Lecture 8

1. Associated Primes and Primary decomposition for Modules

We would like to describe in more detail the structure of submodules of a finitely generated module M over a Noetherian ring A. We will do this by investigating special submodules (called primary) and special primes of A called associated primes.

Throughout this lecture, A will be a ring and M, N etc will be A-modules.

Definition 1.1. *Let $a \in A$. Then $\theta_a : M \to M$ defined by $\theta_a(m) = am$, for $m \in M$ is called the homothety of ratio a.*

Definition 1.2. *Let A be a Noetherian ring and M an A-module. We say that M is coprimary if for all $a \in A$, θ_a is either injective or nilpotent.*

Proposition 1.3. *If M is coprimary, then*

$$\{a : \theta_a \text{ is nilpotent}\} = \sqrt{\mathrm{Ann}_A(M)}$$

is a prime ideal.

Proof. Let $ab \in \sqrt{\mathrm{Ann}_A(M)}$ and assume that $a \notin \sqrt{\mathrm{Ann}_A(M)}$. So, θ_a is not nilpotent, hence it is injective. That is, $am = 0$ implies $m = 0$.

But, since there exists n such that $(ab)^n M = 0$, we get $a^n b^n M = 0$ and, hence, $b^n M = 0$ or $b \in \sqrt{\mathrm{Ann}_A(M)}$. □

Definition 1.4. *If M is coprimary and if $P = \{a : \theta_a \text{ is nilpotent}\}$, then we say that M is P-coprimary.*

Proposition 1.5. *Let M be a P-coprimary finitely generated A-module and $0 \neq N$ a submodule of M. Then N is P-coprimary.*

Proof. It follows directly from the definition that N is coprimary. Let us prove that $\sqrt{\mathrm{Ann}_A(N} = P$.

Since $N \subseteq M$ then $\mathrm{Ann}_A(M) \subseteq \mathrm{Ann}_A(N)$ and then $P = \sqrt{\mathrm{Ann}_A(M)} \subseteq \sqrt{\mathrm{Ann}_A(N)}$.

Now let $a \in \sqrt{\mathrm{Ann}_A(N)}$ and n minimal such that $a^n N = 0$. Since $a^{n-1}N \neq 0$ this means that θ_a is not injective on M. Hence it is nilpotent on M and so $a \in P$.

□

Theorem 1.6. *Let M be a P-coprimary module. Let Q be a prime ideal of A.*

Then A/Q injects into M as an A-submodule, i.e. there exists an A-linear injection

$$A/Q \to M,$$

if and only if $Q = P$.

Proof. First let us show that we can embed A/P into M (via an A-linear map). It is enough to show that A/P embeds into a cyclic submodule of M.

Now, let us note that every cyclic submodule Am with $m \neq 0$ is P-coprimary by the above result.

So we can assume that $M = Am$, for some $m \neq 0$. Let $I = \mathrm{Ann}_A(m)$. Then $A \to M$ via $\phi(a) = am$ has kernel equal to I and so A/I is isomorphic to $Am = M$.

We will show that A/P can be embedded in A/I. Let n minimal with the property $P^n \subseteq I$. This n exists because A/I is P-coprimary.

Let $x \in P^{n-1} \setminus I$. Consider the map $f : A \to A/I$ $f(x) = \overline{ax}$, for $a \in A$. Let us prove that $\mathrm{Ker}(f) = P$. Since $Px \subseteq P^n \subseteq I$, we see that P is in the kernel. Now let a such that $ax \in I$. Then $axm = 0$. But $xm \neq 0$ as $x \notin I$. Then θ_a is not injective on M and so θ_a must be nilpotent. In other words, $a \in P$.

Therefore $A/P \to A/I$ via f, and this shows that A/P embbeds into M.

Now assume that A/Q embeds into M, then A/Q is P-coprimary. But $\mathrm{Ann}_A(A/Q) = Q$ is prime and hence $Q = \sqrt{Q} = P$.

□

Definition 1.7. *Let $N \leq M$. Then N is called primary if for all $a \in A, m \in M$ such that $am \in N$ and $m \notin N$, then $\exists$ n such that $a^n M \subseteq N$. In other words, if a is a zerodivisor of M/N, then $a \in Rad(Ann_A(M/N))$.*

Remark 1.8. $I \leq A$ is a primary ideal if and only if I is a primary submodule of A.

Example 1.9. In $A = k[x, y]$, consider $(xy, x^2) = (x) \cap (x^2, y)$ where the first ideal is prime and the latter is primary. We can also write $(xy, x^2) = (x) \cap (x, y)^2 = (x) \cap (x^2, xy, y^2)$. Note that $\sqrt{(x^2, y)} = \sqrt{(x^2, xy, y^2)} = (x, y)$. As one can note, the number of terms in the decompositions as well as the radicals of the ideals that are part of the decompositions are the same. This is no coincidence, and the primary decomposition theory will explains this phenomenon.

Definition 1.10. *If A is a ring, M is an A-module, and $P \in Spec(A)$. Then P is called an associated prime if $\exists\ 0 \neq m \in M$ such that $P = Ann_A(m)$. The set of associated primes is denoted $Ass_A(M)$.*

Remark 1.11. *$Am \cong \dfrac{A}{Ann_A(m)}$. If $Ann_A(m) = P$, then $Am \cong A/P \hookrightarrow M$ since $Am = A \cdot m \subseteq M$. On the other hand, if $A/P \hookrightarrow M$, then $Ann_A(\overline{1}) = P$, and $A \cdot \overline{1} \hookrightarrow M$, hence $P \in Ass(M)$. So $P \in Ass(M) \Leftrightarrow A/P \hookrightarrow M$.*

Proposition 1.12. *Take $P \in Ass(M)$. Denote $F = Am \cong A/P \hookrightarrow M$, and $P = Ann_A(m)$. If $0 \neq y \in F$, then $Ann_A(y) = P$.*

Proof. We know $y = am$, so $P \subseteq \mathrm{Ann}_A(y)$. Now take $b \in A$ such that $by = 0$. This implies $abm = 0$, so $ab \in P$, and since $a \notin P$, we must have $b \in P$. Thus $\mathrm{Ann}_A(y) \subseteq P$. □

Remark 1.13. *If $M = \bigcup_{i \in I} M_i$, then $Ass(M) = \bigcup_{i \in I} Ass(M_i)$.*

Remark 1.14. *Let M be a finitely generated module over a Noetherian A. The following equivalence will be a consequence of the results we will prove in this lecture as well as Lecture 9: $|Ass(M)| = 1$ if and only if M is A-coprimary. The reader should observe that Theorem 1.6 shows that if M is P-coprimary, then there is only associated prime of M, namely P. The converse will need more work.*

Proposition 1.15. *Take M an A-module such that $M \neq 0$. Let $\mathcal{P} = \{Ann_A(m) \mid 0 \neq m \in M\}$. The maximal elements of $\mathcal{P}$ are prime.*

Proof. Let P be such a maximal element. Let $ab \in P$, so $abm = 0$. We know $bm \neq 0$ if and only if $b \notin P$, so $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(bm)$. But $P = \mathrm{Ann}(m)$ is maximal, so $P = \mathrm{Ann}(m) = \mathrm{Ann}(bm)$. Since $a \in \mathrm{Ann}(bm)$, we have $a \in P$, and have shown that P is prime. □

Corollary 1.16. *If $0 \neq_A M$ with A Noetherian, then $Ass(M) \neq \emptyset$. Moreover, every zerodivisor of M lives in some associated prime of M. More precisely,*

$$ZD(M) = \{a \in A \mid \exists\, m \neq 0, m \in M\ am = 0\} = \bigcup_{P \in Ass(M)} P.$$

Proof. For the first part, if A is Noetherian, then $\mathcal{P}$ has maximal elements. For the second, let a be a zerodivisor of M. So $am = 0$ for some $m \neq 0$, so $a \in \mathrm{Ann}_A(m)$. But for all ideals $I = \mathrm{Ann}_A(m)$, one can find a maximal element of $\mathcal{P}$ that contains I. By the previous theorem, this element is an associated prime, say Q. So $a \in \mathrm{Ann}(m) \subseteq Q \in \mathrm{Ass}(M)$. So $ZD(M) \subseteq \displaystyle\bigcup_{P \in \mathrm{Ass}(M)} P$.

For the reverse implication, if $a \in \bigcup_{P \in \mathrm{Ass}(M)} P$ then $a \in P \in \mathrm{Ass}(M)$, so $P = \mathrm{Ann}_A(m)$ for some nonzero $m \in M$. But then $am = 0$ which implies a is a zerodivisor of M. □

Proposition 1.17. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, then $Ass(M) \subseteq Ass(M') \cup Ass(M'')$.*

Proof. Denote by $\mu : M' \to M$ and $\sigma : M \to M''$ the maps that appear in our short exact sequence. For the first inclusion, if $P \in \mathrm{Ass}(M')$ then $\exists\, 0 \neq m'$ such that $P = \mathrm{Ann}(m')$. But $P = \mathrm{Ann}(\mu(m'))$ since μ is injective, and $\mu(m') \neq 0$. So $P \in \mathrm{Ass}(M)$. For the second inclusion, let $P \in \mathrm{Ass}(M)$. Then $P = \mathrm{Ann}(m)$ for some $0 \neq m \in M$. If $Am \cap \mathrm{Im}(\mu) \neq 0$, then $\exists\, m_0 = \mu(m_0') = am \neq 0$, with $m_0' \in M'$. Remember that $P = \mathrm{Ann}(m_0) = \mathrm{Ann}(\mu(m_0')) = \mathrm{Ann}(m_0')$ since μ is injective. Thus $P \in \mathrm{Ass}(M')$. If $Am \cap \mathrm{Im}(\mu) = 0$, then $P = \mathrm{Ann}(\sigma(m))$. If $a \in P$ then $am = 0$ which implies $a\sigma(m) = 0 \Rightarrow \sigma(am) = 0$, so $am \in \ker\sigma = \mathrm{Im}(\mu)$, so $am \in \mathrm{Im}(\mu)$, so $am = 0$ which gives $a \in P$. Also, $\sigma(m) \neq 0$, since if not, then $m \in \ker\sigma = \mathrm{Im}(\mu)$ which says $m \in Am \cap \mathrm{Im}(\mu) = 0$, a contradiction. □

Corollary 1.18. *$Ass(\displaystyle\bigoplus_{i=1}^{n} M_i) = \bigcup_{i=1}^{n} Ass(M_i)$.*

Proof. We show for $n = 2$, which is the main case that implies the general statement. Consider the exact sequence $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$. By the above theorem, we have that both $\mathrm{Ass}(M_1)$ and $\mathrm{Ass}(M_2)$ belong to $\mathrm{Ass}(M_1 \oplus M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$. □

2. The Prime Filtration

Proposition 2.1. *Let M be finitely generated over A, and A be a Noetherian ring. Then there is a chain of submodules of M $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} \cong A/P_i$ where $P_i \in Spec(A)$.*

Proof. We will build the filtrations recursively. Assume we have chosen $M_0, ..., M_{i-1}$ with $i \geq 1$. If $M_{i-1} \neq M$, then $M/M_{i-1} \neq 0$, so $\exists\, P_i \in \mathrm{Ass}(M/M_{i-1}) \subseteq \mathrm{Spec}(A)$. Recall that $A/P_i \hookrightarrow M/M_{i-1}$. Take $M_i \subseteq M$ such that $M_i/M_{i-1} = A/P_i$ which is the image of the inclusion. So we have M_i as needed, and the procedure stops at $M_n = M$, for some n, since M is Noetherian. □

The above result often allows one to reduce statements about finitely generated arbitrary A-modules to modules of the type A/P, P prime ideal, by using techinques involving short exact sequences whenever possible.

Corollary 2.2. *If A is Noetherian, and M is finitely generated over A, then $Ass(M)$ is finite.*

Proof. Look at our prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$. Then $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_{n-1}) \cup \mathrm{Ass}(M/M_{n-1})$ since as we have the following short exact sequence $0 \to M_{n-1} \to M \to M/M_{n-1} \to 0$. Repeat for M_{n-1}, and obtain that $\mathrm{Ass}(M) \subset \bigcup_{i=1}^{n} \mathrm{Ass}(M_i/M_{i-1}) = \bigcup_{i=1}^{n} \mathrm{Ass}(A/P_i) = \{P_1, ..., P_n\}$. □

Remark 2.3. *Under the conditions of the Proposition we have that $Ass(M) \subset \{P_1, .., P_n\}$ as the proof of the Corollary shows.*

If ${}_AM$ is Noetherian and Artinian, then by definition one can produce a filtration (called composition series) as in the Proposition 2.1 where the factors are simple modules, and hence quotients of A by maximal ideals. The important fact is that Proposition 2.1 holds for a much larger class of modules.