

Our main goal is to present a coherent description of Frobenius stability and establish a clear relationship between the F-stability in the sense of Fedder-Watanabe and the stable part of the local cohomology as in Hartshorne-Speiser and Lyubeznik. With this goal in mind, we will carefully discuss various aspects of Frobenius stability and their relevance to the aforementioned papers hoping to make our exposition valuable to the reader interested in a unitary presentation of these aspects.

Before stating our main contributions, we need to introduce a few notations and related concepts which are used throughout our paper. Everywhere in this note  $(R, \mathfrak{m}, k)$  will denote a local Noetherian ring of positive characteristic  $p$ , where  $p$  is prime, and Krull dimension  $d$ . A Frobenius action on an Artinian  $R$ -module  $M$  is an additive map  $F_M : M \rightarrow M$  such that  $F_M(rm) = r^p F_M(m)$ . We will often drop the subscript “ $-_M$ ” from our notation when there is no danger of confusion. The main example of Artinian  $R$ -modules that we will consider is that of local cohomology modules of  $R$  with support in the maximal ideal  $\mathfrak{m}$ . Let  $\underline{x} = x_1, \dots, x_d$  be a system of parameters for  $R$ . The  $i$ th local cohomology module of  $R$  equal the  $i$ th cohomology module of the Čech complex

$$0 \rightarrow R \rightarrow \bigoplus_{i=1}^d R_{x_i} \rightarrow \cdots \rightarrow \bigoplus_{i=1}^d R_{\mathbf{x}_i} \rightarrow R_{\mathbf{x}} \rightarrow 0,$$

where  $\mathbf{x}_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_d$ , and  $\mathbf{x} = x_1 \cdots x_d$ .

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The Frobenius  $F$  acts on  $R$  and its localizations and hence induces an action to the cohomology modules of the Čech complex. We will denote this induced Frobenius action on  $H_{\mathfrak{m}}^i(R)$  by  $F$ .

The particular case of  $H_{\mathfrak{m}}^d(R)$  is very important, as this is the only nonzero local cohomology module of  $R$  when  $R$  is Cohen-Macaulay. An element of  $\eta \in H_{\mathfrak{m}}^d(R)$  is denoted by  $\eta = [\frac{z}{\mathbf{x}^s}]$  and the Frobenius action  $F$  sends  $\eta$  to  $F(\eta) = [\frac{z^p}{\mathbf{x}^{ps}}]$ .

The following alternate description of  $H_{\mathfrak{m}}^d(R)$  will be useful later in the paper. Let  $x_1, \dots, x_d$  be a system of parameters in  $R$ . The local cohomology module  $H_{\mathfrak{m}}^d(R)$  can be obtained as a direct limit of  $R/(x_1^t, \dots, x_d^t)$  where the maps of the direct system are given by

$$R/(x_1^t, \dots, x_d^t) \xrightarrow{\mathbf{x}^{l-t}} R/(x_1^l, \dots, x_d^l),$$

where  $l \geq t$  and  $\mathbf{x} = x_1 \cdots x_d$ . With this description, an element  $\eta \in H_{\mathfrak{m}}^d(R)$  will be described by  $\eta = [z + (x_1^t, \dots, x_d^t)]$ , where  $z \in R$ . The Frobenius action  $F$  sends  $\eta$  to  $F(\eta) = [z^p + (x_1^{pt}, \dots, x_d^{pt})]$ .

**Definition 1.1.** Let  $(R, \mathfrak{m}, k)$  be a local Noetherian ring of positive characteristic  $p$ ,  $p$  prime. Then  $R$  is  $F$ -injective if  $F$  acts injectively on  $H_{\mathfrak{m}}^i(R)$  for all  $i$ .

In 1989 Fedder and Watanabe have defined the notion of  $F$ -stability for local cohomology modules of  $R$  with support in the maximal ideal  $\mathfrak{m}$  of  $R$  and studied it in the case of  $F$ -injective rings, [5]. This definition can be extended to an Artinian  $R$ -module  $M$  endowed with a Frobenius action. For such modules, Harthorne and Speiser have defined in 1977 [6], in the case when  $R$  contains a coefficient field  $k$ , a natural  $k$ -vector space  $M_s \subseteq M$  called the *stable part* of  $M$ . See Section 1 for precise definitions. Our main contribution in this paper is to establish a clear relationship between the  $F$ -stability of an Artinian module  $M$  and its stable part.

More precisely, we prove the following

**Theorem 1.2.** *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring containing a coefficient field  $k$ . Let  $M$  be an Artinian  $R$ -module which admits an injective Frobenius action.*

*Then  $M$  is  $F$ -stable if and only if  $M_s \neq 0$ .*

This result allows us to establish a connection between the set of prime ideal  $P$  in  $R$  for which  $R_P$  is  $F$ -stable and a set of primes discovered by Lyubeznik in his work on  $F$ -modules. These primes are naturally related to the notion of  $F$ -depth and we explore this relationship. This is done in Section 3. Section 4 presents a counterexample to a natural question on the behavior of  $F$ -injectivity under flat local maps with regular fibers and shows that complete  $F$ -injective 1-dimensional domains with algebraically closed residue field are regular.

We would like to review some of the basic definitions and facts from tight closure theory that will be needed in our paper.

We use  $q$  to denote a power of  $p$ , so  $q = p^e$  for  $e \geq 0$ . For  $I \subseteq R$  set  $I^{[q]} = (i^q : i \in I)$ . Let  $R^\circ$  be the complement in  $R$  of the minimal primes of  $R$ . We say that  $x$  belongs to the *tight closure* of  $I$  and write  $x \in I^*$  if there exists  $c \in R^\circ$  such that for all  $q \gg 0$ ,  $cx^q \in I^{[q]}$ . We say that  $x$  is in the Frobenius closure of  $I$ ,  $I^F$ , if there exists a  $q$  such that  $x^q \in I^{[q]}$ , and say that  $I$  is *Frobenius closed* if  $I = I^F$ . When  $R$  is reduced then  $R^{1/q}$  denotes the ring of  $q$ th roots of elements of  $R$ . When  $R^{1/q}$  is module-finite over  $R$   $R$  is called  *$F$ -finite*. We call  $R$  weakly  $F$ -regular if every ideal of  $R$  is tightly closed. A weakly  $F$ -regular ring is always normal, and

under mild hypotheses is Cohen-Macaulay. A ring  $R$  is  $F$ -regular if every localization of  $R$  is weakly  $F$ -regular.

We call an ideal  $I = (x_1, \dots, x_n)$  a *parameter ideal* if  $\text{ht}(I) \geq n$ . The ring  $R$  is  $F$ -rational if every parameter ideal is tightly closed. We note that  $F$ -rational and Gorenstein rings are  $F$ -regular.

A ring  $R$  for which  $F : R \rightarrow R$  is a pure homomorphism is called  $F$ -pure. An  $F$ -pure ring is  $F$ -injective and moreover an excellent and reduced ring  $R$  is  $F$ -pure if and only if  $I^F = I$  for all ideals  $I$  in  $R$ .

When  $R$  is Cohen-Macaulay, then  $R$  is  $F$ -injective if and only if some (equivalently, every) ideal generated by a system of parameters is Frobenius closed.

## 2. FROBENIUS STABILITY OF ARTINIAN MODULES

Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of characteristic  $p$ , where  $p$  is a prime number, and dimension  $d$ . Let  $M$  be an Artinian  $R$ -module.

Assume that  $M$  admits a Frobenius action  $F = F_M : M \rightarrow M$ , i.e. an additive map with the property that  $F(rm) = r^p F(m)$  for all  $m \in M$  and  $r \in R$ .

Let  $\text{Soc}_R(M) = \{x \in M : \mathfrak{m} \cdot x = 0\}$ . This is a  $R$ -submodule of  $M$  which is naturally a vector space over  $R/\mathfrak{m} = k$ . In fact,  $\text{Soc}_R(M)$  is finite dimensional over  $k$ .

Our main examples of Artinian  $R$ -modules that admit a Frobenius action are the local cohomology modules of  $R$ ,  $H_{\mathfrak{m}}^i(R)$  with  $0 \leq i \leq d$ . However we find it helpful to present the notions related to Frobenius stability in the more general context of Artinian modules and then to apply them to local cohomology.

The following definition is inspired by Fedder and Watanabe who considered it only in the case of local cohomology modules.

**Definition 2.1** (Fedder-Watanabe). Let  $M$  be an Artinian  $R$ -module that admits a Frobenius action. Let  $S = \text{Soc}_R(M)$  be the socle of  $M$ . Denote  $F^e(S) = \{F^e(m) : m \in S\}$ .

We say that  $M$  is  $F$ -unstable if there exists  $N > 0$  such that  $S \cap F^e(S) = 0$  for all  $e \geq N$ . Note that the zero module is  $F$ -unstable. If  $M$  is not  $F$ -unstable, then it will be called  $F$ -stable.

In general, we say that  $R$  is  $F$ -unstable if  $H_{\mathfrak{m}}^i(R)$  is  $F$ -unstable for every  $i$ .

The reader should be aware that a submodule  $N$  of  $M$  is sometimes called  $F$ -stable if  $F(N) \subseteq N$ . We will call such submodules  $N \subseteq M$   $F$ -invariant to avoid any possible confusion.

The following reformulation can be established in the case of an injective Frobenius action on  $M$ . We decided to include its proof for the convenience of the reader.

**Proposition 2.2.** (Fedder-Watanabe) Let  $(R, \mathfrak{m}, k)$  be a local ring. Let  $M$  be an Artinian  $R$ -module that admits an injective Frobenius action  $F : M \rightarrow M$ . Let  $S = \text{Soc}(M)$  be the socle of  $M$ .

If  $S \cap F^e(S) \neq 0$  holds for infinitely many  $e > 0$ , then there exists  $0 \neq \eta \in S$  such that  $F^e(\eta) \in S$  for every  $e \geq 0$ .

*Proof.* Assume that  $S \cap F^e(S) = 0$  for infinitely many  $e$ . Denote  $M_e = S \cap F^e(S)$ .

Claim:  $M_{e+1} \subset F(M_e)$ .

Indeed, take  $m \in M_{e+1}$ , that is  $m \in S$  and  $m = F^{e+1}(a)$ , for some  $a \in S$ . So,  $0 = \mathfrak{m}^{[p^j]}m = F^j(\mathfrak{m}F^{e+1-j}(a))$  so  $F^{e+1-j}(a) \in S$ , for all  $1 \leq j \leq e+1$ .

So  $F^e(a) \in S$  and obviously  $F^e(a) \in F^e(S)$ , since  $a \in S$ , proving the claim.

One can consider  $M_e$  as a vector space over  $R/\mathfrak{m} = k$  with the multiplication  $l * m = l^{[p^e]}m$ , for  $l \in k$  and  $m \in M_e$ .

Moreover,  $F : M_e \rightarrow F(M_e)$  is in fact an injective  $k$ -linear map, and hence bijective. So  $\dim_k(M_e) = \dim_k(F(M_e))$  and hence  $\dim_k(M_{e+1}) \leq \dim_k(M_e)$  with equality if and only if  $F^e(M_e) = M_{e+1}$ .

But from the finite dimensionality of  $S$ , we get that  $\dim(M_e)$  is constant for  $e$  large enough. Since  $M_e \neq 0$  for infinitely many  $e$ , we know that this constant must be non-zero. So, there exist  $N$  and  $\gamma \in M_N \neq 0$ , where  $\dim M_e = \dim M_N$  for all  $e \geq N$  and  $F^l(M_N) = M_{N+l}$  for all  $l \geq 0$ . We have that, for all  $e \geq N$ ,  $F^{e-N}$  is an isomorphism from  $M_N$  to  $M_e$ . Moreover, there exists  $\eta \in S$  such that  $F^N(\eta) = \gamma$ .

Let us check that  $F^e(\eta) \in S$  for all  $e$ : if  $e \leq N$ , then let us first notice that  $\mathfrak{m} \cdot \gamma = 0$ . so,  $\mathfrak{m} \cdot F^N(\gamma) = 0$ . This implies that  $F^{N-e}(\mathfrak{m} \cdot F^e(\gamma)) = 0$  and using the  $F$ -injective of  $R$  we get our statement. For  $e \geq N$ ,  $F^e(\eta) = F^{e-N}(\gamma) \in M_e \subset S$ . So,  $F^e(\eta) \in S$ , for all  $e$ .  $\square$

Let  $m \in M$  and set  $C_m = \langle F^e(m) : e \geq 0 \rangle_R$  the  $R$ -submodule of  $M$  generated by all  $F^e(m)$  with  $m \in M$ . Denote  $F\text{-ann}(m) := \text{Ann}_R(C_m) = \{r \in R : rF^e(m) = 0, \text{ for all } e \geq 0\}$ .

A number of papers have considered these ideals from various points of view, [3, 4, 9]. An important result, independently obtained by Sharp on one hand and Enescu and Hochster on the other, explains the main properties of these ideals in the case of an injective Frobenius action on  $M$ .

A family  $\Gamma$  of radical ideals is *closed under primary decomposition* if for any ideal  $I \in \Gamma$  and any irredundant intersection  $I = P_1 \cap \dots \cap P_n$ , where  $P_1, \dots, P_n$  are prime ideals, it follows that  $P_i \in \Gamma$ , for all  $i = 1, \dots, n$ .

**Theorem 2.3** (Sharp; Enescu-Hochster). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of characteristic  $p$  and let  $M$  be an Artinian  $R$ -module that admits an injective Frobenius action denoted  $F$ . Denote  $\Gamma = \{\text{Ann}_R(N) : N \subseteq M \text{ such that } F(N) \subseteq N\}$ .*

*Then  $\Gamma$  is a finite set of ideals, consists of radical ideals and is closed under primary decomposition.*

**Remark 2.4.** Under the conditions of Theorem 2.3 (or 2.2),  $M$  is  $F$ -stable if and only if  $\mathfrak{m} \in \Gamma$  if and only if there exists  $0 \neq m \in M$  such that  $\mathfrak{m} = F\text{-ann}(m)$ . This follows immediately from Theorem 2.2.

Now it is time to explain the relationship between the  $F$ -stability of  $M$  and a stability concept introduced by Hartshorne and Speiser and refined by Lyubeznik.

**Definition 2.5** (Hartshorne-Speiser). Assume that  $R$  contains a coefficient field  $k$ . Let  $M$  be an Artinian  $R$ -module such that the Frobenius  $F$  acts on  $M$ . For all  $j$ , let  $F^j(M) = \{F^j(m) : m \in M\}$  and  $\langle F^j(M) \rangle_k$  be the  $k$ -vector space spanned by  $F^j(M)$  in  $M$ . The *stable part* of  $M$  is  $M_s := \bigcap_j \langle F^j(M) \rangle_k$ .

While this definition depends on the choice of the coefficient field  $k$ , Lyubeznik has shown that the dimension of  $M_s$  as a  $k$ -vector space is independent of  $k$  (see [7], Corollary 4.11). Moreover, Hartshorne and Speiser proved that, in the case that  $k$  is perfect,  $M_s$  is finite dimensional over  $k$  and the naturally induced Frobenius action is bijective on  $M_s$ .

To be more precise, let  $k \subset K$  and let  $R^K = K \hat{\otimes}_k R$ , the complete tensor product. The ring  $R^K$  is complete and local, with maximal ideal equal to  $K \hat{\otimes}_k R$  and residue field  $K$ .

For an  $R$ -module  $M$ , we let  $M^K = R^K \otimes_R M$ . Since  $M$  is Artinian over  $R$ , we have that  $M^k = K \otimes_k M$ . If  $M$  admits a Frobenius action  $F_M : M \rightarrow M$ , then this induces a Frobenius action on  $M^K$  by  $F_{M^K}(l \otimes m) = l^p \otimes F_M(m)$ , for  $l \in K, m \in M$ . Moreover,  $\text{Soc}_{R^K}(M^K) = K \otimes_k \text{Soc}_R(M)$ .

Let us denote  $M_n = \{m \in M : \text{there exists } e \text{ such that } F^e(m) = 0\}$  and  $M_r = M/M_n$ . Obviously, if  $F$  acts injectively on  $M$  then  $M_n = 0$ . In general,  $M_n$  is invariant under  $F$  and  $F$  acts injectively on  $M_r$ .

**Theorem 2.6** (Harthorne-Speiser; Lyubeznik). *Assume that  $R$  contains a coefficient field  $k$  and let  $M$  be an Artinian  $R$ -module with a Frobenius action  $F$  on it. Then  $M_s$  is finite dimensional over  $k$ ,  $F : M_s \rightarrow M_s$  is injective and the  $k$ -vector subspace of  $M$  spanned by  $F(M_s)$  coincides with  $M_s$ . Moreover, if  $K$  is the perfect closure of  $k$ ,  $M^K$  is Artinian over  $R^K$ , Frobenius acts on it,  $(M^K)_s = K \otimes_k M_s$  and  $\dim_k M_s = \dim_K (M^K)_s$ .*

**Proposition 2.7** (Harthorne-Speiser). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring containing a perfect coefficient field  $k$ .*

- (1) *Let  $M$  be an Artinian  $R$ -module and let  $F = F_M$  be a Frobenius action on  $M$ . If  $F$  acts injectively on  $M$  then  $M_s \subseteq \text{Soc}_R(M)$ .*
- (2) *Let*

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0,$$

*be a short exact sequence of  $R$ -Artinian modules that admit compatible Frobenius actions, i.e. the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow F_N & & \downarrow F_M & & \downarrow F_L & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

*Moreover, assume that  $F_L$  acts injectively on  $L$ .  
Then*

$$0 \rightarrow N_s \rightarrow M_s \rightarrow L_s$$

*is a short exact sequence of  $k$ -vector spaces.*

We are now in position to state the main result of this section.

**Theorem 2.8.** *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring containing a coefficient field  $k$ . Let  $M$  be an Artinian  $R$ -module which admits an injective Frobenius action.*

*Then  $M$  is  $F$ -stable if and only if  $M_s \neq 0$ . Also,  $M_s \subseteq \text{Soc}_R(M)$ .*

*Proof.* Assume that  $M$  is  $F$ -stable. Then there exists  $m \in M$  such that  $\mathfrak{m} = F - \text{ann}(m)$ . Therefore,

$$\mathfrak{m} \cdot F^e(m) = 0,$$

for every  $e \geq 0$ .

Let  $S = \text{Soc}(M)$  and denote  $M_e = S \cap \langle F^e(M) \rangle_k$ .

Note that  $M_e$  is a  $k$ -vector subspace of  $S$  and it can be easily checked that  $M_{e+1} \subseteq M_e$ .

But  $F^e(m) \subseteq \langle F^e(M) \rangle_k$  and since  $\mathfrak{m} \cdot F^e(m) = 0$  we conclude that  $F^e(m) \in M_e$ . Moreover,  $F$  acts injectively on  $M$  hence  $F^e(m) \neq 0$  for all  $e$ . So,  $M_e \neq 0$ .

Since  $S$  is a finite dimensional  $k$ -vector space and  $\{M_e\}_e$  forms a descending chain of  $k$ -vector subspaces, there exists  $e_0$  such that  $M_e = M_{e_0}$  for all  $e \geq e_0$ . But  $M_e \neq 0$  for all  $e$ . Hence

$$0 \neq M_{e_0} = \bigcap_e M_e \subseteq \bigcap_e F^e(M) \simeq_k M_s,$$

proving the first part of the theorem.

Let  $K$  be the perfect closure of  $k$  and  $M^K = R^K \otimes_R M = K \otimes_k M$  where  $R^K = R \hat{\otimes}_k K$ . We have that  $M_s \subset (M^K)_s$ .

Consider the following short exact sequence:

$$0 \rightarrow (M^K)_n \rightarrow M^K \rightarrow M^K/(M^K)_n \rightarrow 0,$$

where the Frobenius actions on each module are naturally compatible. Note that the Frobenius action on  $M^K/(M^K)_n$  is injective. Applying Proposition 2.7 we get the following exact sequence

$$0 \rightarrow (M^K)_s \rightarrow (M^K/(M^K)_n)_s.$$

Since  $F$  acts injectively on  $M^K/(M^K)_n$  then Proposition 2.7 (1) applies and we get

$$(M^K/(M^K)_n)_s \subseteq \text{Soc}_{R^K}(M^K/(M^K)_n).$$

Now assume  $M_s \neq 0$  and let  $\eta \in M_s$  arbitrarily chosen. Then  $1 \otimes \eta \in (M^K)_s$ .

Using the inclusions

$$(M^K)_s \subseteq (M^K/(M^K)_n)_s \subseteq \text{Soc}_{R^K}(M^K/(M^K)_n),$$

we obtain that  $\mathfrak{m}^K \cdot (1 \otimes \eta) \in (M^K)_n$ .

However,  $(1 \otimes \mathfrak{m}) \cdot (1 \otimes \eta) \subset \mathfrak{m}^K \cdot (1 \otimes \eta) \in (M^K)_n$ . Hence there exists  $e$  such that  $F^e(1 \otimes \mathfrak{m}\eta) = 0 \in M^K$ . But  $F^e(1 \otimes \mathfrak{m}\eta) = 1 \otimes F^e(\mathfrak{m}\eta)$ .

Moreover,  $M \subset M^K$  and hence  $F^e(\mathfrak{m}\eta) = 0$  in  $M$ . But  $F$  acts injectively on  $M$ . Therefore,  $\mathfrak{m}\eta = 0$  or  $\eta \in \text{Soc}(M)$ . This shows that  $M_s \subseteq \text{Soc}(M)$ .

To finish the proof, let  $0 \neq \eta \in M_s$ . Then since  $F$  acts injectively on  $M$  we have that  $0 \neq F^e(\eta)$  and  $F^e(\eta) \in M_s$ . So,  $F^e(\eta) \in \text{Soc}_R(M)$ , or  $\mathfrak{m} \cdot F^e(\eta) = 0$  for all  $e$ . Therefore  $M$  is  $F$ -stable, since  $\mathfrak{m} = F - \text{ann}(\eta)$ .  $\square$

**Corollary 2.9.** *Let  $(R, \mathfrak{m}, k)$  be a local Noetherian  $F$ -injective ring.*

*Then the following assertions are equivalent:*

- (1)  $R$  is  $F$ -stable;
- (2) there exists  $i$  and  $0 \neq \eta_i \in H_{\mathfrak{m}}^i(R)$  such that  $\mathfrak{m} = F - \text{ann}(\eta_i)$ ;
- (3) there exists  $i$  such  $H_{\mathfrak{m}}^i(R)_s \neq 0$ .

### 3. F-STABLE PRIMES

In this section we will apply the results of the first section to the local cohomology module of a Cohen-Macaulay local ring.

Let  $(R, \mathfrak{m}, k)$  be a local Noetherian ring of positive characteristic  $p$  and dimension  $d$ . Then  $R$  admits a natural Frobenius action  $F : R \rightarrow R$ , defined by  $F(r) = r^p$  for all  $r \in R$ . This action induces a Frobenius action on the local cohomology modules  $H_{\mathfrak{m}}^i(R)$ ,  $i = 0, \dots, d$ .

If  $R$  is Cohen-Macaulay then there is only one nonzero local cohomology module, namely  $H_{\mathfrak{m}}^d(R)$ . As defined in Section 2, in this case  $R$  is  $F$ -stable if and only if  $H_{\mathfrak{m}}^d(R)$  is  $F$ -stable.

There are two interesting sets of prime ideals in  $R$  that one can define in relation to the action of Frobenius on  $H_{\mathfrak{m}}^d(R)$ .

**Definition 3.1.** Let  $R$  be a Cohen-Macaulay ring of positive characteristic  $p$ .

Let

$$\mathcal{A} = \mathcal{A}(R) = \{P \in \text{Spec}(R) : R_P \text{ is F-stable}\}.$$

If  $(R, \mathfrak{m}, k)$  is local then let

$$\mathcal{B} = \mathcal{B}(R) = \{P \in \text{Spec}(R) : \text{there exists } \eta \in H_{\mathfrak{m}}^d(R) \text{ such that } P = \text{F-ann}(\eta)\}.$$

We remark that Theorem 2.3 implies that when  $R$  is F-injective and Cohen-Macaulay then  $\mathcal{B}(R)$  is finite and  $R$  is F-stable if and only if  $\mathfrak{m} \in \mathcal{B}(R)$  (due to Remark 2.4).

Let  $x_1, \dots, x_d$  be a system of parameters in  $R$ . The local cohomology module  $H_{\mathfrak{m}}^d(R)$  can be obtained as a direct limit of  $R/(x_1^t, \dots, x_d^t)$  where the maps of the direct system are given by

$$R/(x_1^t, \dots, x_d^t) \xrightarrow{\mathbf{x}^{l-t}} R/(x_1^l, \dots, x_d^l),$$

where  $l \geq t$  and  $\mathbf{x} = x_1 \cdots x_d$ .

Let  $x \in R$  and define  $I(x) := \{c \in R : cx^q \in I^{[q]}, \text{ for all } q \gg 0\}$ .

Let  $\bar{x}$  be the class of  $x$  in  $R/(x_1, \dots, x_d)$  and let  $\eta \in H_{\mathfrak{m}}^d(R)$  be the image of  $\bar{x}$  via

$$R/(x_1, \dots, x_d) \rightarrow H_{\mathfrak{m}}^d(R).$$

Then  $\text{F-ann}(\eta) = I(x)$ .

Conversely, consider  $0 \neq \eta \in H_{\mathfrak{m}}^d(R)$  and let  $x$  and  $t$  such that  $\eta$  is the image of  $\bar{x} \in R/(x_1^t, \dots, x_d^t)$  under the natural inclusion

$$R/(x_1^t, \dots, x_d^t) \rightarrow H_{\mathfrak{m}}^d(R).$$

Then let  $J = I_t = (x_1^t, \dots, x_d^t)$ , and note that  $J(x) = I_t(x) = \text{F-ann}(\eta)$ .

**Remark 3.2.** Using the notation just introduced we have

$$\mathcal{B}(R) = \{P \in \text{Spec}(R) : I(x) = P \text{ for some } I = (x_1, \dots, x_d) \text{ generated by parameters and } x \in R\}.$$

These comments allow one to remark to state Propositions 2.6 and 2.7 proven in [3] in the following concise form.

**Theorem 3.3.** *Let  $(R, \mathfrak{m}, k)$  be a local Cohen-Macaulay F-injective complete ring of dimension  $d$ .*

*Then  $\mathcal{B} \subseteq \mathcal{A}$  and  $\text{Max}(\mathcal{A}) = \text{Max}(\mathcal{B})$ .*

It is useful to know how F-stability behaves under a flat local ring extension that has nice fibers.

**Theorem 3.4.** *Let  $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, k)$  be a flat local ring homomorphism of Cohen-Macaulay F-injective rings.*

*Assume that the closed fiber  $S/\mathfrak{m}S$  is regular and  $R$  is F-stable. Then  $\mathfrak{m}S \in \text{Max}(\mathcal{B}(S))$ .*

*In particular if  $(R, \mathfrak{m}, k)$  is local Cohen-Macaulay F-injective and F-stable then  $\widehat{R}$  is local Cohen-Macaulay F-injective and F-stable.*

*Proof.* Since  $R$  is F-stable, then  $\mathfrak{m} = \text{F-ann}(\eta) = I(u)$ , for some  $I = (x_1, \dots, x_d)$  and  $u \in R$ . Since  $I(u) \subseteq I(ru)$ , for any  $r \in R$ , we can arrange that the image of  $u$  in  $R/I$  belongs to  $\text{Soc}(R/I)$ .

Let  $z_1, \dots, z_n$  be a regular system of parameters for  $S/\mathfrak{m}S$ . Note that for any basis of  $\text{Soc}(R/I)$  say  $\overline{u_1}, \dots, \overline{u_l}$ , where  $u_1, \dots, u_l$  belong to  $R$ , their images  $\overline{u_1}, \dots, \overline{u_l}$  in  $S/(I, z_1, \dots, z_n)S$  form a basis for its socle.

Let  $J = (x_1, \dots, x_d, z_1, \dots, z_n)S$ . Note that  $0 \neq \overline{u} \in \text{Soc}(S/J)$ . It is clear that  $\mathfrak{m} \subseteq J(u)$  and so  $\mathfrak{m}S \subseteq J(u)$ .

Let us consider  $c \in S$  such that  $cu^q \in J^{[q]}S$  for all  $q$ .

The induced ring homomorphism  $R \rightarrow S/(z_1^q, \dots, z_n^q)S$  is still flat. Let  $S_q = S/(z_1^q, \dots, z_n^q)S$ .

Let  $cu^q \in J^{[q]}S$  and map this further to  $S_q$ . Keeping the same notations for convenience we see that  $c \in (I^{[q]} :_{S_q} u^q)$ . Since  $R \rightarrow S_q$  is flat we get that  $c \in (I^{[q]} :_R u^q)S_q$  for all  $q$ .

Now,  $\mathfrak{m} = I(u) = \bigcap_q (I^{[q]} : u^q) \subseteq I^{[q]} : u^q \subseteq \mathfrak{m}$ . So,  $\mathfrak{m} = I^{[q]} : u^q$  for all  $q$ .

In conclusion  $c \in \mathfrak{m}S_q$ , for all  $q$ . Pulling back to  $S$  we get that  $c \in \mathfrak{m}S + (z_1^q, \dots, z_n^q)S$  for all  $q$ . So,  $c \in \mathfrak{m}S$  by taking intersection over all  $q$ . □

**Remark 3.5.** In our previous Proposition is necessary to assume that  $S$  is F-injective. This condition cannot be deduced from the other hypotheses as our Proposition 4.2 shows.

In what follows we will present a connection between the prime ideals discussed in Definition 3.1 and a set of prime ideals discovered by Lyubeznik in his work on Frobenius depth. So we will move our attention to the concept of Frobenius depth considered by Harthorne-Speiser, see Definition 6.1 in [6] and Lyubeznik, see Definition 4.12 in [7] and Definition 4.1 in [8].

Assume that  $R$  is Noetherian and let  $P \in \text{Spec}(R)$ . We set  $\text{coht}(P) = \dim(R/P)$ . Let  $k = k(P)$  be the residue field of  $R_P$ . Consider a copy of  $k$  as a coefficient field of  $\widehat{R}_P$  and let  $K = K(P)$  be its perfect closure. We let  $R(P)$  be  $\widehat{R}_P \otimes_k K$  which is a local ring with a perfect residue field. Note that  $R(P)^K$  is the completion of  $R(P)$  at the maximal ideal  $P\widehat{R}_P \otimes K$ . The ring  $R(P)^K$  is Noetherian.

Clearly  $H_{PR_P}^i(R_P) = H_{P\widehat{R}_P}^i(\widehat{R}_P)$  and  $H_{PR(P)}^i(R(P)) = H_{P\widehat{R}_P}^i(\widehat{R}_P) \otimes_k K$ .

Also since  $R(P)^K$  is the completion of  $R(P) \otimes_k K$  we also have that

$$H_{PR(P)^K}^i(R(P)^K) = H_{PR(P)}^i(R(P)).$$

Since  $H_P^i(\widehat{R}_P)$  is Artinian we in fact have that

$$H_{P\widehat{R}_P}^i(\widehat{R}_P)^K = H_{P\widehat{R}_P}^i(\widehat{R}_P) \otimes_k K,$$

which gives

$$H_{P\widehat{R}_P}^i(\widehat{R}_P)^K = H_{PR(P)}^i(R(P)) = H_{PR(P)^K}^i(R(P)^K),$$

and hence are in position to apply results of Theorem 2.6. In conclusion,  $H_{PR(P)}^i(R(P))_s \neq 0$  if and only if  $H_{P\widehat{R}_P}^i(\widehat{R}_P)_s \neq 0$ .



**Definition 3.6** (Hartshorne-Speiser). Let  $R$  be a Noetherian ring with  $\dim(R) < \infty$ . Using the notation introduced above, we say that the Frobenius depth of  $R$ , denoted  $\text{F-depth}_{HS}(R)$ , is

$$\text{F-depth}_{HS}(R) = \max\{r : H_{PR(P)}^i(R(P))_s = 0 \text{ for all } i < r - \text{coht}(P), \text{ for all } P \in \text{Spec}(R)\}.$$

The considerations above allows us to simplify this to

$$\text{F-depth}_{HS}(R) = \max\{r : H_{P\widehat{R}_P}^i(\widehat{R}_P)_s = 0 \text{ for all } i < r - \text{coht}(P), \text{ for all } P \in \text{Spec}(R)\}.$$

To be able to parse through this concept more easily it is helpful to first introduce a local concept of Frobenius depth.

**Definition 3.7.** Let  $(R, \mathfrak{m})$  be a local ring. Then the F-depth of  $R$  at  $\mathfrak{m}$ , denoted  $\text{F-depth}(\mathfrak{m}, R)$ , is

$$\text{F-depth}(\mathfrak{m}, R) = \max\{r : H_{\mathfrak{m}}^i(\widehat{R})_s = 0 \text{ for all } i < r\} = \min\{r : H_{\mathfrak{m}}^r(\widehat{R})_s \neq 0\}.$$

It is easy to conclude that

**Remark 3.8.**  $\text{F-depth}(R) = \text{Min}\{\text{F-depth}(PR_P, R_P) + \text{coht}(P) : P \in \text{Spec}(R)\}.$

Note that  $\text{coht}(\mathfrak{m}) = 0$ , so the inequality above implies that  $\text{F-depth}(\mathfrak{m}, R) \leq \text{F-depth}(R)$ .

Whenever  $R$  is zero dimensional, we can see that  $H_{\mathfrak{m}}^0(R) = R$  and  $R_s = k \neq 0$ . So, in this case  $\text{F-depth}(\mathfrak{m}, R) = 0$ .

Assume that  $P$  is a minimal prime ideal of  $R$ ,  $R_P$  is a local ring of dimension zero and we obtain  $0 \leq \text{F-depth}(R) \leq \dim(R/P)$ .

If  $R$  is Cohen-Macaulay, then its localizations at prime ideals as well as their completions are Cohen-Macaulay, so  $\text{F-depth}(P, R_P)$  either equals  $\dim(R_P)$  or  $\infty$ . This implies that  $\text{F-depth}(R) = \dim(R)$  if  $R$  is Cohen-Macaulay.

Using his theory of  $F$ -modules, Lyubeznik has proven the following theorem.

**Theorem 3.9.** (Lyubeznik) *Let  $R$  be a homomorphic image of a finite type algebra over a regular local ring.*

*Then there exist only a finite number of prime ideals  $P$  in  $R$  such as  $H_{P(\widehat{R}_P)}^k(\widehat{R}_P)_s \neq 0$ .*

*Proof.* Proposition 4.14 in [7] states that there exist only finitely many prime ideals  $P$  such that  $H_{P\widehat{R}_P}^i(\widehat{R}_P^K)_s \neq 0$

But as noticed  $H_{P\widehat{R}_P}^i(\widehat{R}_P^K)_s \neq 0$  if and only if  $H_{P\widehat{R}_P}^k(\widehat{R}_P)_s \neq 0$ .

□

This leads us to consider the following set.

**Definition 3.10.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian ring of positive characteristic  $p$ .

Let  $\mathcal{C}(R) = \{P \in \text{Spec}(R) : \text{there exists } k \text{ such that } H_{P(\widehat{R}_P)}^k(\widehat{R}_P)_s \neq 0\}.$

The investigations of Section 1 allow us to state the following result.

**Proposition 3.11.** *Let  $R$  be a Cohen-Macaulay  $F$ -injective ring.*

*Then*

$$\mathcal{A}(R) = \mathcal{C}(R).$$

*Proof.* Since the F-injectivity property localizes, we conclude that  $R_P$  is F-injective for any prime ideal  $P$  of  $R$ . Also,  $R_P$  is Cohen-Macaulay as well.

Let  $P \in \mathcal{A}(R)$ , that is,  $R_P$  is F-stable. Then  $\widehat{R}_P$  is F-stable as well. The ring  $\widehat{R}_P$  is complete and we can apply Theorem 2.8 to it. Assume that  $\text{ht}(P) = k$ . Then  $H_P^k(\widehat{R}_P)_s \neq 0$ . Therefore,  $P \in \mathcal{C}(R)$ .

Conversely, if for a prime ideal  $P$  we have that  $H_P^k(\widehat{R}_P)_s \neq 0$  then  $k = \text{ht}(P)$  and  $\widehat{R}_P$  is F-stable as a consequence of Theorem 2.8. This implies that  $R_P$  is F-stable and hence  $P \in \mathcal{A}(R)$ . □

**Corollary 3.12.** *Let  $R$  be a Cohen-Macaulay F-injective ring. Assume that  $R$  is a homomorphic image of an algebra of finite type over a regular local ring.*

*Then  $\mathcal{A}(R)$  is finite.*

The concept of Frobenius depth can appear a little technical at a first glance. Harthorne and Speiser used it to give answers to an important problem stated by Grothendieck: Let  $A$  be a commutative ring and let  $I \subset A$  be an ideal. If  $n$  is an integer, find conditions under which  $H_i^I(M) = 0$  for all  $i > n$  and all  $A$ -modules  $M$ . For their answer this problem we refer the reader to [6].

More recently Lyubeznik introduced the following variant of Frobenius depth, Definition 4.1 in [8] and proved that it coincides with the earlier introduced concept of Hartshorne-Speiser under mild conditions.

**Definition 3.13** (Lyubeznik). Let  $(R, \mathfrak{m}, k)$  be a local Noetherian ring of positive characteristic. The  $F$ -depth of  $R$  is the smallest  $i$  such that  $F^s$  does not send  $H_{\mathfrak{m}}^i(R)$  to zero for any  $s$ . We will denote this number by  $\text{F-depth}_L(R)$ .

**Theorem 3.14** (Lyubeznik). *Let  $R$  be a local ring which is a homomorphic image of a regular local ring. Then*

$$\text{F-depth}_{HS}(R) = \text{F-depth}_L(R).$$

Lyubeznik has also given interesting characterizations for the cases  $\text{F-depth}_L(R) \leq 1$ . Singh and Walther have added to these results by proving the following interesting theorem. We will show later that one cannot replace the hypothesis  $k$  algebraically closed and hope to obtain the same result.

**Theorem 3.15** (Singh-Walther). *Let  $(R, \mathfrak{m}, k)$  be a complete local ring of positive characteristic. Assume that the residue field  $k$  is algebraically closed. Then the number of connected components of the punctured spectrum  $\text{Spec}(R) \setminus \{\mathfrak{m}\}$  is*

$$1 + \dim_k(H_{\mathfrak{m}}^1(R))_s.$$

#### 4. EXAMPLES

The reader can note that the F-injectivity assumption is crucial in our treatment of F-stability in Section 2. At times one needs to enlarge the residue field of a ring to its perfect closure and this brings up a natural question which is of interest in own right: is F-injectivity preserved under those circumstances?

To be more precise, let us formulate the following question.

**Question 4.1.** Let  $(R, \mathfrak{m}, k)$  be a local Noetherian F-injective ring of positive characteristic  $p$ . Assume that  $k$  is a coefficient field of  $R$ , that is the composition of the maps  $k \hookrightarrow R \rightarrow R/\mathfrak{m} = k$  is the identity map. Let  $k \rightarrow k'$  be a purely inseparable extension. Is the ring  $k' \otimes_k R$  F-injective? Is  $k' \widehat{\otimes}_k R$  F-injective?

Let  $K$  be the perfect closure of  $k$ , that is  $K = k^{1/p^\infty}$ . The ring  $S = K \otimes_k R$  is local with maximal ideal  $\mathfrak{m} \otimes_k K$ . By Dumitrescu (Theorem 4.8 [2]), the ring  $S$  is local and Noetherian with maximal ideal  $\mathfrak{m}_e = \mathfrak{m} \otimes_k K$ . Since  $K \widehat{\otimes}_k R$  is the completion of  $S$  at the maximal ideal it is Noetherian as well. Also by flat base change we see that  $k' \otimes_k R \subset K \otimes_k R$  is a flat local extension and hence the ring  $k' \otimes_k R$  is Noetherian as well.

Since F-injectivity commutes with completion (see for example Lemma 2.7 in [4]) we will examine whether  $S$  is F-injective when  $R$  is.

We will present an example that provides a negative answer to the question. The question is however still open if we further assume that  $R$  is normal.

Let  $k \subset L$  be an algebraic field extension and  $x$  an indeterminate. Consider  $R = k + xL[[x]]$  which is a local Noetherian with maximal ideal  $\mathfrak{m} = xL[[x]]$  and residue field equal to  $k$ . This ring is one dimensional complete domain. This example was considered in [4], Example 2.16.

In [4] it was proven that  $R$  is F-injective if and only if

$$L^p \cap k = k^p.$$

Our claim is that for a suitable extension  $k \subset L$ , the ring  $k^{1/p} \otimes_k R$  is not F-injective. In fact, it is not even reduced.

Now note that  $k^{1/p} \subset K$  and hence  $k^{1/p} \otimes_k R \subset K \otimes_k R$  and therefore  $K \otimes_k R$  cannot be reduced when  $k^{1/p} \otimes_k R$  is not reduced.

**Proposition 4.2.** *Let  $k \subset L$  be a finite algebraic extension such that  $L^p \cap k = k^p$  and  $k \subset L$  is not separable.*

*Let  $x$  be an indeterminate and consider  $R = k + xL[[x]]$ .*

- (1)  *$R$  is F-injective local complete F-stable 1-dimensional ring.*
- (2)  *$R \otimes_k k^{1/p}$  is not reduced, hence not F-injective.*

*Proof.* (1) It is immediate that  $R$  is local, one dimensional and complete. For a proof of the F-injectivity our  $R$  we refer to Example 2.16 in [4].

Note that  $x$  is a parameter for  $R$ . Let  $I = xR$ . We will show that there exist  $u \in R$  such that  $\mathfrak{m} = I(u) = \{c \in R : cu^q \in I^{[q]} \text{ for all } q\}$ . If this holds then, according to Remark 3.2,  $\mathfrak{m} \in \mathcal{B}(R)$  which is equivalent to the F-stability of  $R$ , as  $R$  is Cohen-Macaulay.

Let  $a \in L \setminus k$ . Then  $ax \in \mathfrak{m} = xL[[x]]$ , but it is not in  $I$ .

Let  $c \in \mathfrak{m}$  arbitrary. Then  $c(ax)^q$  is a formal power series of order at least  $q + 1$  which means that it belongs to  $I^{[q]} = x^q R$ . So if we let  $u = ax$  then  $\mathfrak{m} \subseteq I(u)$ . But  $I(u) = R$  if and only if  $u \in I^F = I$  which is not the case, as  $a \in L \setminus k$ .

(2) Since  $k \subseteq L$  is not separable, then  $k^{1/p} \otimes_k L$  is not reduced.

Let  $u = \sum_{i=1}^h a_i^{1/p} \otimes_k b_i$ , where  $a_i \in k, b_i \in L, i = 1, \dots, h$  such that  $u^p = 0$  but  $u \neq 0$ . Note that  $u^p = \sum_{i=1}^h a_i \otimes_k b_i^p = 1 \otimes \sum_{i=1}^h a_i b_i^p$ , which is equivalent to  $\sum_{i=1}^h a_i b_i^p = 0$  in  $L$ .

Consider  $v = \sum_{i=1}^h a_i^{1/p} \otimes_k (b_i \cdot x)$  as an element of  $k^{1/p} \otimes_k R$ . Now,  $v^p = 1 \otimes [(\sum_{i=1}^h a_i b_i^p) x^p] = 0$ . Let us argue that  $v \neq 0$ . Regard  $L \cdot x$  as a  $k$ -vector space and note that  $k^{1/p} \otimes_k L \simeq k^{1/p} \otimes_k L \cdot x$  as  $k$ -vector spaces. Moreover,  $L \cdot x \subset R$  as  $k$ -vector spaces and hence  $k^{1/p} \otimes_k L \cdot x \subset$

$k^{1/p} \otimes_k R$ . Under the isomorphism the nonzero element  $u \in k^{1/p} \otimes_k L$  corresponds to  $v \in k^{1/p} \otimes_k L \cdot x \subset k^{1/p} \otimes_k R$ . In conclusion,  $v$  is nonzero in  $k^{1/p} \otimes_k R$  as well.  $\square$

We recall the Example 2.14 in [4] exhibits a finite extension  $k \subset L$  such that  $L^p \cap k = k^p$  and  $k \subset L$  is not separable. Let  $F$  be an infinite perfect field of characteristic  $p$  and set  $k = F(u, v)$ . Let  $L = k[y]/(y^2p + y^pu - v)$ . In [4] it was proven that  $L^p \cap k = k^p$  and  $[L : k(L^p)] \geq p > 1$ . It is known that a finite extension  $k \subset L$  is separable if and only if  $k(L^p) = L$ .

Using the notations in the preceding Proposition, it is interesting to note that  $R \rightarrow k^{1/p} \otimes_k R$  is a flat local map with closed fiber equal to  $k^{1/p}$ , a field. It in this example the F-injectivity of  $R$  does not pass to  $k^{1/p} \otimes_k R$ . Much of this has to do with the fact that  $k^{1/p}$  is not separable over  $k$ , where  $k$  is the residue field of  $R$ . Indeed, whenever the residue field extension is separable F-injectivity is preserved under flat local maps. The following result is essentially contained in [1], Theorem 4.2, although not stated for F-injective rings. We present the argument here in our context.

Let  $F$  be a  $k$ -algebra where  $k$  is a field. We say that  $F$  is *geometrically F-injective* over  $k$ , if, for every finite field extension  $k \subseteq k'$  the ring  $k' \otimes_k R$  is F-injective. Note that a field extension  $L$  of  $k$  is geometrically F-injective if and only if  $k \subset L$  is separable.

**Theorem 4.3** (Aberbach-Enescu). *Let  $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, L)$  be a flat local map such that  $S/\mathfrak{m}S$  is Cohen-Macaulay and geometrically F-injective over  $k$ . If  $R$  is F-injective and Cohen-Macaulay, then  $S$  is F-injective and Cohen-Macaulay.*

*Proof.* The proof follows as in Theorem 4.2 (1) in [1] with minor changes: replace tight closure by Frobenius closure and use  $c = 1$ .  $\square$

We would like to end by showing that Theorem 3.15 cannot be extended to rings with non-algebraically closed residue field. Also a result on one dimensional F-injective rings is provided as well.

**Remark 4.4.** Let  $k \subseteq L$  be a field extension that satisfies the conditions of Prop 4.2. Then  $R = k + xL[[x]]$  is a domain and hence the punctured spectrum is connected, while  $\dim_k H_{\mathfrak{m}}^1(R)_s \neq 0$ . The latter claim follows since  $R$  is F-injective and F-stable hence by Theorem 2.8  $H_{\mathfrak{m}}^1(R)_s \neq 0$ .

**Proposition 4.5.** *Let  $(R, \mathfrak{m}, k)$  be a local complete F-injective ring of dimension 1. Assume that  $k$  is algebraically closed and  $R$  is domain. Then  $R$  is regular.*

*Proof.* The punctured spectrum  $\text{Spec}(R) \setminus \{\mathfrak{m}\}$  is connected since  $R$  is domain. According to Theorem 3.15, we have that  $H_{\mathfrak{m}}^1(R)_s = 0$ . But Theorem 2.8 implies that  $R$  is not F-stable. This implies that  $R$  is F-rational, hence regular as  $\dim(R) = 1$ .  $\square$

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