Lecture 7

1. Modules of fractions

Let $S \subseteq A$ be a multiplicative set, and ${}_AM$ an A-module. In what follows, we will denote by s, t, u elements from S and by m, n elements from M.

Similar to the concept of rings of fractions, we can define

$$S^{-1}M = \{\frac{m}{s} \mid m \in M,\ s \in S\}.$$

This set is the set of equivalence classes of the equivalence relation $\equiv$ on $M \times S$ defined by

$$(m, s) \equiv (n, t)$$

if and only if $\exists\ u \in S$ such that $u(mt - sn) = 0$. The equivalence class of (m, s) will be denoted by $\frac{m}{s}$.

Therefore $\dfrac{m}{s} = \dfrac{n}{t}$ if $\exists\ u \in S$ such that $u(mt - sn) = 0$. This set with the natural addition and multiplication with scalars from $S^{-1}A$ defines an $S^{-1}A$-module: the $S^{-1}A$-module structure is defined by $\dfrac{a}{s} \cdot \dfrac{m}{t} = \dfrac{am}{st}$. By restriction of scalars to A, $S^{-1}M$ is an A-module as well.

The natural map $\phi_M : M \to S^{-1}M$, $\phi_M(m) = \frac{m}{1}$ is an A-module homomorphism.

This construction has numerous similarities to that of rings of fractions. We will present them without proof.

Definition 1.1. *Let $a \in A$. Let $\theta_a : M \to M$ defined by $\theta_a(M) = am$. This defines an A-linear map on M, called the homothety (of ratio) a on M. Obviously, θ_a is injective if and only if a is nonzerodivisor on M.*

Definition 1.2. *Let S be a multiplicative set in A and M an A-module. We say that M is S-torsion free if θ_s is injective, for all $s \in S$. We say that M is S-divisible if θ_s is surjective for all $s \in S$. The module M is said to be of S-torsion if $M = \cup_{s \in S} Ker(\theta_s)$.*

If A is domain and $S = A \backslash \{0\}$, we refer to the M as torsion free, divisible, respectively torsion.

Proposition 1.3. *Let $S \subseteq A$ be a multiplicative set. Then $S^{-1}M$ is S-torsion-free and S-divisible.*

Proof. Let $s \in S$ and $\frac{m}{t} \in S^{-1}M$ such that $s \cdot \frac{m}{t} = 0$. Then $\frac{sm}{t} = 0$ and so there exist $u \in S$ such that $usm = 0$. But this shows that $\frac{m}{t} = 0$ as well.

The second property can also be easily checked in a similar fashion.

□

Theorem 1.4 (Universal Property)**.** *Let A be a ring, M an A-module and S a multiplicatively set in A. Let $\phi_M : M \to S^{-1}M$ be the natural map and $f : M \to N$ a A-linear map.*

If N is an S-torsion free and S-divisible, then there exists a unique A-module homomorphism $g : S^{-1}M \to N$ such that $g \circ \phi_M = f$.

Let $N \in \mathcal{L}_A(M)$. We say that N is S-saturated if for $s \in S, m \in M$, $sm \in N$ implies that $m \in N$.

Proposition 1.5. *Every submodule can be embedded in a saturated one.*

Indeed, let

$$Sat_S(N) = \{m \in M : \text{there exists } s \in S \text{ such that } sm \in N\}.$$

Then $Sat_S(N)$ is an S-saturated submodule of M and contains N.

Observation 1.6. *$S^{-1}N = S^{-1}Sat_S(N)$, and if $S^{-1}N = S^{-1}K$, where K is a submodule of M, then $K \subseteq Sat_S(N)$.*

Indeed, let $\frac{m}{s} \in S^{-1}Sat_S(N)$. Then there exists $t \in S$ such that $tm \in N$ and so

$$\frac{m}{s} = \frac{tm}{ts} \in S^{-1}N.$$

If $k \in K$ then $\frac{k}{1} = \frac{n}{s}$, and so there exists $u \in S$ such that $usk = un \in N$ which means that, in particular, $k \in Sat_S(N)$.

Theorem 1.7. *Let S be a multiplicative set in A and M an A-module. Let $\phi_M : M \to S^{-1}M$ be the natural map. The map $N' \to \phi_M^{-1}(N') = N' \cap M$ establishes a bijection between $\mathcal{L}_{S^{-1}A}(S^{-1}M)$ and the subset of $\mathcal{L}_A(M)$ consisting of S-saturated submodules. This bijection preserves inclusions and arbitrary intersections.*

Proof. Let us shows that $N := \phi_M^{-1}(N')$ is S-saturated. Indeed, if $sm \in N$ then $\phi_M(sm) \in N'$, so $s\phi_M(m) \in N'$. This implies that $\phi_M(m) = \frac{1}{s} \cdot s\phi_M(m)$ belongs to N' as well, so $m \in N = \phi_M^{-1}(N')$.

Let us check that $S^{-1}N = N'$. Since by definition, $\phi(N) \subseteq N'$, and N' is an $S^{-1}A$-module we get that $S^{-1}N \subseteq N'$:

$$\frac{n}{s} = \frac{1}{s} \cdot \frac{n}{1} = \frac{1}{s} \cdot \phi_M(n) \in N'.$$

Conversely, let $\frac{n}{s} \in N'$. Then $\frac{n}{1} = s \cdot \frac{n}{s} \in N'$ as well so $n \in \phi_M^{-1}(N') = N$. But then $\frac{n}{s} \in S^{-1}N$ by definition.

The rest of statements involve similar checks and we leave them to the reader.

□

2. Brief Review of Tensor Products

Let M, N, and E be A-modules.

Definition 2.1. *The map $f : M \times N \to E$ is called A-bilinear if f is linear on each term of the direct product. Specifically, this means that $f(am, n) = af(m, n) = f(m, an)$, and that $f(-, n)$ and $f(m, -)$ are A-linear $\forall\, m \in M$, $n \in N$.*

Note that an A-linear map f where $M \times N$ is regarded as an A-module implies, in particular, that $af(m, n) = f(am, an)$. This shows that the concept of an A-bilinear map on $M \times N$ is different from that of an A-linear map on $M \times N$.

The tensor product $M \otimes_A N$ is an A-module with elements of the form $\sum_{i=1}^{k} m_i \otimes n_i$, where $m_i \in M, n_i \in N$ and $k \in \mathbb{N}$. The elements $m \otimes n$ are called **tensor monomials** and are subject to the following rules:

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$

$$a(m \otimes n) = am \otimes n = m \otimes an,$$

for all $a \in A, m, n_1, m_2 \in M, n, n_1, n_2 \in N$.

Moreover, we have as a consequence that $0 \times m = n \otimes 0 = 0$, $(-m) \otimes n = m \otimes (-n) = -(m \otimes n)$, $rm \otimes n = m \otimes rn = r(m \otimes n)$ for all $r \in \mathbb{Z}$.

There exists a natural bilinear map $i : M \times N \to M \otimes_A N$, defined by $i(m, n) = m \otimes n$.

Theorem 2.2 (The Universal Property of Tensor Products)**.** *For every E and every $f : M \times N \to E$ that is bilinear, there exists a unique A-linear map g, with $g : M \otimes_A N \to E$ such that $g \circ i = f$.*

2.1. **Facts about Tensor Products.**

Proposition 2.3. *The following are true:*

(1) *$M \otimes_A N \cong N \otimes_A M$.*

(2) *$A \otimes_A M \cong M_A \cong M \otimes_A A$.*

(3) *$(M \otimes_A N) \otimes_A E \cong M \otimes_A (N \otimes_A E)$.*

(4) *$(\bigoplus_{i\in I} E_i) \otimes_A M \cong \bigoplus_{i\in I} E_i \otimes_A M$. Note that this is not true for infinite products and for Hom.*

(5) *If M and N are A-free with $\{m_i\}$ and $\{n_j\}$ for bases, then $M \otimes_A N$ is free with $\{m_i \otimes n_j\}_{i,j}$ as a basis.*

These statements can be proved using the universal property of the tensor product. We will end our short treatment with the following proposition that shows the reader how to use the universal property to establish an isomorphism that involves the tensor product.

Proposition 2.4. *Let S be a multplicative system in A, and M and A-module. The there exists an isomorphism of A-modules*

$$g : S^{-1}A \otimes M \to S^{-1}M,$$

such that $g(\frac{a}{s} \otimes m) = \frac{am}{s}$, for all $s \in S, A \in A, m \in M$.

Proof. First let us show that such a map g exists. We will use the universal property of the tensor product of modules, so let us note first that that the map

$$f : S^{-1}A \times M \to S^{-1}M,$$

such that $g(\frac{a}{s} \otimes m) = \frac{am}{s}$, for all $s \in S, A \in A, m \in M$ is A-bilinear.

Therefore, this induces an A-linear map g as in the statement of the theorem. It remains to show that g is an isomorphism.

To show that g is injective let $z \in S^{-1}A \times M$ such that $g(z) = 0$, Then $z = \sum_{i=1}^{k} \frac{a_i}{s_i} \otimes m_i$ with $a_i \in A, s_i \in S, m_i \in M$.

Let $s = s_1 \cdots s_k, t_i = \prod_{j\neq i} s_j$. Rewrite $z = \sum_{i=1}^{k} \frac{a_i t_i}{s} \otimes m_i = \frac{1}{s} \otimes (\sum_{i=1}^{k} a_i t_i m_i)$.

Let $m = \sum_{i=1}^{k} a_i t_i m_i$. Since $g(z) = \frac{m}{s}$, we get that there exists $u \in S$ such that $um = 0$. But then

$$z = \frac{1}{s} \otimes m = \frac{u}{su} \otimes m = \frac{1}{su} \otimes um = 0.$$

This shows that g is injective.

The surjectivity of g is trivial.

An alternative to showing that g is bijective is defining an A-linear map $g : S^{-1}M \to S^{-1} \otimes M$ by $h(m/s) = 1/s \otimes m$.

□

3. Extension and restriction of scalars

Let B an A-algebra. This means that B is a ring and that there exists a ring homomorphism $\phi : A \to B$.

Let E be an B-module. We can consider E as an A-module naturally by letting $a \cdot e := \phi(a)e$, for all $a \in A, e \in E$. One says that the A-module E is obtained by *restriction of scalars*. It should be remarked that often the map ϕ is dropped from notation and instead of $\phi(a)e$ one writes ae. It is obvious that every $f : E \to F$ B-linear map of B-module induces an A-linear map between the A-modules E, F obtained by restriction of scalars.

Let M be an A-module. Then $B \otimes_A M$ is an A-module which can be endowed with a natural B-module structure as follows: $b \cdot (c \otimes m) := (bc) \otimes m$, for all $b, c \in B, m \in M$. One says that this B-module $B \otimes_A M$ is obtained by *extension of scalars*.

One can easily check that any A-linear map $f : M \to N$ between A-modules can be lifted to a B-linear map $1 \otimes f : B \otimes_A M \to B \otimes_A N$ such that $(1 \otimes f)(b \otimes m) = b \otimes f(m)$, for all $b \in B, m \in M$. Often the newly constructed map $1 \otimes f$ is still denoted by f.

An easy check establishes the following

Corollary 3.1. *The map in Propositio 2.1 is an $S^{-1}A$-module isomorphism, where $S^{-1}A \times M$ is regarded as an $S^{-1}A$-module by extension of scalars.*