

LECTURE 21

1. THE HILBERT-SAMUEL FUNCTION

This next result clarifies the degree of the polynomial describing the Hilbert function of a graded module.

Theorem 1.1. (*Hilbert polynomial*)

Let M be a finitely generated graded over $R = R_0[R_1]$ where R is Noetherian, R_0 is Artinian, and $\dim M = d$. Then there exists a polynomial HP of degree $d - 1$, called the Hilbert polynomial of M , such that for all $n \gg 0$ $H_M(n) = HP(n)$ (the zero polynomial will have degree 1 by convention here).

Proof. We will use induction on d .

The case $d = 0$ is immediate since in this case M is Artinian and so $M_n = 0$ for all $n \gg 0$, hence $H_M(n) = 0$ for $n \gg 0$. First let us prove the statement for the case $M = R/p$ where p is a graded prime ideal of R .

Assume $d > 0$ and so let x be a homogeneous element of R/P of degree 1 (this is possible because R is standard graded). We have the following short exact sequence:

$$0 \rightarrow (R/P)(-1) \xrightarrow{x} R/P \rightarrow R/(xR + P) \rightarrow 0.$$

Apply the additivity of the length function in each degree n and get

$$H_M(n) - H_M(n - 1) = H_N(n),$$

where $N = R/(xR + P)$. Since the dimension of N is $d - 1$ and H_N is a function of polynomial type of degree $d - 2$ it follows from the previous equality that H_M is of polynomial type of degree $(d - 2) + 1 = d - 1$.

For the general case, note that, similar to the ungraded case, there is prime filtration for M : $0 = M_0 \subset M_1 \subset M_i \subset M_n = M$, where $M_{i+1}/M_i \simeq (R/p_i)(a_i)$ where p_i are

graded prime ideals in R and $a_i \in \mathbb{Z}$. The additivity of the length function gives that $H_M(n) = \sum_{i=1}^n H_{(R/p_i)(a_i)}$. Note that d is indeed the maximum of all dimensions of R/p_i and the Hilbert polynomials have nonnegative leading coefficients since they define a nonnegative function. In conclusion, the case R/p implies the general case. \square

Definition 1.2. Let M be a finitely generated graded over $R = R_0[R_1]$ where R is Noetherian, R_0 is Artinian, and $\dim M = d$. The Hilbert iterated functions are defined as $H_{M,0} : \mathbb{N} \rightarrow \mathbb{Z}$, where $H_{M,0} = H_M$, and $H_{M,i} = \sum_{j \leq n} H_{M,i-1}(j)$. We take it as fact that the latter is a function of polynomial type with degree $d + i - 1$ (the reader can try it as an exercise based upon Theorem 1.1).

We write $HP(n) = \frac{e}{(d-1)!} n^{d-1} + \text{lower degree terms}$. By definition the multiplicity of M , $e(M)$, is e if $d > 0$ and $l(M)$ if $d = 0$.

Note that as a consequence $H_{M,1}(n) = \sum_{i=0}^d a_i \binom{n+i}{i}$, for $n \gg 0$, with $a_d = e(M)$, and the sum is of polynomial type with degree d .

Example 1.3. The curve $x^2 - y^2 - y^3 = 0$ defines a one-dimensional ring $\frac{k[x, y]}{(x^2 - y^2 - y^3)}$. The ring is not graded, however we still want to find a natural way to define the multiplicity of the curve at the origin.

Now we will lead up to the Hilbert Samuel function of a finitely generated module M over a semi-local (has finitely many maximal ideals), Noetherian ring R . First, we need some definitions.

Definition 1.4. Let $\mathfrak{m} = \text{Jac}(R) = \bigcap_{m \in \text{Max}(R)} m$. Remember that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Let $I \leq R$. Then I is said to be an **ideal of definition** for R if $\exists \theta > 0$, integer, such that $\mathfrak{m}^\theta \subset I \subset \mathfrak{m}$.

Definition 1.5. Let $gr_I(M) = \bigoplus_{n=0}^{\infty} \frac{I^n M}{I^{n+1} M}$ over $gr_I(R) = \bigoplus_{n=0}^{\infty} \frac{I^n}{I^{n+1}}$ where $I^n = R$ for $n = 0$. Let M be finitely generated over R , and $\dim M = d$. The **Hilbert-Samuel function** is $HS_M(n) = \ell \left(\frac{M}{I^{n+1} M} \right)$.

Consider the exact sequence

$$0 \rightarrow \frac{I^n M}{I^{n+1} M} \rightarrow \frac{M}{I^{n+1}} \rightarrow \frac{M}{I^n M} \rightarrow 0.$$

The exactness implies $\ell\left(\frac{I^n M}{I^{n+1} M}\right) = \ell\left(\frac{M}{I^n M}\right) - \ell\left(\frac{M}{I^{n+1} M}\right)$, which says $\text{HS}_M(n) = \text{Hgr}_{I(M),1}(n)$. This can be shown through the following string:

$$\text{Hgr}_{I(M),1}(n) = \sum_{j \leq n} \text{Hgr}_{I(M)}(j) = \sum_{j \leq n} \ell([\text{gr}_I(M)]_j) = \sum_{j \leq n} \ell\left(\frac{I^j M}{I^{j+1} M}\right) = \ell\left(\frac{M}{I^{n+1} M}\right) = \text{HS}_M(n).$$

Hence by Theorem 1.1 and above considerations, the Hilbert-Samuel function of M is of polynomial type with degree d , where $d = \dim(\text{gr}_I(M))$.

Let R be semilocal and Noetherian, $\mathfrak{m} = \text{Jac}(R)$. Take $I \leq R$ such that $\exists \theta > 0$ with $\mathfrak{m}^\theta \subseteq I \subseteq \mathfrak{m}$. Let M be a finitely generated R -module. We want to show $\deg HSP_{M,I} = \dim M$, where $PS_{M,I}$ is the Hilbert-Samuel polynomial of M with respect to I . We know $HP_{\text{gr}_I(M)}$ has degree equal to $\dim \text{gr}_I(M) - 1$, and hence $HSP_{M,I}$ has degree equal to $\dim \text{gr}_I(M)$. So it suffices to show $\dim M = \dim \text{gr}_I(M)$.

Denote the degree of $HSP_{M,I}$ by the number \tilde{d}_M . This does not depend on I , our ideal of definition. In fact, $\mathfrak{m}^{n\theta+1} \subseteq I^{n+1} \subseteq \mathfrak{m}^{n+1}$ for all n . So $HS_{M,\mathfrak{m}}(n) \leq HS_{M,I}(n) \leq HS_{M,\mathfrak{m}}(n\theta)$. This comes from the fact that the maps in

$$\frac{M}{\mathfrak{m}^{n\theta+1}} \rightarrow \frac{M}{I^{n+1}} \rightarrow \frac{M}{\mathfrak{m}^{n+1}}$$

are surjective. Thus the degree of $HSP_{M,\mathfrak{m}}$ equals that of $HSP_{M,I}$.

Proposition 1.6. *Let R, I be as above, and M', M, M'' be finitely generated R -modules such that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact. Then $\tilde{d}_M = \max\{\tilde{d}_{M'}, \tilde{d}_{M''}\}$, and the polynomials $HSP_{M'',I}, HSP_{M,I} - HSP_{M',I}$ have the same leading coefficient.*

Proof. We can think of $M'' = M/M'$. Consider $\frac{M''}{I^n M''} = \frac{M/M'}{I^n M/M'} = \frac{M}{I^n M + M'}$. This implies

$$0 \rightarrow \frac{I^n M + M'}{I^n M} \rightarrow \frac{M}{I^n M} \rightarrow \frac{M}{I^n M + M'} \rightarrow 0$$

is exact. Thus

$$\ell\left(\frac{M}{I^n M}\right) = \ell\left(\frac{M''}{I^n M''}\right) + \ell\left(\frac{M' + I^n M}{I^n M}\right) = \ell\left(\frac{M''}{I^n M''}\right) + \ell\left(\frac{M'}{M' \cap I^n M}\right).$$

We write $\phi(n) = \ell\left(\frac{M'}{M' \cap I^{n+1} M}\right)$, so $\phi(n)$ is of polynomial type, and $\phi(n) \geq 0$ for all $n \geq 0$. So $\tilde{d}_M = \max\{\tilde{d}_{M''}, \deg \phi\}$. It now suffices to show $\deg \phi = \tilde{d}_{M'}$. The Artin-Rees lemma implies $I^i(I^k M \cap M') = I^{k+i} M \cap M'$ for some k , and all $i \geq 0$. We will

now show the claim that $I^{n+1}M' \subseteq M' \cap I^{n+1}M \subseteq I^{n-k+1}M'$ for some large n and some fixed k . From the lemma result, we get $I^{k+1}M \cap M' \subseteq I(IM \cap M') \subseteq IM'$, and $I^{k+2}M \cap M' \subseteq I^2(I^kM \cap M') \subseteq I^2M'$. This implies $I^{m+k}M \cap M' \subseteq I^mM'$ for all m . Thus $I^{n+1}M \cap M' \subseteq I^{n-k+1}M'$ for every $n \geq k-1$, and the claim is established. Therefore all the maps in

$$\frac{M'}{I^{n+1}M'} \rightarrow \frac{M'}{M' \cap I^{n+1}M} \rightarrow \frac{M'}{I^{n-k+1}M'},$$

are surjective. So $HS_{M',I}(n) \geq \phi(n) \geq HS_{M',I}(n-k)$ which implies $\tilde{d}_{M'} = \deg HSP_{M',I} = \deg \phi$, so the first part of the proposition holds. For the leading coefficients, $LC(\phi) = LC(HSP_{M',I})$, but $HSP_{M,I} - HSP_{M',I} = HSP_{M'',I}$, so the second part holds. \square

Remark 1.7. Let (R, \mathfrak{m}) be local and Noetherian, and let ${}_R M$ be finitely generated. Let $s = \min\{k \mid \exists x_1, \dots, x_k \in \mathfrak{m} \text{ with } \ell(M/(x_1, \dots, x_k)M) < \infty\}$. Then $s = \dim M$.

Proof. Similar to the case $M = R$. \square

Theorem 1.8. Let (R, \mathfrak{m}) be local and Noetherian, ${}_R M$ be finitely generated, and I an \mathfrak{m} -primary ideal of R . Let $\tilde{d}_M = \dim gr_I(M)$. Then $\dim M = \tilde{d}_M$.

Proof. Using the interpretation of the dimension as the degree of the Hilbert-Samuel polynomial we can replace I by \mathfrak{m} .

First we show $\tilde{d}_M \geq \dim M$. We proceed by induction on \tilde{d}_M . Assume first that $M = R$. If $\tilde{d}_R = 0$, then $gr_{\mathfrak{m}}(R)$ is Artinian which implies $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Now, by NAK, we have $\mathfrak{m}^n = 0$, which gives $\dim R = 0$. Next, let $\tilde{d}_R > 0$, and take $P_0 \subseteq \dots \subseteq P_n$ a chain of prime ideals in R . Let $x \in P_1 \setminus P_0$, and let $B = \frac{R}{P_0 + xR}$. Note that $\dim(B) \geq n-1$. Then the sequence

$$0 \rightarrow R/P_0 \rightarrow R/P_0 \rightarrow B \rightarrow 0$$

is exact with the first nontrivial map being multiplication by x . This gives $d_{R/P_0} = \max(d_{R/P_0}, \tilde{d}_B)$, so $\tilde{d}_B \leq d_{R/P_0}$, but in fact we have a strict inequality by a leading coefficient argument using the previous theorem. Since R surjects onto R/P_0 , we have R/\mathfrak{m}^n surjecting onto $\frac{R}{\mathfrak{m}^n + P_0} = \frac{R/P_0}{\mathfrak{m}^n(R/P_0)}$, so $\tilde{d}_B < \tilde{d}_{R/P_0} \leq \tilde{d}_R$. By the induction hypothesis, we have $n-1 \leq \dim B \leq \tilde{d}_B \leq \tilde{d}_R - 1$. This implies $\tilde{d}_R \geq n$.

For a general M the proof goes through with the help of prime filtrations and Proposition 1.6 as in the proof of the existence of the Hilbert polynomial.

Next we show $\dim M \geq \tilde{d}_M$ and proceed by induction on $\dim M$. If $\dim M = 0$, then M is Artinian which implies $\ell(M) < \infty$. Then $\ell\left(\frac{M}{\mathfrak{m}^n M}\right)$ are bounded by $\ell(M)$ for all n since M surjects onto $\frac{M}{\mathfrak{m}^n M}$, so $\tilde{d}_M = 0$. Now let $\dim M = s > 0$. By the remark $\exists x_1, \dots, x_s \in \mathfrak{m}$ such that $\ell\left(\frac{M}{(x_1, \dots, x_s)M}\right) < \infty$. Let $M_i = \frac{M}{(x_1, \dots, x_i)M}$ for each $1 \leq i \leq s$. Note that $\dim M_i = \dim M - i = s - i$. By the second isomorphism theorem, $\ell\left(\frac{M_1}{\mathfrak{m}^n M_1}\right) = \ell\left(\frac{M}{x_1 M + \mathfrak{m}^n M}\right)$, and since the sequence

$$0 \rightarrow \frac{xM + \mathfrak{m}^n M}{\mathfrak{m}^n M} \rightarrow \frac{M}{\mathfrak{m}^n M} \rightarrow \frac{M}{x_1 M + \mathfrak{m}^n M} \rightarrow 0$$

is exact, the length equation is equal to $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) - \ell\left(\frac{x_1 M}{x_1 M \cap \mathfrak{m}^n M}\right)$, and considering

$(\mathfrak{m}^n M : x_1) = \{m \in M \mid x_1 m \in \mathfrak{m}^n M\}$, the equation now becomes $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) -$

$\ell\left(\frac{M}{(\mathfrak{m}^n M : x_1)}\right)$. Consider the map $\varphi : M \rightarrow \frac{x_1 M}{x_1 M \cap \mathfrak{m}^n M}$ which acts by multiplication by x_1 , and then by the appropriate modulo. We have $\ker \varphi = \{m \in M \mid \overline{x_1 m} = 0\} = \{m \mid x_1 m \in \mathfrak{m}^n M\} = (\mathfrak{m}^n M : x_1)$. But $\mathfrak{m}^{n-1} M \subseteq (\mathfrak{m}^n M : x_1)$ since $x_1 \mathfrak{m}^{n-1} M \subseteq \mathfrak{m}^n M$ because $x_1 \in \mathfrak{m}$. Thus $\frac{M}{\mathfrak{m}^{n-1} M}$ surjects onto $\frac{M}{(\mathfrak{m}^n M : x_1)}$ which implies that the

length equation above is greater than or equal to $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) - \ell\left(\frac{M}{\mathfrak{m}^{n-1} M}\right)$. Therefore

$\text{HSP}_{M_1}(n) \geq \text{HSP}_M(n) = \text{HSP}_M(n-1)$ (the leading terms on the right hand side cancel out), so $\tilde{d}_{M_1} \geq \tilde{d}_M - 1$. But $\dim M_1 = \dim M - 1$. Applying the induction hypothesis, $\dim M - 1 = \dim M_1 \geq \tilde{d}_{M_1} \geq \tilde{d}_M - 1$, so $\dim M \geq \tilde{d}_M$. Thus we have equality, and the theorem is established. \square

Remark 1.9. *The same statement is true for R a semilocal Noetherian ring with $\mathfrak{m} = \text{Jac}(R)$, I an ideal of definition for R , and M a finitely generated R -module.*

Note that $d = \dim M$ if $I \subseteq \mathfrak{m} = \text{Jac}(R)$ an ideal of definition for M . The leading coefficient of the Hilbert-Samuel polynomial is $\frac{e(\text{gr}_I(M))}{d!}$.

Definition 1.10. We let $e(I, M) := e(\text{gr}_I(M))$ denote the *multiplicity of M with respect to I* . The *multiplicity of M* is the multiplicity of M with respect to $\mathfrak{m} = \text{Jac}(R)$, but most often this is used for the case when R is local.

This Theorem allows us to work with the multiplicity of an ideal I primary to the maximal ideal in a local ring R , without necessarily mentioning the associated graded

ring of R with respect to I . One can do this by thinking of the multiplicity as a certain limit, as described in the previous lecture.

Corollary 1.11. $\lim_{n \rightarrow \infty} d! \cdot \frac{\ell(M/I^n M)}{n^d}$ exists and equals $e(I, M)$.

Proof. Consider $\ell\left(\frac{M}{I^{n+1}M}\right) = \frac{e(\text{gr}_I(M))}{d!}n^d +$ lower degree terms. Then, dividing by $(n+1)^d$ and letting $n \rightarrow \infty$, the limit exists. \square

Corollary 1.12. $e(I^r, M) = e(I, M)r^d$.

Proof. Left as a simple exercise. \square