LECTURE 18

1. Flatness and completion

Let M be an A-module. We say that M is *A-flat*, respectively *A-faithfully flat* if, for all sequences of A-modules $E \to F \to G$, the sequence is exact implies, respectively is equivalent to, that the sequence $E \otimes_A M \to F \otimes_A M \to G \otimes_A M$ is exact.

For an A-algebra B, we say that B is a flat A-algebra if it is flat as an A-module. A ring homomorphism $f : A \to B$ is called a flat homomorphism if B is an A-flat algebra.

We note that $S^{-1}A$, with S a multiplicatively closed set, and $A[x_1, \ldots, x_n]$ are A-flat algebras. In fact any free A-module, including polynomial rings over A, are faithfully flat.

Using the fact that the tensor product is right exact, we note that flatness can be checked by only considering exact sequences of A-modules of the type $0 \to E \to F$.

The following two propositions list some simple consequences of flatness and faithful flatness which can be proven easily by manipulating the definitions.

Proposition 1.1. *Let B be an A-algebra and M a B-module. Then*

1. *B is A-flat (respectivey A-faithfully flat) and M is B-flat (respectively B-faithfully flat) then M is A-flat (respectively A-faithfully flat);*
2. *M is B-faithfully flat and M is A-flat (respectively A-faithfully flat) then B is A-flat (respectively A-faithfully flat) .*

Proposition 1.2. *Let B be an A-algebra and M an A-module.*

Then M A-flat (respectively A-faithfully flat) implies $M \otimes_A B$ is B-flat (respectively B-faithfully flat).

The first important result on flatness shows that checking flatness can be performed locally.

Theorem 1.3. *Let $f : A \to B$ be a ring homomorphism and M a B-module. Then M is A-flat if and only if for every prime ideal P of B , M_P is flat over A_p where $p = P \cap A$ (or the same condition for every maximal ideal P of B).*

Proof. Note that if N is an $S^{-1}A$-module, where S is a multiplicatively closed set in A, then $S^{-1}A \otimes_A N = N$.

The map $A \to B$ naturally gives the homomorphism $A_p \to B_P$, and M_P is naturally an A_p-module.

Note that $N \otimes_{A_p} M_P = N \otimes_{A_p} A_p \otimes_A M_P = N \otimes_A M_P = N \otimes_A M \otimes_B B_P$. From this we see immediately that M flat over A together with the observation that B_P is B-flat implies that M_P is flat over A_p.

For the converse, let $0 \to E \to F$ be an injective map and consider $0 \to K \to E \otimes_A M \to F \otimes_A M$. The plan is to show that $K = 0$. Localizing at P, we get an exact sequence $0 \to K_P \to E \otimes_A M \otimes_B B_P = E \otimes_A M_P \to F \otimes_A M \otimes_B B_P = F \otimes_A M_P$.

But note that $E \otimes_A M_P = E \otimes_A A_p \otimes_{A_p} M_P = E_p \otimes_{A_p} M_P$ (and similarly for F).

So, $0 \to K_P \to E_p \otimes_{A_p} M_P \to F_p \otimes_{A_p} M_P$ is exact.

But $0 \to E_p \to F_p$ is injective for any prime ideal p in A. When tensoring with M_P over A_p, exactness is preserved. So, $K_P = 0$. This condition for all maximal (respectively prime) ideals in B implies that $K = 0$.

□

Theorem 1.4. *Let A be a ring and M an A-module. the following assertions are equivalent:*

(1) *M is faithfully flat over A;*
(2) *M is flat over A and $N \otimes M \neq 0$ for all nonzero A-modules N.*
(3) *M is flat over A and $M \neq \mathfrak{m}M$ for all maximal ideal $\mathfrak{m}$ of A.*

Proof. (1) implies (3) Let $0 \to A/\mathfrak{m} \to 0$. Assume that $\mathfrak{m}M = M$. This implies that $0 \to A/\mathfrak{m} \otimes M \to 0$ is exact so the original sequence must be exact. Hence $A = \mathfrak{m}$ which is a contradiction.

(3) implies (2)

Let $0 \neq x \in N$ and consider a maximal ideal $\mathfrak{m}$ containing $Ann(x)$. But then $Ax \simeq A/Ann(x)$ and then $Ax \otimes M = A/Ann(x) \otimes M = M/Ann(x)M$ which is nonzero because $Ann(x)M \subset \mathfrak{m}M \neq M$. Now, we have an R-linear map which is injective $0 \to Ax \to N$ and by flatness of M we get that $Ax \otimes M$ injects into $N \times M$, so the latter is nonzero as well.

(2) implies (1)

Let f be an A-linear map between two modules $E \to F$. We claim that $Ker(f) \otimes M = Ker(f \otimes 1)$ and $Im(f) \otimes M = Im(f \otimes 1)$. Indeed, $Ker(f) \to E \to F$ and $E \to F \to F/Im(f)$ are exact, so they remain exact after tensoring with M.

Consider a sequence of A-modules $N' \to N \to N''$ such that $N' \otimes M \stackrel{f\otimes 1}{\to} N \otimes M \stackrel{g\otimes 1}{\to} N'' \otimes M$ is exact.

So $(g \circ f) \otimes 1 = 0$ hence $g \circ f = 0$. In conclusion $Im(f) \subset Ker(g)$.

Consider now $H = Ker(g)/Im(f)$. Then by flatness $H \otimes M = (Ker(g) \otimes M)/(Im(f) \otimes M) = 0$. Therefore $H = 0$.

□

Corollary 1.5. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ a local homomorphism of rings (that is f is a ring homomorphism and $f(\mathfrak{m}) \subset \mathfrak{n}$). Then B is A-flat if and only if B is A-faithfully flat.*

Proof. Since $f(\mathfrak{m}) \subset \mathfrak{n}$ we get that $\mathfrak{m}B \subset \mathfrak{n} \neq B$. □

Proposition 1.6. (1) *Let A be a ring and M an A-flat module. Let N_1, N_2 be two submodules of M. Then*

$$(N_1 \cap N_s) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M),$$

where the objects are regarded as submodules of $N \otimes_A M$.

(2) *Therefore, if $A \to B$ is flat then for any ideals I, J of A, we have $(I \cap J)B = IB \cap JB$. If J is finitely generated, then $(I : J)B = (IB : JB)$.*

(3) *If $f : A \to B$ is faithfully flat, then for any A-module M the natural map $M \to M \otimes_A B$ is injective. In particular f is injective. In particular, for any ideal $I \subset A$, $IB \cap A = I$.*

Proof. For (1), consider the exact sequence of A-modules $0 \to N_1 \cap N_2 \to N \to N/N_1 \oplus N/N_2$, and tensor with M. The resulting exact sequence gives the statement.

For (2), let $N = A$, $N_1 = I, N_2 = J$, and $M = B$. For the second part, let $J = (a_1, \ldots, a_k)$. But then $I : J = \cap_{i=1}^k (I : Aa_i)$.

Fix i, and let $0 \to (I : Aa_i) \to A \stackrel{\cdot a_i}{\to} A/I$ which is exact. Since B is A-flat we get that the sequence stays exact after tensoring with B. This gives us $0 \to (I : Aa_i)B \to B \stackrel{\cdot a_i}{\to} B/IB$. Therefore, $(I : Aa_i)B = (IB : Ba_i)$ by computing the kernels in two ways.

Therefore, $(I : J)B = (\cap_{i=1}^k (I : Aa_i))B$ which equals $(\cap_{i=1}^k (I : Aa_i)B)$ by the first part of (2). But this last term equals $\cap_{i=1}^k (IB : Ba_i) = IB : JB$.

Finally, let $m \in M$ such that $m \otimes 1 = 0$ in $M \otimes_A B$. We need $m = 0$, so let us assume that $m \neq 0$. But then $0 \neq Am \subset M$ and therefore, since B is A-faithfully flat, we get that $0 \neq Am \otimes_A B$ in $M \otimes_A B$. On the hand $m \otimes 1 = 0$ so $Am \otimes_A B = 0$ as well. Contradiction. The final statement is obtained by letting $M = B$.

□

Lemma 1.7. *Let $i : E \to F$ be an injective A-linear map. Let M be an A-module and consider $u \in ker(1_M \otimes i) \subset E \otimes_A M$, where $1_M \otimes i : E \otimes_A M \to F \otimes_A M$. Then there exists N finitely generated submodule of M and $v \in ker(1_N \otimes i)$ such that v maps to u under the canonical map $E \otimes N \to E \otimes M$.*

Proposition 1.8. *A module M is flat over A if all its finitely generated submodules are flat over A.*

Proof. This is a straightforward application of the Lemma. If there exists an R-linear injection $i : E \to F$ and for any element $u \in Ker(i \otimes_A 1_M)$, we can find a finitely generated submodule N of M and $v \in ker(i \otimes_A 1_N)$ such that v maps onto u under the canonical map. But N is flat so $v = 0$ which gives $u = 0$. □

Proposition 1.9. *Let A be a domain. Then every flat A-module is torsion free. The converse holds, if A is a PID.*

Proof. Let $a \neq 0$ in A. Then multiplication by a is injective on A ($0 \to A \xrightarrow{a} A$ is exact), hence it is stays injective after tensoring with M: $0 \to M \xrightarrow{a} M$ is exact, so M is torsion-free.

For the converse, by Proposition 1.8 it is sufficient to show that any finitely generated submodule N of M is flat. But such an N is torsion-free, so by the structure theorem of finitely generated modules over a PID, N must be free hence flat. □

The following theorems also hold, but their proofs are more difficult so we will not include them.

Theorem 1.10. *Let A be a ring and M be an A-module. Then M is A-flat if and only if for all finitely generated modules $E \subset F$ the map $E \otimes M \to F \otimes M$ is injective.*

Theorem 1.11. *Let A be a ring and M a module over A. Then M is A-flat if and only if for any finitely generated ideal I of A we have $I \otimes_A M \to IM$ is injective, and therefore bijective.*

Theorem 1.12. *Let*

$$0 \to N \to M \to P \to 0$$

be a short exact sequence of A-modules. If N, P are A-flat then M is A-flat as well.

Theorem 1.13. *Let M be a finitely generated A-module, where $(A, \mathfrak{m})$ is a local Noetherian ring. Then M is flat if and only if M is projective if and only if M is free.*

2. I-adic completion

Definition 2.1. *Let A be a commutative ring. Let I be a partially ordered set. A pair $((M_i)_i, \{p_{ji}\}_{i \geq j})$ where M_i are A-modules for all $i \in I$, and $p_{ji} : M_i \to M_j$ are A-linear for all $j \leq i \in A$ such that*

1. $p_{ii} = 1_{M_i}$
2. $p_{ij} \circ p_{jk} = p_{ik}$ *for all* $i \leq j \leq k$.

is called an inverse (or projective) system of A-modules.

Definition 2.2. *Let I be a partially order set and $((M_i)_i, \{p_{ji}\}_{i\geq j})$ an inverse system of A-modules. A module $M = \varprojlim M_i$ together with a family of A-linear maps $q_i : M \to M_i$, $i \in I$, is called the inverse limit of the system if*

(1) *$p_{ij}q_i = q_j$, for all $i \leq j$*
(2) *for every A-module X and any A-linear maps $f_i : X \to M_i$, $i \in A$, such that $p_{ij}f_i = f_j$ for all $j \leq i$, there exists a unique A-linear map $F : X \to M$ such that $q_iF = f_i$ for all $i \in I$.*

Theorem 2.3. *Let A be a ring, I a partially ordered set and $((M_i)_i, \{p_{ji}\}_{i\geq j})$ an inverse system of A-modules. Then the inverse limit $\varprojlim M_i$ exists.*

Proof. Consider the A- submodule M of the direct product $\prod_i M_i$ defined by $\{(m_i)_i : p_{ji}(m_i) = m_j$, for all $j \leq i \in I\}$. It is routine to check that $M = \varprojlim M_i$. The maps $q_i : M \to M_i$ are the canonical projections.

□

Let A be a ring and I an ideal of A. We say that a sequence of elements $\{x_n\}_n$ is *Cauchy in the I-adic topology* if for all n there exists N such that $x_i - x_j \in I^n$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from A converges to 0 if for all n there exists N such that $x_i \in I^n$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in A$ such that $\{x_n - x\}_n$ converges to zero. We say that A is *complete in the I-adic topology* if every Cauchy sequence in A converges to an element in A.

Note that A/I^n together with the natural projections $p_{mn} : A/I^n \to A/I^m$ for $n \geq m$ form an inverse system. The I-adic completion of A is by definition $\hat{A}^I := \varprojlim A/I^n$ which is a natural A-algebra. We will generally drop the symbol I from our notation when the ideal is understood from the context. Note that we have a natural A-algebra homomorphism $i : A \to \hat{A}$ with kernel equal to $\cap_n I^n$. We say that A is *separated in the I-adic topology* if $\cap_n I^n = 0$.

Proposition 2.4. *The map $i : A \to \hat{A}$ is a ring isomorphism if and only if A is complete in the I-adic topology.*

Proof. Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Since i is an isomorphism we can find an element $x \in A$ such that $x - x_n \in I^n$ for all n which implies easily that $\{x_n\}_n$ converges to x.

Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Therefore for all n, $x_{n+1} - x_n \in I^n$ for all n. Hence $x_i - x_j \in I^n$ for all $i, j \geq n$ which gives that $\{x_n\}$ is Cauchy in A. Hence it is convergent to x an element in A. Therefore, it is enough to show that if $\{x_n\}_n$ is a sequence that converges to zero, then the corresponding element in $\hat{A}$ is zero as well. By definition, $x_j - x_i \in I^j$ for all $j \leq i$. For all n there exists N such that $x_i \in I^n$ for all $i \geq N$. But $x_i - x_n \in I^n$ for $i \geq n$ so $x_n \in I^n$ for all n. □

Remark 2.5. *A Cauchy sequence in A defines a unique canonical element in $\hat{A}$. One can check that the difference between a Cauchy sequence and a subsequence defines a sequence that converges to zero. This can be used to show that that given a Cauchy sequence $\{x_n\}_n$ in A, we can replace it by a sequence $\{y_n\}_n$ that gives the same canonical element in $\hat{A}$ with the additional property that $y_{n+1} - y_n \in I^n$ for all n. This observation is often useful in computations.*

Proposition 2.6. *Let A, B be two rings and let I be an ideal of A, respectively J be an ideal of B. Consider $f : A \to B$ be a ring homomorphism such that $f(I) \subset J$. Then there is a canonical ring homomorphism $\hat{f} : \hat{A}^I \to \hat{B}^J$.*

Moreover, if f is surjective such that $f(I) = J$, then $\hat{f}$ is surjective.