Lecture 9

1. Associated Primes and Localization

Remark 1.1. Consider the natural map $A \to S^{-1}A$, we see that by restriction of scalars, an $S^{-1}A$-module N is an A-module under the multplication $an := \frac{a}{1}n$, if $n \in N, a \in A$.

There is a natural correspondence between the sets: $\mathrm{Spec}(S^{-1}A)$ and $\mathrm{Spec}(A)$. Indeed, take P such that $P \cap S = \emptyset$: then $P \in \mathrm{Spec}(A)$ corresponds to $S^{-1}P \in \mathrm{Spec}(S^{-1}A)$.

We will show below that $\mathrm{Ass}_{S^{-1}A}(N) = \mathrm{Ass}_A(N)$ when N is an $S^{-1}A$-module.

Theorem 1.2. *Let $S \subset A$ be a multiplicative set, and N be an $S^{-1}A$-module. Then $Ass_A(N) = Ass_{S^{-1}A}(N)$.*

If A is Noetherian, and M is an A-module, then

$$Ass_{S^{-1}A}(S^{-1}M) = Ass_A(S^{-1}M) = Ass_A(M) \cap Spec(S^{-1}A),$$

where we identify $Spec(S^{-1}A)$ with $\{P \in Spec(A) : P \cap S = \emptyset\}$.

Proof. Let $Q \in \mathrm{Ass}_{S^{-1}A}(N)$. Then $Q = S^{-1}P$, where $P \in \mathrm{Spec}(A)$, $P \cap S = \emptyset$. Then $P = Q \cap A$. By definition, $Q = \mathrm{Ann}_{S^{-1}A}(x)$ for some $0 \neq x \in N$. We claim that $P = Q \cap A = \{a \in A \mid ax = 0\} = \mathrm{Ann}_A(x)$. For forward inclusion, take $a \in Q \cap A \Rightarrow \frac{a}{1} \in Q$ and $\frac{a}{1}x = 0$ which is equivalent to $ax = 0$ since N is A-module by restriction of scalars via $A \to S^{-1}A$. For the reverse inclusion, if $ax = 0$, then $\frac{a}{1}x = 0$, so $\frac{a}{1} \in Q$, so $\frac{a}{1} \in Q \cap A$, and the claim is verified.

Let $P \in \mathrm{Ass}_A(N)$, so $P = \mathrm{Ann}_A(x)$ for some $0 \neq x \in N$. Note that $P \cap S = \emptyset$ (since if $a \in P \cap S$ then $x = \frac{1}{a}ax = \frac{1}{a}0 = 0$, a contradiction). So $S^{-1}P = PS^{-1}A \in \mathrm{Spec}(S^{-1}A)$. Let $\frac{a}{s} \in S^{-1}P$, so $a \in P$ and hence $\frac{a}{1}x = 0$. This implies that $\frac{a}{s} \in \mathrm{Ann}_{S^{-1}A}(x)$. Conversely, let $\frac{a}{s} \in \mathrm{Ann}_{S^{-1}A}(x)$. Then $\frac{a}{s}x = 0$ which implies $\frac{a}{1}x = s\frac{a}{s}x = 0$, and so $a \in P$. So, $S^{-1}P = \mathrm{Ann}_{S^{-1}A}(x) \in \mathrm{Ass}_{S^{-1}A}(N)$.

For the second statement of the Theorem, let us note that the first equality is an immediate consequence of what has been proved so far and it does not need the Noetherian assumption.

Let $P \in \mathrm{Ass}_A(M) \cap \mathrm{Spec}(S^{-1}A)$, i.e., such that $P \cap S = \emptyset$. So $P = \mathrm{Ann}_A(x)$ for some $0 \neq x \in M$. If $\frac{a}{s}x = 0$ then $\exists\, t \in S$ such that $tax = 0$. Note that $t \notin P$, so $ta \in P = \mathrm{Ann}(x)$, and $t \notin P$ implies that $a \in P$. This gives that $\mathrm{Ann}_{S^{-1}A}(x) \subseteq S^{-1}P$. It is obvious that $S^{-1}P \subseteq \mathrm{Ann}_{S^{-1}A}(x)$. Therefore $\mathrm{Ann}_{S^{-1}A}(x) = S^{-1}P \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M) = \mathrm{Ass}_A(S^{-1}M)$.

Now let us take $P = \mathrm{Ann}_A(y)$ for some $0 \neq y \in S^{-1}M$, and write $y = \frac{x}{s}$. We claim that $P \cap S = \emptyset$ since if $s \in P \cap S$, then $\frac{x}{1} = \frac{s}{1}\frac{x}{s} = s\frac{x}{s} = 0$, but $\frac{x}{1} = 0$ implies $y = \frac{1}{s} \cdot \frac{x}{1} = 0$, a contradiction.

(We remark here that it can be proved that $P = \mathrm{Ann}_A(\frac{x}{1})$, but this does not imply that $P = \mathrm{Ann}_A(x)$, because in general $\frac{m}{1} = 0$ does not imply $m = 0$, for $m \in M$.)

Note that $\forall\, a \in P$, $a\frac{x}{1} = \frac{a}{1}\frac{x}{1} = \frac{sa}{1}\frac{x}{s} = s \cdot a \cdot \frac{x}{s} = 0$, so $\exists\, t \in S$ such that $tax = 0$.

But $P = (a_1, ..., a_n)$ since A is Noetherian, so $\forall\, a_i$, take $t_i \in S$ such that $t_i a_i x = 0$. Let $t_0 = t_1 \cdots t_n$, and we claim that $\mathrm{Ann}_A(t_0 x) = P$. This final claim concludes the proof of the theorem. For the reverse inclusion stated in the final claim, if $a \in P$, then $a = \sum_{i=1}^{n} \lambda_i a_i$, with $\lambda_i \in A$. Then $at_0x = \sum_{i=1}^{n} \lambda_i t_i \cdots t_n a_i x = 0$, which implies $a \in \mathrm{Ann}_A(t_0 x)$.

For the forward inclusion, if $at_0x = 0$ then $a\frac{x}{1} = a\frac{t_0 x}{t_0} = \frac{at_0x}{t_0} = \frac{0}{t_0} = 0$. □

Corollary 1.3. *If A is Noetherian, $P \in Spec(A)$, and M is an A-module, then $P \in Ass(M)$ if and only if $PA_P \in Ass(M_P)$.*

Theorem 1.4. *If M is finitely generated over A, a Noetherian ring, then the following are true:*

1. *$Ass(M)$ is finite and nonempty.*
2. *$Ass(M) \subseteq \{P \in Spec(A) \mid M_P \neq 0\} := Supp(M)$. We say that M is supported at P if $P \in Supp(M)$.*
3. *$\min(Ass(M)) = \min(Supp(M))$. In particular, for $I \leq A$, there are finitely many prime ideals that are minimal over I.*

Proof. The first assertion has been proved already. For the second, it is enough to note that if P is associated to M then A/P injects into M and so A_P/PA_P injects into M_P. Hence M_P is nonzero.

For the third assertion, we need to show that if M is supported at P and P is minimal with this property, then P is associated to M. The A_P-module M_P is nonzero and hence it has at least an associated prime. Let $Q \subset P$ such that QA_P is associated to M_P. Then Q is associated to M and hence in the support of M. The minimality of P now shows that $Q = P$ and so $P \in \mathrm{Ass}(M)$.

□

Proposition 1.5. *Let ${}_AM$ be finitely generated. Then $Supp(M) = V(Ann_A(M)) = \{P \in Spec(A) \mid P \supseteq Ann_A(M)\}$.*

Proof. Let $M = Am_1 + \cdots + Am_r$ with $m_i \in M$. Then $P \in \mathrm{Supp}(M)$ is equivalent to $M_P \neq 0$, which happens if and only if $\exists\, i$ such that $\frac{m_i}{1} \neq 0$ in M_P, which is equivalent to $\mathrm{Ann}(m_i) \subseteq P$ (since $\frac{m_i}{1} = \frac{m_i}{s}\frac{s}{1} = 0$, if $s \notin P$ and $s \in \mathrm{Ann}(m_i)$). This holds if and only if $\mathrm{Ann}_A M = \bigcap_{i=1}^{r} \mathrm{Ann}(m_i) \subseteq P$. □

Corollary 1.6. *If $I \leq A$, with A Noetherian, then the set of minimal prime ideals containing I is finite.*

Proof. Take $M = A/I$, then $\mathrm{Supp}(M) = V(I) = \{P \in \mathrm{Spec}(A) \mid P \supseteq I\}$. So $\min(I) := \{P \in \mathrm{Spec}(A) \mid P \text{ minimal over } I\}$ is finite because $\min(I) = \min(\mathrm{Supp}(A/I))$ which equals $\min(\mathrm{Ass}(A/I))$. By the above theorem this set is indeed finite. □

Definition 1.7. *If $n \geq 1$, and $P \in Spec(A)$, then $P^{(n)} = P^n A_P \cap A = \{a \mid a \in P^n A_P\} = \{a \mid \exists\, s \notin P,\ sa \in P^n\}$. This is called* the n^{th} symbolic power *of P. It is the preimage in A of the primary ideal $P^n A_P$ in A_P so it is a primary ideal itself. Its radical equals P so $P^{(n)}$ is P-primary in A.*

Definition 1.8. *Let M be a finitely generated module over a Noetherian ring A. If $Supp(M) = V(Ann_A(M))$, and $\min(Supp(M)) = \{P_1, ..., P_r\}$, then $P_1, ..., P_r$ are called isolated associated primes of M. The remaining primes are called embedded associated primes of M. If $M = A/I$, with $I \leq A$, one talks of isolated (embedded) associated primes of I. Associated primes of I were once called prime divisors of I.*

Definition 1.9. *Let M be a finitely generated module over a Noetherian ring A, and let $N \leq M$. If $N = \bigcap_{i=1}^{n} N_i$, with $N_i \leq M$, then this equality is called a finite decomposition.*

Definition 1.10. *N is called* irreducible *if whenever $N = N_1 \cap N_2$ implies $N_1 = N$ or $N_2 = N$.*

Proposition 1.11. *In a Noetherian module M, each $N \leq M$ has an irreducible decomposition, i.e., a finite decomposition with all terms being irreducible.*

Proof. Let $\mathcal{F} = \{N \mid N \text{ is not a finite intersection of irreducible submodules}\}$. Assume $\mathcal{F} \neq \emptyset$. Take N_0 a maximal element of $\mathcal{F}$ (this exists since M is Noetherian). Then N_0 is reducible, so $N = N_1 \cap N_2$, with $N_i \supset N_0$. But then N_1 and N_2 are finite intersections of irreducible submodules, which implies $N_0 = N_1 \cap N_2$ is a finite intersection of irreducible submodules, a contradiction. Thus $\mathcal{F} = \emptyset$. □

Definition 1.12. *A primary decomposition $\bigcap_{i=1}^{n} N_i$ is a decomposition where all the N_i are primary. If $\bigcap_{i=1}^{n} N_i$ is a primary decomposition, and $N \subset \bigcap_{i \neq j} N_i$ for all j, then this decomposition is called irredundant. Note that $\sqrt{Ann(M/N_i)} = P_i$ are prime ideals, (since $\{P_i\} = Ass_A(M/N_i)$), so if $i \neq j$, and if $P_i \neq P_j$ and $N = \bigcap_{i=1}^{n} N_i$ is irredundant, then the decomposition is called minimal.*

Exercise: Show that if N_1, N_2 are P-primary, then $N_1 \cap N_2$ is P-primary.

So given a primary decomposition, we can find an irredundant one, and then, by using the claim of above exercise, obtain a minimal one.

Theorem 1.13. *Let M be a finitely generated module over a Noetherian ring A. Then the following are true:*

(1) *An irreducible submodule $N \leq M$ is primary.*
(2) *If $N = N_1 \cap \cdots \cap N_r$ with $Ass(M/N_i) = \{P_i\}$ for $1 \leq i \leq r$ is an irredundant primary decomposition of $N \leq M$, then $Ass(M/N) = \{P_1, ..., P_r\}$. This says that the P_i's are uniquely determined by N and M.*
(3) *Every $N \leq M$ has a primary decomposition, in fact a minimal primary decomposition. If $P \in Ass(M/N)$ with P minimal in $Ass(M/N)$ then the P-primary component of N is $\phi_P^{-1}(N_P)$ where ϕ_P is the localization map $M \to M_P$. (If*

$N = \bigcap_{i=1}^{n} N_i$ *is a minimal primary decomposition and* $\{P_i\} = Ass(M/N_i)$ *then* N_i *is called the* P_i*-primary component and so it is unique.)*

Proof. (1) N irreducible in M is equivalent to 0 irreducible in M/N, since for $K \subseteq N$, N is primary if and only if N/K is primary in M/K. We claim that if 0 is irreducible, then 0 is primary. If not, then $\text{Ass}(M) \supseteq \{P_1, P_2\}$ with $P_1 \neq P_2$ prime ideals. (If there were only one associated prime of M, then M would be coprimary, so 0 would be primary.) Then we have

$$A/P_1 \cong K_1 \hookrightarrow M$$

$$A/P_2 \cong K_2 \hookrightarrow M.$$

But $K_1 \cap K_2 = 0$ since if $x \in K_1 \cap K_2$, then $P_1 x = 0$ which says $P_1 \subseteq \text{Ann}(x)$. However, we proved that for a cyclic module with only one associated prime (such as K_1 and K_2) the annihilator of the generator equal the annihilator of any nonzero element in the module. Therefore $P_1 \subseteq \text{Ann}(x) = P_2$. We have $P_2 \subseteq P_1$ by symmetry, and their equality is a contradiction. So 0 is primary in M/N which implies N is primary in M.

(2) Without loss of generality, assume $N = 0$, so $0 = N_1 \cap \cdots \cap N_r$ is irredundant, and $\text{Ass}(M/N_i) = \{P_i\}$. Consider the map $\phi : M \to \bigoplus_{i=1}^{r} M/N_i$ which acts by $m \mapsto \oplus_i \overline{m_i}$. Since $\ker(\phi) = (\bigcap N_i) = 0$, the map is injective, so $\text{Ass}(M) \subseteq \text{Ass}(\bigoplus M/N_i) = \bigcup_{i=1}^{r} \text{Ass}(M/N_i) = \{P_1, ..., P_r\}$. The irredundancy implies $N_2 \cap \cdots \cap N_r \neq 0$, so take $0 \neq x \in N_2 \cap \cdots \cap N_r$. Since $0 = N_1 \cap \cdots \cap N_r$, we can easily see that $\text{Ann}(x) = N_1 : 0$. M/N_1 is P_1-primary and this implies that $\text{Ann}(M/N_1)$ has its radical equal to P_1. hence, there is k such that $P_1^k \subset \text{Ann}(M/N_1)$. Hence, $P_1^k \cdot M \subset N_1$, in particular $P_1^k \cdot x \subset N_1$ which implies that $P_1^k \cdot x = 0$. Let $i \geq 0$ such that $P_1^i \cdot x \neq 0$ and $P_1^{i+1} x = 0$ (i can be equal to zero). Let $0 \neq y \in P_1^i x$ and so $y \in N_2 \cap \cdots \cap N_r$. Clearly $P_1 \cdot y = 0$. On the other hand $y \notin N_1$, because $y \neq 0 = N_1 \cap \cdots \cap N_r$. Then $0 \neq \overline{y} \in M/N_1$, and so $\text{Ann}(y) \subseteq P_1$. On the other hand, $P_1 \subset \text{Ann}(y)$ and in conclusion, $P_1 = \text{Ann}(y)$ which says that $P_1 \in \text{Ass}(M/N)$.

(3) Let $N = N_1 \cap \cdots \cap N_r$, and take $P \in \text{Ass}(M/N)$ a minimal prime. Then if N_1 is the P-primary component, we have $N_1 = \phi_P^{-1}(N_P)$ where $\phi_P : M \to M_P$. Then

$N_P = (N_1)_P \cap \cdots \cap (N_r)_P$. We claim that for $i \neq 1$, $(M/N_i)_P = 0$. We know $\frac{\overline{m}}{a} = 0$ if and only if $\exists\, b \in P$ such that $b\overline{m} = 0$ (Take $b \in P_i - P$). Since $P_i \subseteq P$, the claim is true. This is equivalent to $M_P/(N_i)_P = 0$ which is true if and only if $M_P = (N_i)_P$. Now localize $N = N_1 \cap \cdots \cap N_r$ at P and get that $N_P = (N_1)_P$. We leave showing $N_1 = \phi_P^{-1}(N_P)$ as an exercise.

$\square$