

LECTURE 19

1. PROPERTIES OF COMPLETION; ARTIN-REES LEMMA

It is helpful to note that any element of \hat{A}^I is given by a sequence $\{x_n\}$ such that $x_{n+1} - x_n \in I^n$. Hence we can find $a_{n+1} \in I^n$ for all $n \geq 0$ such that $x_n = a_1 + \cdots + a_n$ for all $n \geq 1$.

It can easily be checked that if $A = R[x_1, \dots, X_n]$ is a polynomial ring over a ring R , and $I = (X_1, \dots, X_n)$, then $\hat{A}^I = R[[X_1, \dots, X_n]]$.

Proposition 1.1. *Let A, B be two rings and let I be an ideal of A , respectively J be an ideal of B . Consider $f : A \rightarrow B$ be a ring homomorphism such that $f(I) \subset J$. Then there is a canonical ring homomorphism $\hat{f} : \hat{A}^I \rightarrow \hat{B}^J$.*

Moreover, if f is surjective such that $f(I) = J$, then \hat{f} is surjective.

Proof. We have natural maps $A/I^n \rightarrow B/J^n$, so $\varprojlim A/I^n \rightarrow B/J^n$ for all n which implies, by applying the universal property of the inverse limit, the first part.

For the second part, consider a sequence of elements in B , say $\{b_n\}$ such that $b_{n+1} \in J^n$, $y_n = b_1 + \cdots + b_n$ and let $\{y_n\}_n$ give an elements in \hat{B}^J . But I^n maps onto J^n via f , so we can find a sequence of elements $a_{n+1} \in I^n$ mappint onto b_{n+1} . Set $x_n = a_1 + \cdots + a_n$, for $n \geq 1$. Then $\{x_n\}_n$ gives an element in \hat{A}^I that maps onto the element corresponding to $\{y_n\}_n$ in \hat{B}^J . □

Corollary 1.2. *If A is Noetherian and $I = (r_1, \dots, r_n) \subset A$, then \hat{A}^I is Noetherian.*

Remark 1.3. *In fact, we hane $\hat{A}^I = \frac{A[[X_1, \dots, X_n]]}{(X_i - r_i, \dots, X_n - r_n)}$. This fact will be proved later.*

Proof. Indeed, let $I = (r_1, \dots, r_n)$. Map $R[[X_1, \dots, X_n]]$ onto A by sending $X_i \rightarrow r_i$. This maps (X_1, \dots, X_n) onto I and hence we obtain \hat{A}^I as a quotient of the Noetherian ring $A[[X_1, \dots, X_n]]$. □

Theorem 1.4. *Let A be a ring and I an ideal of A . Let $\pi : \hat{A}^I \rightarrow A/I$ the natural projection. Then $\text{Ker}(\pi) \subseteq \text{Jac}(\hat{A}^I)$. This implies that there is a one-to-one correspondence between the maximal ideals in \hat{A}^I and the maximal ideal of A/I . In particular, the completion of a local ring (A, \mathfrak{m}) at its maximal ideal is a local ring as well.*

Proof. Let $\{x_n\}$ an element x of \hat{A}^I that belongs to $\text{Ker}(\pi)$: $x_n \in I$ for all n . We will show that $1 + x$ is invertible in \hat{A}^I . Consider $y_n = \sum_{i=0}^{n+1} (-1)^i x_n^i$. It is clear that y_n define a Cauchy sequence in A which therefore gives an element y of the completion. But then $z_n = 1 - (1 + x_n)y_n = x_n^{n+2} \in I^{n+2}$. This implies that $\{z_n\}_n$ is 0 in \hat{A}^I and then $1 = (1 + x)y$ in \hat{A}^I . \square

Definition 1.5. Let A be a ring, I an ideal of A and M an A -module. We say that a sequence of elements $\{x_n\}_n$ in M is *Cauchy in the I -adic topology* if for all n there exists N such that $x_i - x_j \in I^n M$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from M converges to 0 if for all n there exists N such that $x_i \in I^n M$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in M$ such that $\{x_n - x\}_n$ converges to zero in M . We say that M is *complete in the I -adic topology* if every Cauchy sequence in M converges to an element in M . We say that M is *I -adically separated* if $\bigcap_{n=1}^{\infty} I^n M = 0$.

Definition 1.6. *The I -adic completion of M is $\hat{M}^I : \varprojlim M/I^n M$. It can be checked that \hat{M}^I is a \hat{A}^I -module and there exists a natural A -module homomorphism $M \rightarrow \hat{M}^I$ with kernel $\bigcap_n I^n M$.*

We say that a filtration of submodules of M say $\{N_n\}$ is cofinal with the filtration $\{I^n M\}$ if for all n there exists m such that $N_m \subseteq I^n M$ and for all t there exists s such that $I^s M \subseteq N_t$. It can be checked that $\varprojlim M/N_n \simeq \varprojlim M/I^n M$ (in fact, the filtrations define the same linear topology on M).

Moreover, we can see that a Cauchy sequence and a subsequence of it define the same element in \hat{M}^I , so we assume that every element $m \in \hat{M}$ is defined by a sequence $\{m_n\}$ such that $m_{n+1} - m_n \in I^n$. Therefore there exists $z_{n+1} \in I^n$ such that for $y_n = z_0 + \dots + z_n$ we have that $\{y_n\}_n$ gives m .

Proposition 1.7. *Let A be a ring, I an ideal of A . Then*

- (1) *Any A -linear map $f : M \rightarrow N$ of A -modules induces an \hat{A}^I -linear map $\hat{f} : \hat{M}^I \rightarrow \hat{N}^I$. Moreover, f surjective implies that \hat{f} is surjective.*

- (2) *There exists a natural isomorphism of \hat{A}^I -modules $\hat{M}^I \oplus \hat{N}^I \simeq \hat{M}^I \oplus \hat{N}^I$, for any two A -modules M, N .*
- (3) *The multiplication by an element $M \xrightarrow{a} N$ defines a natural map \hat{A}^I -linear map $\hat{M}^I \rightarrow \hat{N}^I$ given by the multiplication by the image of $a \in \hat{A}^I$.*

Proof. The proof of the first part follows the ring case mutatis mutandis. The last two parts are straightforward \square

Let $N \subset M$ be a pair of A -modules. In what follows we need to compare the I -adic topology on N with the topology induced by the I -adic topology on M restricted to N . In essence we will show that $\varprojlim N/I^n M \cap N = \hat{N}^I$. To prove this we need to develop some considerations on filtrations of modules and in fact we will be proving a statement that is more general.

Definition 1.8. Let M be an A -module and I an ideal of A . Let $\mathcal{M} = \{M_n\}_n$ be a filtration of submodules of M , i.e. $M_{n+1} \subset M_n$ and $M_0 = M$. We say that \mathcal{M} is an I -filtration if $IM_n \subset M_{n+1}$ for all $n \geq 0$. The filtration \mathcal{M} is called I -stable if $I^n M_m = M_{n+1}$ for $n \gg 0$.

An example of an I -stable filtration is the one given by $\{I^n M\}_n$. The case of $M = A$ is particularly important because we can associate the following object to the filtration $\{I^n\}_n$: $gr_I(A) := I^n/I^{n+1}$ which is an A -module naturally. In fact this object, which is called the *associated graded ring* with respect to the ideal I is a ring with multiplication defined as follows: $\overline{ab} = \overline{a}\overline{b}$ for any two elements $a \in I^n, b \in I^m$. It can be checked that this is well-defined and that it extends via distributivity to a multiplication on $gr_I(A)$.

Now consider an I -filtration \mathcal{M} . We can define the following A -module $gr_{\mathcal{M}}(M) : \oplus_{n \geq 0} M_n/M_{n+1}$. An important feature of it is that this object is in fact an $gr_I(A)$ -module. For $\overline{a} \in I^n/I^{n+1}$ and $\overline{m} \in M_k/M_{k+1}$ we let $\overline{am} : \overline{am} \in M_{n+k}/M_{n+k+1}$. It can be checked that definition is well-defined. By distributivity, we can extend this to a scalar multiplication on $gr_{\mathcal{M}}(M)$ with elements from $gr_I(A)$ and we call it *the associated graded module* of M with respect to \mathcal{M} .

Proposition 1.9. *Let A be a ring, I be an ideal of A , M be an A -module, and \mathcal{M} be an I -filtration on M . Then*

- (1) If $A[It]$ is a finitely generated A -algebra, if I is a finitely generated ideal.
- (2) If $A[It]/IA[It] \simeq gr_I(A)$ as A -algebras.

Proof. For (1), let $I = (a_1, \dots, a_r)$. Then $A[X_1, \dots, X_n]$ maps onto $A[It]$ under $X_i \rightarrow a_i$. For (2), let $A[It] \rightarrow gr_I(A)$ that sends at to $\bar{a} \in I/I^2$, for any $a \in I$. It can be easily check that this is an well-defined A -algebra homomorphism with kernel equal to $IA[It]$. □

Proposition 1.10. *Let A be a ring, I be an ideal of A , M be a finitely generated module over A , and \mathcal{M} be an I -stable filtration on M composed of finitely generated submodules. Then $gr_{\mathcal{M}}(M)$ is a finitely generated module over $gr_I(A)$.*

Proof. Since the filtration is I -stable so $I^k M_N = M_{N+k}$ for some $N \geq 0$ and all $k \geq 0$. Therefore $\frac{I}{I^2} \cdot \frac{M_n}{M_{n+1}} = \frac{M_{n+1}}{M_{n+2}}$, for all $n \geq N$.

This shows that $gr_{\mathcal{M}}(M)$ is generated by the union of all the generators of M_n/M_{n+1} for $n \leq N$. This is a finite set which proves the claim. □

Definition 1.11. Let I be an ideal in A . The A -algebra $R_I(A) = A[It] \subset A[t]$ is called the *Rees algebra*, or the *blowup algebra*, of A with respect to I . Note that $A[It] = \bigoplus_{n \geq 0} I^n$.

Similarly, for an I -filtration \mathcal{M} on an A -module M , we can define the *Rees module* of M with respect to \mathcal{M} by $R_{\mathcal{M}}(M) := \bigoplus_{n \geq 0} M_n t^n = \bigoplus_{n \geq 0} M_n$. Note that $R_{\mathcal{M}}(M)$ is a module over $A[It]$ in a natural way.

Theorem 1.12. *Let A be a ring, I be an ideal of A , M be an A -module with I -filtration \mathcal{M} consisting of finitely generated A -submodules of M . Then the filtration \mathcal{M} is I -stable if and only if $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$ -module.*

Proof. If \mathcal{M} is I -stable, then $M_{N+k} = I^k M_N$ for some $N \geq 0$ and for all $k \geq 0$. Then $R_{\mathcal{M}}(M)$ is finitely generated by the union of the generators of M_i , with $I \leq N$.

If $R_{\mathcal{M}}(M)$ is finitely generated over $A[It]$, there exists $N \geq 0$ such that all generators belong to the union of M_i , $i \leq N$. But $\bigoplus_{k > 0} M_{N+k}$ is finitely generated as an $A[It]$ -module (since it is a homomorphic image of $R_{\mathcal{M}}(M)$). Using thsi and the fact that \mathcal{M} is an I -filtration we derive that $I^k M_N = M_{N+k}$, for all $k \geq 0$. □

Corollary 1.13 (Artin-Rees Lemma). *Let M be a finitely generated A -module, where A is Noetherian. Assume that I is an ideal of A and let N an A -submodule of M .*

- (1) *Let \mathcal{M} be an I -stable filtration on M . Then $\{M_n \cap N\}_n$ is an I -stable filtration on N .*
- (2) *The filtration $\{I^n M \cap N\}_n$ is I -stable that is there exists $c > 0$ such that*

$$I^n M \cap N = I^{n-c}(I^c M \cap N),$$

for all $n \geq c$.

Proof. It suffices to prove (1). Let \mathcal{M}' the filtration with terms $I^n M \cap N$. Clearly $R_{\mathcal{M}'}(N)$ is an $A[It]$ -submodule of $R_{\mathcal{M}}(M)$.

Note that $A[It]$ is a Noetherian A -algebra and $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$ -module by Theorem 1.12. So, $R_{\mathcal{M}'}(N)$ is a finitely generated $A[It]$ -module, hence by Theorem 1.12 we get that \mathcal{M}' is I -stable on N . \square

Theorem 1.14. *Let A be a Noetherian ring and I an ideal of A .*

- (1) *If $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ is a short exact sequence of finitely generated A -modules then*

$$0 \rightarrow \hat{N}^I \rightarrow \hat{M}^I \rightarrow \hat{P}^I \rightarrow 0$$

is a short exact sequence of \hat{A}^I -modules.

- (2) *The universal property of the tensor product implies that there is a natural \hat{A}^I -map $\hat{A}^I \otimes_A M \rightarrow \hat{M}^I$ for any A -module M . For every finitely generated A -module M , $\hat{A}^I \otimes_A M \simeq \hat{M}^I$ is an isomorphism.*
- (3) *\hat{A}^I is a flat A -algebras which is faithfully flat if (A, \mathfrak{m}) is a local ring.*

Proof. To be included later.

\square