BRIANÇON-SKODA FOR NOETHERIAN FILTRATIONS

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Abstract. In this note the Briançon-Skoda theorem is extended to Noetherian filtrations of ideals in a regular ring. The method of proof couples the Lipman-Sathaye approach with results due to Rees.

Let A be a regular ring of dimension d and I an ideal of A. Let $\overline{I}$ denote the integral closure of I. The Briançon-Skoda theorem asserts that if I is generated by l elements then $\overline{I^{n+l}} \subseteq I^{n+1}$, for all nonnegative integers n. Moreover, $\overline{I^{n+d}} \subseteq I^{n+1}$ for all nonnegative integers n. Both statements were proven by Lipman-Sathaye [4]. These results have generated a considerable number of papers in commutative algebra and algebraic geometry, for a general discussion see for example Chapter 13 in [7] and Chapter 9 in [3].

In this paper, our aim is to prove a Briançon-Skoda type theorem for Noetherian filtrations. Our treatment will follow classical arguments by Lipman and Sathaye, and, respectively, Rees.

Let $\mathcal{F} = \{I_n\}_n$ be a filtration of ideals of A: that is, $I_0 = A$, $I_{n+1} \subseteq I_n$, and $I_n I_m \subseteq I_{n+m}$, for all nonnegative n, m. For any nonnegative integer k, let $\mathcal{F}(k) = \{I_{n+k}\}_n$ (technically this is not a filtration according to the above definition since on the zeroth spot we have I_k and not A, but this will not affect what follows). Also, given two filtrations $\mathcal{F} = \{I_n\}_n$ and $\mathcal{G} = \{J_n\}_n$, we write $\mathcal{F} \subseteq \mathcal{G}$ if $I_n \subseteq J_n$ for all n.

The filtration $\mathcal{F}$ is called *Noetherian* if its Rees algebra $R = \oplus_{n \geq 0} I_n t^n$ is Noetherian over A. This holds if and only if its extended Rees algebra $S = R[t^{-1}] \subset A[t^{-1}, t]$ is Noetherian.

There are various definitions of Noetherian filtrations in the literature. We follow the terminology used by Rees in [6], although the reader should be aware that for sake of readability we will avoid the interpretation of filtrations as special real valued functions on the ring A. What we call here Noetherian filtration is called in some papers an essentially power filtration, Definition (2.1.2) and Remark (2.2) in [1].

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We note that the filtration given by the powers of a single ideal in a Noetherian ring forms itself a Noetherian filtration.

Another characterization of Noetherian filtration is as follows: A filtration $\mathcal{F} = \{I_n\}_n$ is Noetherian if and only if there exists $m \geq 1$ such that for all n, $I_n = \sum I_1^{e_1} \cdots I_m^{e_m}$, where the sum ranges over all nonnegative integers $e_1, \ldots, e_m$ such that $e_1 + 2e_2 + \cdots + me_m = n$. This was proven by Ratliff, see [5] and [1], Remark (2.2).

The integral closure of R in $A[t]$ is a $\mathbf{N}$-graded ring, $\overline{R} = \oplus_{n \geq 0} J_n t^n$. As in [6], the integral closure of $\mathcal{F}$ is then defined by $\overline{\mathcal{F}} = \{J_n\}_n$. This is equivalent to saying that $x \in J_n$ if and only if there exist elements $a_i \in I_{ni}$ and a positive integer m such that

$$x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m = 0.$$

Some authors call the filtration $\mathcal{G} = \{\overline{I_n}\}_n$ the integral closure of the filtration $\mathcal{F} = \{I_n\}_n$. It should be noted that $\mathcal{G} \subseteq \overline{\mathcal{F}}$ with our definition. Note that J_n belongs to the radical of I_n, so the filtration and its integral closure share the same radical (the radical of a filtration is the radical of any of its components).

A reduction for $\mathcal{F} = \{I_n\}$ is a filtration $\mathcal{G} = \{L_n\}_n$ such that $\mathcal{G} \subseteq \mathcal{F}$ and with the same integral closure filtration. A reduction is called basic if its Rees algebra is generated over A by the least number of elements.

Recently, Küronya and Wolfe have studied extensions of the Briançon-Skoda theorem to graded systems of ideals. A family of ideals of A, $\mathbf{a}_\bullet = (\mathbf{a}_n)_n$ is called a graded system of ideals if $\mathbf{a}_n \mathbf{a}_m \subseteq \mathbf{a}_{n+m}$ for all nonnegative n, m. One should note that the family of ideals is not assumed descending. Küronya and Wolfe established a Briançon-Skoda type theorem for a particular type of graded systems, named stable graded system of ideals, that arise in algebraic geometry. Their generalization states that for a stable graded system of ideals $\mathbf{a}_\bullet$, there exists a positive constant C such that, for all $n \gg 0$, $\overline{\mathbf{a}_{Cn}} \subseteq \mathbf{a}_n$ (see Corollary 3.4 in [2]). Our statement will be stronger than this but under different hypotheses. The authors obtain in fact a statement regarding multiplier ideals of an graded system of ideals, as defined in Lazarsfeld [3]. For details on this statement and the definition of stable graded systems of ideals we refer the reader to [2].

Given a filtration $\mathcal{F} = \{I_n\}_n$, we call $a_1 t^{k_1}, \ldots, a_h t^{k_h}$ a system of generators for $\mathcal{F}$ if $a_1 t^{k_1}, \ldots, a_h t^{k_h}, u = t^{-1}$ generate the extended Rees algebra $(\oplus_{n \geq 0} I_n t^n)[u] = A[u, I_n t^n : n \geq 0]$. It can be arranged that $a_i \in I_{k_i} \setminus I_{k_i+1}$, $i = 1, \ldots, h$. The numbers k_i are referred to as degrees of the generators.

Before we state the main result, we need to introduce more notations.

For a filtration $\mathcal{G} = \{L_n\}_n$, let $g_1t^{k_1}, \ldots, g_ht^{k_h}$ be a minimal set of generators of $\mathcal{G}$. Let $k :=$ the least common multiple of $k_1, \ldots, k_s = [k_1, \ldots, k_s]$.

Theorem. *Let A be a d-dimensional regular ring and let $\mathcal{F} = \{I_n\}$ be a Noetherian filtration of ideals of A.*

For a reduction of $\mathcal{F}$, let l denote the sum of the degrees of the generators of the reduction. Also, consider $\mathcal{G} = \{L_n\}_n$ a basic reduction of $\mathcal{F}$, and let k defined as above corresponding to $\mathcal{G}$.

Then

$$\overline{\mathcal{F}}(m) \subset \mathcal{F},$$

where $m := \min\{l-1, kd-1\}$.

It is clear that in the case $\mathcal{F} = \{I^n\}_n$, with I ideal of A, we obtain the standard Briançon-Skoda theorem since in this case l can be taken to equal the number of generators of I and $k = 1$.

In his Strong Valuation Theorem, Rees has already shown that there exist a positive integer k such that $\overline{\mathcal{F}}(k) \subset \mathcal{F}$ Theorem 5.33, [6]. However, the integer k produced by this result depends upon the degrees of the generators of the integral closure $\overline{R}$ over R which are hard to estimate even in the classical case of a filtration of the type $\{I^n\}_n$, for an ideal I of A.

Proof. The proof of the theorem will follow closely the Lipman-Sathaye proof of the standard Briançon-Skoda theorem. We found the exposition in [7] particularly useful and we will follow it for the first part of the proof.

First we will show that $\overline{\mathcal{F}}(l-1) \subseteq \mathcal{F}$.

For a finitely generated extension of rings $A \subseteq B$, we will write $J_{B/A}$ for the Jacobian ideal of B over A.

For a reduction $\mathcal{F}' = \{I'_n\}$ of $\mathcal{F}$, we have that $\overline{\mathcal{F}'} = \overline{\mathcal{F}}$ and $\mathcal{F}' \subseteq \mathcal{F}$, so if show that $\overline{\mathcal{F}'}(l-1) \subseteq \mathcal{F}'$, then this implies that $\overline{\mathcal{F}}(l-1) \subseteq \mathcal{F}' \subseteq \mathcal{F}$.

We will work with the reduction $\mathcal{F}'$ and, by relabeling, we will still call it $\mathcal{F}$.

We can localize and assume hence that our regular ring A is local.

For the Noetherian filtration $\mathcal{F}$, call its minimal generators $f_1t^{l_1}, \ldots, f_rt^{l_r}$, with $f_i \in I_{l_i}$.

Consider the extended Rees algebra of our filtration $S = A[u, I_nt^n : n \geq 0]$ where $u = t^{-1}$. Clearly we can rewrite $S = A[u, f_1t^{l_1}, \ldots, f_rt^{l_r}] = A[u, f_1/u^{l_1}, ..., f_r/u^{l_r}]$. Now let $B = A[u]$. Since $S = B[f_1/u^{l_1}, ..., f_r/u^{l_r}]$, a standard argument allows us to conclude that $u^{l_1+\cdots+l_r} \in J_{S/B}$ (see Lemma 13.3.1 in [7]). Let $l = l_1 + \ldots + l_r$.

Let $\overline{S}$ be the integral closure of S. Since S is finitely generated over $A[u] = B$, B is regular and the fraction field of S is separable over B, we get that $\overline{S}$ is module finite over S.

We need to apply the following important result

Theorem (Lipman-Sathaye, Theorem 2, [4] or Theorem 12.3.10, [7])**.** *Let R be a Cohen-Macaulay domain with field of fractions K. Let S be a domain that is finitely generated R-algebra. Assume that the field of fractions of S is separable and finite over K and that the integral closure $\overline{S}$ of S is a finitely generated S-module. Assume that for all prime ideal Q in S of height one, $R_{Q\cap R}$ is a regular local ring. Then*

$$\overline{S} :_L J_{\overline{S}/R} \subseteq S :_L J_{S/R}.$$

In particular, $J_{S/R}\overline{S} \subseteq S$.

The fraction field of S is the fraction field of B, so Lipman-Sathaye Theorem applied to B and S gives that $J_{S/B}\overline{S} \subset S$. In particular $u^l\overline{S} \subset S$.

At this stage we need another difficult result by Lipman-Sathaye:

Proposition (Lipman-Sathaye, Lemma, [4] or Theorem 13.3.2, [7])**.** *Let R be a regular domain with field of fractions K. Let L be finite separable field extension of K and S be a finitely generated R-algebra in L with integral closure T. Let $0 \neq t$ be such that R/tR is regular. Then if $ts \cap R \neq tR$, then $J_{T/R} \subseteq tT$.*

We can check that $I_1 \subset uS \cap B$, but not in uB, and B/uB is regular. The above Proposition applies and gives $J_{\overline{S}/B} \subset u\overline{S}$.

So, $u^{-1}\overline{S} \subset \overline{S} : J_{\overline{S}/B} \subset S : J_{S/B}$ by 12.3.10 ($\overline{S}$ is module finite over S) so $u^{-1}J_{S/B}\overline{S} \subset S$ which gives $u^{l-1}\overline{S} \subset S$.

But $\overline{S} = \oplus_n K_n t^n$ so $K_n t^{n-l+1} \subseteq I_{n-l+1}t^{n-l+1}$. In particular

$$J_{n+l-1} \subseteq K_{n+l-1} \subseteq I_n,$$

or

$$\overline{\mathcal{F}}(l-1) \subseteq \mathcal{F}.$$

Now we will show that $\overline{\mathcal{F}}(kd-1) \subseteq \mathcal{F}$.

For every positive integer k and every ideal J of A one can define a filtration denoted kJ in the following way:

$$(kJ)_n = J^{\lceil n/k \rceil},$$

for all nonnegative n.

We can localize at a prime ideal containing the radical of the filtration $\mathcal{F}$, we can assume that we are in the local case.

Rees has proved that given a Noetherian filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ there exists a positive integer k and an ideal J such that $\mathcal{F}$ and kJ are equivalent, that is they have the same integral closure, Theorem 6.12 and its Corollary, [6].

In fact this number k is obtained by Rees as described in the statement of the theorem. Referring to the notations introduced just above the theorem, one chooses first a basic reduction $\mathcal{G} = \{L_n\}_n$ for $\mathcal{F}$. For $r_i = k/k_i$, let $a_i = g_i^{r_i}$. The ideal J mentioned in the paragraph above is $J = (a_1, \ldots, a_h)$ and moreover the filtration kJ represents a basic reduction for $\mathcal{F}$.

As before, let us denote the integral closure of $\mathcal{F} = \{I_n\}_n$ by $\{J_n\}_n$, and hence, by the above paragraph, this also represents the integral closure of kJ.

According to the definition of the integral closure of a filtration, we see that an element x belongs to J_n if and only if there exist elements $a_i \in (kJ)_{ni}$ and a positive integer m such that

$$x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m = 0.$$

Since $((kJ)_n)^i \subset (kJ)_{ni}$, it follows that $\overline{(kJ)_n} \subset J_n$.

We would like to remark that

$$\lceil \frac{n-k+1}{k} \rceil i \leq \lceil \frac{ni}{k} \rceil :$$

this follows easily, since $\lceil \frac{n-k+1}{k} \rceil i = \lceil \frac{n+1}{k} \rceil i - i$ and $\lceil \frac{n+1}{k} \rceil i - i \leq (\lceil \frac{n}{k} \rceil + 1)i - i \leq \lceil \frac{ni}{k} \rceil$.

With this in mind we see that $(kJ)_{ni} \subseteq ((kJ)_{n-k+1})^i$ which implies that for $n \geq k$, $J_n \subseteq \overline{(kJ)_{n-k+1}}$.

Now we are in position to apply Briançon-Skoda for ideals in regular rings of dimension d:

$J_n \subseteq \overline{(kJ)_{n-k+1}} = \overline{J^{\lceil \frac{n-k+1}{k} \rceil}} \subseteq J^{\lceil \frac{n-k+1}{k} \rceil - (d-1)} = (kJ)_{n-kd+1} \subseteq I_{n-kd+1}$.

Putting everything together,

$$\overline{\mathcal{F}}(kd-1) \subset \mathcal{F}.$$

$\square$

We would like to illustrate our result with an example.

Let $A = k[[x, y]]$ where k is a field, $I_1 = (x, y^2), I_2 = (x,^2, xy^2, y^3)$. Note that $I_1^2 \subset I_2 \subset I_1$. Define $I_n = \sum I_1^{e_1} I_2^{e_2}$, where sum ranges over all nonnegative integers e_1, e_2 such that $e_1 + 2e_2 = n$. The filtration $\mathcal{F} = \{I_n\}_n$ is Noetherian and its extended Rees algebra

$$S = A[t^{-1}, I_n t^n : n \geq 0]$$

is generated by xt, y^2t, y^3t^2.

According to results by Rees, Theorem 6.12 and Lemma 6.13 in [6], we know that a basic reduction of $\mathcal{F}$ must have 2 generators. Note that y^2t is integral over $S : (y^2t)^2 - y(y^3t^2) = 0$. So, if $S' = A[xt, y^3t^2, t^{-1}]$ then the integral closure of S' is S. The generators are xt, y^3t^2 which live in degrees 1 and 2. Applying the Theorem, we get that $m = 2$, so

$$\overline{\mathcal{F}}(2) \subseteq \mathcal{F}.$$

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