Chapter 1: Lecture 5

If A is a ring and M an A-module we let

$$\mathcal{L}_A(M) = \{N \subseteq M : \; N \text{ is an } A-\text{submodule of } M\}.$$

Let N be an A-submodule of M. We refer to it as proper if $N \neq M$. If $N = 0$ or M, then we say that N is trivial.

Theorem 0.1 (Krull). *If ${}_AM$ is finitely generated and L is a proper submodule of M, then there is a maximal proper submodule of M, say N, with $L \leq N$.*

Proof. Let $P = \mathcal{L}_A(M) \setminus \{M\}$ and consider $(P, \subseteq)$ which is a partially ordered set. We need to show that every nonempty totally ordered subset of P has an upper bound. We intend to use Zorn's Lemma so it suffices to show this is true for chains.

Let $N_1 \leq N_2 \leq \cdots$ be a chain in P. Then $N = \bigcup_{i=1}^{\infty} N_i$ is an A-submodule of M which is the desired upper bound. It is proper: if $N = M$, then, since M is finitely generated, say by $x_1, ..., x_r$, and P is a chain, then there exists N_i such that $x_1, \ldots, x_r$ are all contained in N_i. But this forces $N_i = M$ which is not the case.

So, Zorn's Lemma can be applied and the statement of the theorem follows.

□

Definition 0.2. *Let M be an A-module. We call M simple if it does not contain nontrivial A-submodules.*

It is clear that if M is simple then, for each $x \neq 0$ in M, the A-submodule Ax must equal M. Also, a simple A-module M satisfies both the DCC and ACC conditions in trivial fashion. Let M be an arbitrary A-module. Let $N, N' \in \mathcal{L}_A(M)$, with $N \subseteq N'$. By the Correspondence Theorem for quotients, N'/N is A-simple if and only if there are no submodules $K \in \mathcal{L}_A(M)$ such that $N \subsetneq K \subsetneq N'$.

With this observation, we remark that under the notations and hypotheses of the above Theorem, the module M/N is A-simple.

Also, if M happens to contain a minimal nonzero submodule N, then N is A-simple. We can always pick such a submodule if M is Artinian over A.

Finally, let S A-simple, and $x \neq 0$ an element of A. Since $S = Ax$, then the map $A \to Ax$ that sends a to ax gives that $A/\mathrm{Ann}(x) \simeq Ax = S$. Therefore, S simple is equivalent to the condition that $\mathrm{Ann}(x)$ is a maximal ideal of A.

Definition 0.3. *Let M be an A-module. An ascending filtration of M is a family of submodules $\{M_i\}$ such that $M_i \subseteq M_{i+1}$. The A-modules M_{i+1}/M_i are called the* factors *of the filtration. In similar fashion, one can define descending filtrations. Filtrations can be finite or infinite. Obviously, any ascending filtration can be renumbered to become a descending filtration, and conversely.*

Definition 0.4. *Consider ${}_AM$. Then M is said to be of* **finite length** *over A if there is a chain of A-submodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $\dfrac{M_i}{M_{i-1}}$ is A-simple for every i.

Such a chain will be called a composition series *of length n for M, and $\dfrac{M_i}{M_{i-1}}, i = 1, \ldots, n$, are its* factors.

Proposition 0.5. *Let A be a ring and ${}_AM$ a module. Then ${}_AM$ is of finite length if and only if M is Artinian and Noetherian.*

Proof. First assume that M is of finite length and let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a chain of submodules such that $\dfrac{M_i}{M_{i-1}}$ are A-simple for every i, i.e., each one has no proper submodules.

If S is a simple module over A, then S is Noetherian and Artinian. This remark applies to all the factors of the composition series.

Consider

$$0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0.$$

Using the behavior of Noetherian/Artinian modules on short exact sequences, we can show by induction on i that every M_i is Noetherian and so $M = M_n$ is Noetherian.

Conversely, let us take M Artinian and Noetherian and $M_0 = 0$. Let M_1 minimal among the nonzero submodules of M. If $M_1 \neq M$, let M_2 minimal among the submodules of M strictly containing M_1 and so on. For all these steps we use that M is Artinian. We have an ascending chain of submodules $\{M_i\}$ and clearly M_{i+1}/M_i are all simple submodules. M is Noetherian and hence the chain must stabilize. By construction it must stabilize at $M_n = M$, for some n, and this shows that M is of finite length. $\square$

We plan to prove that all composition series of a finite length module have the same length. We will do so by proving a stonger statement on composition series. For this, we first need a lemma.

Lemma 0.6. *Let ${}_AM$ a module and $E, F, G \in \mathcal{L}_A(M)$ such that $E \subsetneq F$ and F/E is A-simple.*

Then $\dfrac{F \cap G}{E \cap G}$ is either zero or A-simple.

Proof. The natural projection $F \to F/E$ induces, by restriction, an A-linear map $F \cap G \to F/E$. The kernel of this map equals $E \cap G$.

Hence $\dfrac{F \cap G}{E \cap G}$ injects into $\dfrac{F}{E}$. Therefore the image is either zero or the entire $\dfrac{F}{E}$ since this is an simple A-module. □

Theorem 0.7 (Jordan-Hölder)**.** *Let ${}_AM$ be a module with two finite composition series:*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

and

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = M.$$

Then $m = n$ and there exists a permuation $\sigma \in S_n$ such that $M_{i+1}/M_i \simeq N_{\sigma(i)+1}/N_{\sigma(i)}$ for all $i = 0, \ldots, n-1$.

Proof. If M is simple then the statement is immediately true. Let us assume that the statement is true for all A-modules that admit at least one composition series of length $n-1$.

Consider the filtration:

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots M_{n-1} \cap N_{m-1} \subseteq M_n \cap N_{m-1} = N_{m-1}.$$

Its factors are either zero or simple by Lemma 0.6. So the induction hypothesis applies to N_{m-1}.

Note that we also have the following filtration

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1}.$$

Its factors are either zero or simple by Lemma 0.6 except possibly the last factor. The induction hypothesis applies to M_{m-1} as well.

Case A: If $M_{n-1} = N_{m-1}$ then it follows that $n-1 = m-1$ and $M_{i+1}/M_i \simeq N_{\sigma(i)+1}/N_{\sigma(i)}$ for all $i = 0, \ldots, n-2$ and some $\sigma \in S_{n-1}$. Note that $M/M_{n-1} = M/N_{n-1}$. Putting this together gets us the statement.

Case B: If $M_{n-1} \neq N_{m-1}$, since M_{n-1}, N_{m-1} are proper maximal submodules of M we see that $M_{n-1} + N_{m-1} = M$.

Note that by using the third isomorphism theorem for modules

$$\frac{M}{N_{m-1}} = \frac{M_{n-1} + N_{m-1}}{N_{m-1}} \simeq \frac{M_{n-1}}{M_{n-1} \cap N_{m-1}}.$$

Therefore $\frac{M_{m-1}}{M_{n-1} \cap N_{m-1}}$ is A-simple as well. This guarantees that

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1}$$

is a composition series of M_{n-1} after removing the redundant terms corresponding to possible zero factors.

Now note that by the induction hypothesis M_{n-1} satisfies the statement of the theorem, and hence all its composition series must have length $n-1$. So, in fact, exactly one term is redundant in the above filtration. After removing this factor we obtain as a consequence that

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1}$$

is a composition series of length $n-2$ and completing it at the end with N_{m-1} we obtain a composition series for N_{m-1} of length $n-1$. But the induction hypothesis applied to N_{m-1} shows that all its composition series muts have length $m-1$, so $n-1 = m-1$ or $n = m$.

For the remaining part the reader should keep in mind that $n = m$.

Let us also note that

$$\frac{M}{M_{n-1}} = \frac{M_{n-1} + N_{m-1}}{M_{n-1}} \simeq \frac{N_{m-1}}{M_{n-1} \cap N_{m-1}}.$$

This (and the similar isomorphism proved earlier) allows us to conclude that the two resulting composition series

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1} \subseteq M_n = M$$

and

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq N_{m-1} \subseteq N_m = M$$

have the same factors up to a permutation.

But

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1} \subseteq M_n = M$$

has the same factors, up to a permutation, as

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

by Case A which was already considered.

Similarly

$$0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subset N_{m-1} \subset N_m = M$$

has the same factors, up to a permutation, as

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M.$$

Putting all these together we get our statement.

□

Example 0.8. Let $A = \dfrac{k[x]}{(x^3)}$. The chain of ideals $0 \subset (x^2) \subset (x) \subset A$ is a composition series for A as an A-module0.

Definition 0.9. *For an A-module M of finite length, the length of any composition series is call the* length *of M. It is denoted by $\lambda_A(M)$ or, simply, $\lambda(M)$.*

An important result is that the length function behave nicely on short exact sequences of finite length modules.

Proposition 0.10. *Let $0 \to N \to M \to K \to 0$ be a short exact sequence of A-modules of finite length. Then*

$$\lambda(M) = \lambda(N) + \lambda(K).$$

Proof. We can consider that N is a submodule of M and note that $K \simeq M/N$. Hence any composition series of K can be lifted back to M resulting in a filtration of $N \subseteq M$ with A-simple factors. We can concatenate it to a composition series for N and obtain a composition series for M. Counting the numbers of factors gives the statement of the Proposition.

□

A few corollaries can be obtained readily.

Corollary 0.11. *Let M_i be A-modules, $i = 1, \ldots, n$, of finite length.*

Then $M = \oplus_{i=1}^n M_i$ is of finite length over A and $\lambda(M) = \sum_{i=1}^n \lambda(M_i)$.

Proof. Consider the following short exact sequences for $k = 2, ..., n$:

$$0 \to \oplus_{i=1}^{k-1} kM_i \to \oplus_{i=1}^{k} M_i \to M_k.$$

Apply Propositions 0.5 and 0.10 to get the result. □

Theorem 0.12. *Let A be an Artinian ring and M and finitely generated A-module. Then M has finite length over A.*

Proof. We will prove this by induction on the number of generators of M.

If M is cyclic then $M = Ax$, $x \in M$ and then $M \simeq A/I$, where $I = \text{Ann}(x)$. Since A is Artinian and Noetherian it follows that A is of finite length over A. Hence A/I is of finite length over A which proves that M is of finite length as well.

Now, let $x_1, ..., x_{n-1}, x_n$ be generators for M, where $n > 1$. Then let $N = Rx_1 + \cdots + Rx_{n-1}$ and consider

$$0 \to N \to M \to M/N \to 0.$$

Remark that the image of x_n in M/N generates M/N. Since $N, M/N$ are both of finite length by the induction hypothesis, we can conclude that M is of finite length.

□