1 Localization

Let A be a ring, and $S \subset A$ a multiplicative set. We will assume that $1 \in S$ and $0 \notin S$. (If $1 \notin S$ then use $\hat{S} = \{1\} \cup S$.)

Definition 1.1. *S is called* **saturated** *if $x \in S$ and $y|x$ implies $y \in S$.*

We regard $S^{-1}A$ as the set of equivalence class of $A \times S$ under the equivalence relation (the reader is invited to check this):

$$(a, s) \sim (b, t),$$

if and only if there exists $u \in S$ such that $u(at - bs) = 0$.

The equivalence class of (a, s) is denoted by $\frac{a}{s}$. Define the addition and multplication in $S^{-1}A$ mimicking the corresponding laws for fractions. One can check that with these definitions $S^{-1}A = \{\frac{a}{s} \,|\, a \in A, \quad s \in S\}$ is a ring.

Clearly, for $s \in S$, we have $s \cdot \frac{1}{s} = 1$ in $S^{-1}A$. Addition and multiplication are well defined, but checking this requires a bit of routine work; we show the addition case here. Let $\frac{a}{s} = \frac{a'}{s'}$ and $\frac{b}{t} = \frac{b'}{t'}$, then $\exists\, u, v \in S$ such that $u(as' - a's) = 0 = v(bt' - b't)$. We need to see that $\frac{at + bs}{st} = \frac{a't' + b's'}{s't'}$; that is, $\exists\, w \in S$ such that $w[(at + bs)s't' - st(a't' + b's')] = 0$. Taking $w = uv \in S$, the equation holds. That multiplication is also a well-defined operation can be verified similarly.

We can also regard $S^{-1}A$ as an A-algebra. The structure morphism is $\phi : A \to S^{-1}A$, a ring homomorphism, that acts by $a \mapsto \frac{a}{1}$. This is equivalent to defining the A-algebra structure on $S^{-1}A$ by $a \cdot \frac{b}{s} := \frac{ab}{s}$.

On a more general note:

If $\phi : A \to B$ is a ring homomorphism, then B is an A-algebra (via the map ϕ). Given

$I \leq A$ an ideal, we can expand I to B using the following notation: $I^e = (\phi(I)) = \phi(I)B$, the latter being linear combinations of elements in $\phi(I)$ with coefficients in B. By abuse of notation one writes $I^e = IB$. If $J \leq B$, then we can consider the contraction of J to A, which turns out to be an ideal of A, as $J^c := \phi^{-1}(J) = J \cap A$, the latter being another abuse of notation.

Theorem 1.2. *Let S be a multiplicative subset of A. Then there is an ideal P that is maximal with respect to inclusion among all the ideals contained in $A \setminus S$ (i.e., $P \cap S = \emptyset$). Furthermore, P is a prime ideal.*

Proof. This is another use of Zorn's Lemma. Let $M = \{I \leq A \,|\, I \cap S = \emptyset\}$. Then M is an (partially) ordered set with respect to inclusion. Let $\{I_i\}_{i \in \Lambda}$ be a chain of ideals in A. Set $I = \bigcup_i I_i$. Then I is an upper bound for our chain, and I is an ideal since $\forall\ i, j$, we know $I_i \subseteq I_j$ or vice-versa, and for set-theoretic reasons, $I \cap S = \emptyset$. Now Zorn's Lemma gives us an ideal P that satisfies the first statement of the theorem. For the second, let $xy \in P$, and assume, by way of contradiction, that $x \notin P$ and $y \notin P$. This implies that $P + Ax \supsetneq P \Rightarrow (P + Ax) \cap S \neq \emptyset$, and similarly for y. Then we can choose $z_1, z_2 \in (P+Ax) \cap S$, such that $z_1 = p_1 + ax$ and $z_2 = p_2 + by$, with $p_1, p_2 \in P$, $a, b \in A$. Now we have $z_1, z_2 \in S$ which implies $z_1 z_2 = p_1 p_2 + p_1 by + p_2 ax + abxy \in S \cap P$, a contradiction to this set being empty. Thus P is prime. $\square$

Theorem 1.3. *Let S be a multiplicative set. Then the following are equivalent:*

1. *S is saturated;*
2. *$A \setminus S = \bigcup_{i \in I} P_i$ where $\{P_i\}$ is the set of prime ideals of A such that $P_i \cap S = \emptyset$.*

Proof. Assume S is saturated, and consider $x \notin S$. Then $(x) \cap S = \emptyset$. By Zorn's Lemma, we can find a P with $(x) \subseteq P$ such that $P \cap S = \emptyset$ and P a prime ideal. This implies $x \in \bigcup P_i$ as defined above. For the other inclusion, let $x \in \bigcup P_i$. Then $\exists\ P$ prime such that $x \in P$. But $P \cap S = \emptyset$ so $x \in A \setminus S$.

Now assume (2). Let $x \in S$ and y be a divisor of x, so $x = ya$ for some $a \in A$. Since $x \in S$, we have $x \in A \setminus P_i$ for all i. Assume $y \notin S$. Then $y \in P_i$ for some i which implies

$x = ay \in P_i$ which would say $x \notin S$, a contradiction. Thus $y \in S$, so S is saturated. ☐

Example 1.4. The set of all units in A is a saturated multiplicative set. In fact, the complement of this multiplicative set equals the union of all maximal ideals in A.

Example 1.5. The set of all non-zero-divisors is a saturated multiplicative set.

Example 1.6. If P is a prime ideal then $A \setminus P$ is a saturated multiplicative set.

Corollary 1.7. *Let $Z(A)$ be the set of zero divisors in A (i.e., the complement of the set in the second example). Then $Z(A)$ can be wriiten as a union of prime ideals.*

Corollary 1.8. $\mathfrak{N}(A) = \bigcap_i P_i$ *where the intersection runs over all P_i prime ideals of A.*

Proof. Let $a \in \mathfrak{N}(A)$. Then $\exists\, n$ such that $a^n = 0$. But $0 \in P$ so $a^n \in P \Rightarrow a \in P$ which gives $\mathfrak{N}(A) \subseteq \bigcap_i P_i$. Conversely, let $a \in \bigcap_i P_i$, and assume $a \notin \mathfrak{N}(A)$. Then $S_a = \{1, a, a^2, ...\}$ is a multiplicative set so $\exists\ P$ a prime ideal such that $P \cap S = \emptyset$. But $a \in P \cap S$, a contradiction. ☐

Corollary 1.9. *If $I \leq A$, then $\sqrt{I} = \bigcap P_i$ where the intersection runs over all P_i prime ideals containing I. In fact, if A is Noetherian, then $\sqrt{I}$ is a finite intersection of prime ideals containing I.*

Proof. To get the first part apply the above Corollary to A/I and the use the correspondence theorem between ideals in A containing I and ideals of A/I.

For the second part, assume $\exists\ I$ such that $\sqrt{I}$ is not a finite intersection of primes. Take I such that $\sqrt{I}$ is maximal with this property (by the Noetherian hypothesis). If $\sqrt{I}$ is prime, we are done, so assume not. Let $a, b \in A$ such that $a, b \notin I$ but $ab \in I$. Consider the ideals (I, a) and (I, b). Then both $\sqrt{(I, a)}$ and $\sqrt{(I, b)}$ are finite intersection of prime ideals, and $\sqrt{I} \subseteq \sqrt{(I, a)} \cap \sqrt{(I, b)}$. Let x be in the intersection $\sqrt{(I, a)} \cap \sqrt{(I, b)}$. Then $\exists\ n, m \in \mathbb{N}$ such that $x^n = i_1 + ar$ and $x^m = i_2 + bs$ for $r, s \in R$ and $i_1, i_2 \in R$. Then $x^n x^m = i_1 bs + i_2 ar + abrs + i_1 i_2 \in I$ since $ab \in I$. Thus $\sqrt{I} = \sqrt{(I, a)} \cap \sqrt{(I, b)}$, a contradiction since the right hand side is a finite intersection. Thus $\sqrt{I}$ must be a finite intersection of primes. ☐

1.1 Universal Property of the Ring of Fractions

Recall the canonical ring homomorphism $\phi : A \to S^{-1}A$, $\phi(a) = a/1$. This is in fact an A-algebra homomorphism.

Then ϕ has the following properties:

1. ϕ is a ring homomorphism and $\phi(S) \subseteq$ units in $S^{-1}A$.
2. If $f : A \to A'$ is a ring homomorphism with $f(S) \subseteq$ units of A' then there exists a unique homomorphism $g : S^{-1}A \to A'$ such that $g \circ \phi = f$ (this is the universal property of the ring of fractions).

Proof. For part (1), $\phi(s) = \frac{s}{1}$ and $\frac{s}{1} \cdot \frac{1}{s} = 1$ imply that $\phi(s)$ is a unit. Secondly, let ϕ and f be as in (2). Define $g(\frac{a}{s}) = f(a)(f(s))^{-1}$. The statement can now be easily verified. $\square$

This says that $S^{-1}A$ (together with the structural homomorphism ϕ) is unique (up to isomorphism) with the properties (1) and (2) (one can say that $S^{-1}A$ is the "smallest" ring with property (1)): let $(B, \tilde{\phi})$ have properties (1) and (2). Then $\exists\ f : S^{-1}A \to B$ and $g : B \to S^{-1}A$, where f and g are inverses of each other. Indeed, the existence of these maps follows from applying properties (1) and (2) to both $S^{-1}A$ and B.

1.2 Properties of $S^{-1}A$

Proposition 1.10. *Let S be a multiplicative set of A. Then,*

1. *Proper ideals of the ring $S^{-1}A$ are of the form $IS^{-1}A = S^{-1}I = (\frac{i}{s} \mid i \in I, s \in S)$ with I ideal in A and $I \cap S = \emptyset$;*
2. *Prime ideals in $S^{-1}A$ are of the form $S^{-1}P$ where P is prime in A and $P \cap S = \emptyset$.*

Proof. 1. Let J be a proper ideal of $S^{-1}A$. Let $I = J \cap A$. Then I is an ideal of A, and assume $I \cap S \neq \emptyset$. Take $s \in S \cap I \Rightarrow \frac{s}{1} \in J \Rightarrow J = A$, a contradiction, so $I \cap S = \emptyset$.

To show that $J = S^{-1}I$, let $\frac{i}{s} \in S^{-1}I$. But $\frac{i}{s} = \frac{i}{1} \cdot \frac{1}{s} \in J$ so one containment is established. Now let $a \in J$. Then $a = \frac{b}{s}$, so we have $\frac{b}{s} \cdot \frac{s}{1} \in J \Rightarrow \frac{b}{1} \in J \Rightarrow b \in I$. Thus $J = S^{-1}I$.

2. Let $I = S^{-1}P$ where $P \cap S = \emptyset$. Note that $P = I \cap A$. Assume I is prime. Let $a, b \in P$ so $\frac{ab}{1} \in I \Rightarrow \frac{a}{1} \in I$ or $\frac{b}{1} \in I$ which gives either $a \in P$ or $b \in P$. In fact, P prime and $P \cap S = \emptyset$ implies $S^{-1}P$ is prime:

   If
   $$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in S^{-1}P,$$
   then we can find $u, v \in S, p \in P$ such that $u(abv - pst) = 0$, which further shows that $uvab \in P$. Since $uv \notin P$ (as $P \cap S = \emptyset$), we get $ab \in P$ so $a \in P$ or $b \in P$ which gives that either $a/s \in S^{-1}P$ or $b/t \in S^{-1}P$.

□

Corollary 1.11. *Let P be prime, $S = A \setminus P$, and define $A_P = (A \setminus P)^{-1}A$. Then PA_P is prime in A_P and in fact, it is maximal. This also implies that A_P is local.*

Example 1.12. Let $A = k[x, y, z]$. Then $P = (x, y)$ is prime and $A/P = k[z]$ is a domain. Then A_P is local, and $\frac{x}{z}, \frac{x}{z+x}$ are examples of elements in A_P.

Example 1.13. Let A be the same as above. Define $S = \{1, z, z^2, ...\}$. Then $S^{-1}A = A_z$. In general, A_z is not local. The fraction $\frac{x}{z^3}$ is an example of an element of A_z.

Theorem 1.14 (Localization commutes with taking quotients). *If A is a ring, $S \subset A$ a multiplicative subset, I an ideal of A, and $\overline{S} = Im(S)$ the natural image of S in A/I, then $S^{-1}A/S^{-1}I \cong \overline{S}^{-1}(A/I)$ as A-algebras.*

Proof. Note that the canonical map $A \to A/I \to \overline{S}^{-1}(A/I)$ induces an A-algebra homomorphism $A \to \overline{S}^{-1}(A/I)$ by $a \to \frac{\overline{a}}{1}$. Under this map, S is obviously mapped to units of $\overline{S}^{-1}(A/I)$ The universal property of the ring of fractions shows then that this map induces an A-algebra homomorphism $S^{-1}A \to \overline{S}^{-1}(A/I)$ by $\frac{\overline{a}}{s} \mapsto \frac{\overline{a}}{\overline{s}}$. Furthermore, under this map, $S^{-1}I$ is clearly mapped to 0 so this map induces an A-algebra homomorphism

$$\phi : S^{-1}A/S^{-1}I \to \overline{S}^{-1}(A/I)$$

by $\dfrac{\overline{a}}{s} \mapsto \dfrac{\overline{a}}{\overline{s}}$.

To show the isomorphism, one can define, by a similar sequence of steps, an A-algebra homomorphism

$$\psi : \overline{S}^{-1}(A/I) \to S^{-1}A/S^{-1}I$$

by $\phi(\frac{\overline{a}}{\overline{s}}) = \overline{\frac{a}{s}}$.

The maps ϕ, ψ are easily seen to be inverses to each other. □

Proposition 1.15. *Let $S \subset A$ be a multiplicative subset of A If $B \subseteq S^{-1}A$ with B a subring of $S^{-1}A$, then $S^{-1}A$ is a ring of fractions of B, more precisely $T^{-1}B$, where T is the image of S in T under the canonical map $A \to S^{-1}A$.*

Proof. If $\dfrac{b}{s} \in T^{-1}B$ then since $b = \dfrac{a}{t}$, we can write $\dfrac{b}{s} = \dfrac{\frac{a}{t}}{s} = \dfrac{a}{ts}$ and hence $T^{-1}B = S^{-1}A$. □

Definition 1.16. *Let A be a domain, and $S = A \setminus \{0\}$. Then we call $S^{-1}A = Q(A)$ the field of fractions of A.*

Corollary 1.17. *Every domain can be embedded in a field.*

Proof. It remains to be shown as an exercise that $\dfrac{a}{s}\dfrac{s}{a} = 1$ if and only if $a \neq 0$. The field that we constructed is the smallest field containing A (and the reader should attempt to prove this by using the universal property of the ring of fractions). □

Remark 1.18. Note that a ring A can be embedded in a field if and only if A is a domain, since a subring of a field is always a domain.

Proposition 1.19. *Let $S \subset A$, $P \in$Spec(A) such that $P \cap S = \emptyset$. Then $(S^{-1}A)_{PS^{-1}A} = A_P = (A \setminus P)^{-1}A$.*

Proof. The proof relies on the simple fact that $\frac{a/s}{b/t} = as/bt$ and is a variant of Proposition 1.15.

□

Corollary 1.20. *If $P \subseteq Q$, with $P, Q \in Spec(A)$, then $(A_Q)_{PA_P} = A_P$.*

Proof. Take $S = A \setminus Q$ in the above proposition, noting that $A \setminus Q \subseteq A \setminus P$. $\square$

2 More on Primes

Proposition 2.1. *Let $I \leq A$ and $\sqrt{I} \in Max(A)$. Then I is primary. (Recall: I primary means that $xy \in I$ implies $x \in I$ or $y \in \sqrt{I}$.)*

Proof. Let $xy \in I$, and assume that $y \notin \sqrt{I} = M \in \mathrm{Max}(A)$. Then $M + Ay = A$ so we can write $1 = ay + z$ for some $z \in M$ and $a \in A$. Then $\exists\, n \in \mathbb{N}$ such that $z^n \in I$. This implies $1 = (ay + z)^n \Rightarrow x = x(ay + z)^n = xyb + xz^n$, with $b \in A$. Since $xy \in I$ and $z^n \in I$ it follows that x is in I. $\square$

Remark 2.2. If $\sqrt{I}$ is prime, this does not imply that I is primary. Remember that if $\sqrt{I}$ is prime and $xy \in I \subset \sqrt{I}$ it follows that $x \in \sqrt{I}$ or $y \in \sqrt{I}$. This is weaker than I primary and examples confirming this can be found easily: let k be a field, and $A = \frac{k[x,y,z]}{(xy-z^2)}$. Show that $I = P^2$ is not primary where $P = (x, z) \subset A$.

2.1 Prime Avoidance Lemma

Lemma 2.3. *Let $\emptyset \neq I \subseteq A$ be a set closed under multiplication and addition, and let $P = \{P_1, ..., P_n\}$ be a set of ideals in A with all but at most two of them prime. Then $I \subseteq \bigcup_{i=1}^{n} P_i$ implies $\exists\, k$ such that $I \subseteq P_k$.*

Proof. We proceed with induction on n. The case $n = 1$ is clear. Now assume I is not contained in $(\bigcup_{i=1}^{n} P_i) \setminus P_k = Q_k$ (otherwise, after removing one ideal, one can apply the induction step). Let $x_k \in I \setminus Q_k$. Note that this means that x_k belongs to P_k and does not belong to any P_i, $i \neq k$.

Case 1: $n = 2$. Let $y = x_1 + x_2 \in I = P_1 \cup P_2$. If $y \in P_1$ then since $x_1 \in P_1$ we get that $x_2 \in P_1$, a contradiction.

Case 2: $n > 2$. Without loss of generality, let P_1 be prime. Set $z = x_1 + x_2 \cdots x_n \in I$. Then $z \notin P_1$ since $x_2, ..., x_n \notin P_1$ implies $x_2 \cdots x_n \notin P_1$. In fact, each $x_i \in P_i$ for $i \geq 2$, so $z - x_1 \in P_i$ for $i \geq 2$. But since $x_1 \notin P_i$ for $i \geq 2$ we must also have that $z \notin P_i$ for $i \geq 2$, a contradiction with $z \in I$.

In conclusion, one cannot choose such elements x_k and hence one of the ideals P_i appears redundantly in the union. By removing it, we get to the case of $n-1$ ideals and the induction step can now be applied.

□