LECTURE 17

1. Characterization of normal rings; Dedekind rings

Theorem 1.1. *A ring R is a DVR if and ony if it is a local one-dimensional Noetherian normal domain.*

Proof. The forward inclusion is easy: we know that a DVR is local normal domain (since it is a valuation ring). The maximal ideal of R is a principal ideal, so it has height one.

For the converse, write $K = Q(R)$ and $\mathfrak{m}$ for the maximal ideal of R. According to Nakayama, $\mathfrak{m} \neq \mathfrak{m}^2$ and so we can choose an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since R has only two prime ideals, 0 and $\mathfrak{m}$, we conclude that $\mathfrak{m}$ is an associated prime of R/xR, that is there exists $y \in R$ such that $\mathfrak{m} = \mathrm{Ann}_R(\hat{y}) = (Rx :_R y)$. Let $a = y/x \in K$ and note that $\mathfrak{m} \cdot a \subseteq R$, which implies that $a \in \mathfrak{m}^{-1}$. But $a \notin R$ and this shows that $R \subsetneq \mathfrak{m}^{-1}$.

Noet that $\mathfrak{m}\mathfrak{m}^{-1}$ is an ideal of R containing $\mathfrak{m}$. If $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$ then $a\mathfrak{m} \subseteq \mathfrak{m}$ and so a is integral over R (by the determinental trick) and so $a \in R$ since R is normal. This is false, so $\mathfrak{m}\mathfrak{m}^{-1} = R$. This implies that $\mathfrak{m}$ is an invertible ideal of R and hence it is principal because it is prime. Since R is Noetherian and $\mathfrak{m}$ is principal, we have that R is DVR.

□

Theorem 1.2. *Let R be a Noetherian normal domain. Then all the associated primes P of principal ideals in R have height one and R_P is DVR. Moreover, $R = \cap_{\mathrm{ht}P=1} R_P$.*

In fact, in a Noetherian domain, if all the associated primes P of principal ideals in R have height one, then $R = \cap_{\mathrm{ht}P=1} R_P$.

Proof. Let $I = Rx$ be a principal ideal of R and P an associated prime of I. There exists $y \in R$ such that $P = (Rx :_R y)$. R_P is normal Noetherian domain as well. Let $a = y/x$ be an element of $Q(R)$. Now, $a \notin R_P$ and $aPR_P \subset R_P$ which implies that either $aPR_P \subseteq PR_P$, and so a is integral over R_P and hence in R_P (false), or $aPR_P = R_P$

which proves that PR_P is an invertible (prime) ideal of R_P. As before, this implies that PR_P is principal and so R_P is a DVR. In particular P has height one.

Let $a = y/x \in Q(R)$ such that $y/x \in R_P$ for all P prime ideals of height 1.

Consider $I = Rx = Q_1 \cap \cdots \cap Q_n$ a primary decomposition of I. Let $P_i = Rad(Q_i)$, and note that we proved that P_i have all height one since they are associated primes of I.

But then, by hypothesis, $y \in xR_{P_i} \cap R = Q_i$, since all P_i are minimal over x by height reasons.

Therefore $y \in Q_1 \cap \cdots \cap Q_n = I = Rx$ and hence $a = y/x \in R$. □

The following is an important characterization of normal rings.

Corollary 1.3. *A Noetherian domain R is normal if and only if*

(1) *for all prime ideals P of R of height 1, R_P is DVR.*
(2) *all associated prime ideals of principal nonzero ideals have height 1.*

Proof. One direction was already proven. We can see that we have seen that (2) implies that $R = \cap_{\mathrm{ht} P=1} R_P$. But R_P is a normal ring with $Q(R_P) = Q(R)$, and so R is normal as well.

□

The following result will introduce the concept of Dedekind rings.

Theorem 1.4. *Let R be a domain. The following assertions are equivalent:*

(1) *Every nonzero ideal of R is invertible.*
(2) *R is normal Noetherian domain of dimension 1, or a field.*
(3) *Every nonzero ideal of R is a product of finitely many prime ideals of R.*

Moreover, the representation in (3) unique.

Proof. (1) implies (2). Since an invertible ideal is finitely generated we get that R is Noetherian. Assume that R is not a field. Now, let P be a nonzero prime ideal. Since P

is invertible we get that $\text{ht}(P) = 1$ and R_P is DVR. This proves that R is one dimensional. The fact that every localization is a normal proves that R is normal as well.

(2) implies (1). R is Noetherian so every ideal is finitely generated. Let I be a nonzero fractional ideal of R. Let P be a maximal ideal of R containing I. Since $\text{ht}(P) = 1$ we get that R_P is DVR, and so IR_P is cyclic. therefore I is invertible.

(1) implies that (3). First note that R is Noetherian. Assume that we proved the statement for all ideals J strictly containing I. We can assume $I \neq R$, and take $I \subseteq \mathfrak{m}$ in a maximal ideal. Therefore, $I \subsetneq I\mathfrak{m}^{-1} \subseteq R$.

The first inclusion is strict since otherwise $I\mathfrak{m} = I$ and Nakayama lemma implies that there exists $r \in I$ such that $(1 + r)\mathfrak{m} = 0$. This is impossible since R is domain.

But then $I \subseteq I\mathfrak{m}^{-1} = P_1 \cdots P_k$ gives $I = P_1 \cdots P_k\mathfrak{m}$.

We will not prove (3) implies (1), and the uniqueness here.

□

Definition 1.5. *A domain satisfying the conditions in the above theorem is called a* Dedekind ring *or* Dedekind domain*.*

One can consider the set of all fractional invertible ideals in a domain R. The multiplication of fractional ideals induces a multiplication such that this set becomes an Abelian group H. Consider the subgroup H of nonzero principal fractional ideals of R. The subgroup G/H is called the Picard group of R and it is denoted by $Pic(R)$. It represents an important notion not only in commmutative algebra, but also in number theory and algebraic geometry. When R is Dedekind, this object is called the ideal class group of R and it is denoted by $Cl(R)$.

Proposition 1.6. *Let R be a Dedekind ring. Then $Cl(R) = 0$ if and only if R is PID.*

Proof. Let I be a nonzero ideal in R. Then I is fractional, and so it is invertible. But $Cl(R) = 0$ hence $I = Rx$ for some $x \in K$, x nonzero.

But $I \subset R$ and so $x \in R$. Hence I is a principal ideal of R.

□

Theorem 1.7. *Let R be a Dedekind ring. Then R is PID if and only if R is UFD.*

Proof. Assume that R is a Dedekind domain and UFD. Then every nonzero prime ideal is of height one so it is principal (due to the UFD property). But every ideal is product of primes, so every ideal is principal. □