Lecture 4

1. General facts

Proposition 1.1. *Let A be a commutative ring, and $\mathfrak{m}$ a maximal ideal. Then TFAE:*

1. *A has only one maximal ideal (i.e., A is local);*
2. *$A \setminus \mathfrak{m}$ consists of units in A;*
3. *For all non-units a and b, $a+b$ is a nonunit.*

Proof. For (1) implies (2), if $u \notin \mathfrak{m}$, then u is a unit.

For (2) implies (3), take a and b to be nonunits. Then $a, b \in \mathfrak{m}$. This implies $a+b \in \mathfrak{m}$, so $a+b$ is a nonunit.

For (3) implies (1), take I to be the set of all nonunits. Then I is an ideal: we only need to show that if a is a nonunit, λa is a nonunit for all $\lambda \in A$. Assume not, and take b such that $\lambda a b = 1$ which implies a is a unit, a contradiction. Also, I is maximal since otherwise we can find a proper ideal $J \leq A$ containing I. But any element of J is a nonunit, since J is proper, so $J \subseteq I$ which is impossible.

Finally since $\mathfrak{m}$ is a proper ideal we have that $\mathfrak{m}$ consists of nonunits, so $\mathfrak{m} \subseteq I$. They are both maximal ideals, so $\mathfrak{m} = I$. □

Corollary 1.2. *If the set of all nonunits is an ideal in A, then A is local and this ideal is the maximal one.*

Definition 1.3. *Let $Jac(A)$ be the intersection of all maximal ideals.*

Proposition 1.4. *$x \in Jac(A)$ if and only if $\forall\, a \in A$ $1 + ax$ is a unit.*

Proof. For the forward implication, if $1 + ax$ is not a unit, then $\exists \mathfrak{m} \in \mathrm{Max}(A)$ with $1+ax \in \mathfrak{m}$. Then $x \in \mathrm{Jac}(A)$ implies $x \in \mathfrak{m}$ which implies $ax \in \mathfrak{m}$, so $1 = 1+ax-ax \in \mathfrak{m}$, a contradiction.

For the reverse, take $\mathfrak{m} \in \mathrm{Max}(A)$ so that $x \notin \mathfrak{m}$. Then $Ax + \mathfrak{m} = A$, so $1 = ax + b$ for some $b \in \mathfrak{m}$. But then $1 - ax = b$, so b is a unit, a contradiction. □

Definition 1.5. *If $A \neq 0$ and $|Max(A)| < \infty$, then A is called* **semilocal**.

Definition 1.6. *Take A a commutative ring and ${}_AM$ a module. Then $\mathfrak{N}_A(M) = \{a \in A \mid \exists\, n \in \mathbb{N}$ such that $a^n M = 0\}$ is called the* **nilpotent radical of** *${}_AM$. This equals* *the set* $\{a \in A \mid \exists\, n \in I\!N$ *such that* $a^n \in Ann_A(M)\}$. *Similarly,* $\mathfrak{N}_A(A) = \{a \in A \mid \exists\, n$ *such that* $a^n = 0\}$. *The latter is often denoted* $Nil(A)$*, or just* $N(A)$*.*

To see another characterization of this, consider the map $\pi : A \to A/Ann_A(M)$. *Then* $\mathfrak{N}_A(M) = \pi^{-1}(J)$ *where* $J = \{\overline{a} \in A/Ann_A(M) \mid \exists\, n$ *such that* $\overline{a}^n = 0\}$. *Note that* $J = \mathfrak{N}(A/Ann_A(M))$. *We can also consider* $\mathfrak{N}_A(A/I)$ *for some* $I \leq A$. *This equals* $\pi^{-1}(\mathfrak{N}_{A/I}(A/I)) = \{a \in A \mid \exists\, n$ *such that* $a^n \in I\}$. *This is denoted* $Rad(I) = \sqrt{I}$ *and it is the* **radical ideal** *of* I*.*

Example 1.7. *Let* $A = k[x, y]$, $I = (x^2, y) \nsubseteq (x, y)$ *but* $\sqrt{I} = (x, y)$. *Note that* $\sqrt{I}$ *is not necessarily a prime ideal in general.*

2. Artinian rings

Lemma 2.1. *Let* K *be a field and* V *a* K*-vector space. Then* V *is Artinian over* K *if and only if* V *is finite dimensional if and only if* V *is Noetherian over* K*.*

Theorem 2.2. *(Akizuki-Hopkins-Levitzki) An Artinian ring is Noetherian.*

Proof. (AHS Theorem) There exist only finitely many maximal ideals of A: If not, let $M_1, M_2, ..$ an infinite collection of distinct maximal ideals. Then we can construct a descending chain that does not stabilize: $A \supseteq M_1 \supseteq M_1M_2 \supseteq M_1M_2M_3....$ Indeed, if $M_1 \cdots M_k = M_1 \cdots M_kM_{k+1}$, then $M_1 \cdots M_k \subset M_{k+1}$. Since M_{k+1} is maximal, and hence prime, we get that there exists $1 \leq i \leq k$ such that $M_i \subseteq M_{k+1}$. But this implies that $M_i = M_{k+1}$, a contradiction.

Now, let $J = \mathrm{Jac}(A)$. We will show that J is nilpotent. Since $\cdots \supset J^k \supset J^{k+1} \supset \cdots$ is a descending chain of ideals of A, $\exists\, s$ such that $J^s = J^{s+1} = \cdots$. Let us consider the following ideal $(0 :_A J^s) = K$. We want $K = A$ since this gives $1 \in K$ and so $J^s = 0$. Assume K is not equal to A. By the minimal property, one can find $K \subsetneq K'$, K' minimal over K. We see that $K' = K + Ax$ for all $x \in K' \setminus K$, by the minimal property of K'. Consider $K' \supseteq K + Jx \supseteq K$, which leads to $K + Jx = K$ or $K + Jx = K'$. We will show the former. Notice that K'/K is an A-module generated by x. So $K'/K = A\bar{x}$. Then $J \cdot K'/K = \dfrac{JK' + K}{K} = \dfrac{J(K + Ax) + K}{K} = \dfrac{JK + JAx + K}{K} = \dfrac{JAx + K}{K} = K'/K$ if we assume that $K' = K + Jx$. By NAK above, we have $K'/K = 0 \Rightarrow K = K'$, a contradiction. Thus $K + Jx = K$ so $Jx \subseteq K$. This gives $JxJ^s = 0$, so $xJ^{s+1} = 0$, so $xJ^s = 0$ which implies $x \in K$, a contradiction. Thus $K = A$, so $J^s = 0$.

Let $\mathrm{Max}(A) = \{M_1, ..., M_k\}$. Then if $I = M_1 \cdots M_k \subseteq J$, then $I^s \subseteq J^s = 0$. We then have the following chain, $A \supset M_1 \supset M_1M_2 \supset M_1M_2M_3 \supset \cdots \supset M_1 \cdots M_k \supset$

$IM_1 \supset \cdots \supset IM_1 \cdots M_k \supset \cdots \supset I^s M_1 \supset \cdots \supset IM_1 \cdots M_k = 0$. A quotient given by two successive factors in this chain is killed by some M_i. This quotient is hence an A/M_i-module with the same structure as an A-module. Since A/M_i is a field and each quotient is Artinian, we see that each quotient in fact a finite dimensional vector space over A/M_i and, hence, of finite length over A. By the Serre class property, A is also of finite length, and thus Noetherian. For clarity, say $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset A$ is a filtration with Artinian factors annihilated by a maximal ideal of A. So each factor is Artinian over the quotient ring modulo the respective maximal ideal. This is a vector space, so it is Noetherian as well, and hence each factor is Noetherian over A.

Then the outer pieces in the exact sequence $0 \to N_1 \to N_2 \to N_2/N_1 \to 0$ are Noetherian, so this implies that N_2 is Noetherian also. We can move up the filtration chain by looking at similar succesive short exact sequences to conclude that A is Noetherian over A, hence a Noetherian ring. □

Corollary 2.3. *If A is Artinian then it has only finitely many prime ideals.*

Proof. We have seen in the proof above that there are finitely many maximal ideals $M_1, \ldots, M_k$ and that there exists s such that $(M_1 \cdots M_k)^s = 0$.

Let P be a prime ideal of A. Since $(M_1 \cdots M_k)^s = 0 \subset P$, there exists $M_i \subseteq P$ which implies $P = M_i$ for some $1 \leq i \leq k$.

□

Corollary 2.4. *If A is Artinian, and M is an Artinian A-module, then M is Noetherian.*

Proof. Assume A is local for simplicity. Let $\mathfrak{m}_A$ be the maximal ideal of A. Then, in our case, $\mathfrak{m}_A = \mathrm{Rad}(A) = \mathrm{Jac}(A)$ is nilpotent (by the proof of the above theorem), so the chain $M \supset \mathfrak{m}_A M \supset \mathfrak{m}_A^2 \supset \cdots \supset \mathfrak{m}_A^s M = 0$ exists. Successive quotients $\mathfrak{m}_A^{i-1} M/\mathfrak{m}_A^i M$ are vector spaces over A/m_A (by arguments similar to end of above proof), they are Artinian and thus Noetherian over A/m_A (or over A since the structure is identical). As in the above given proof, this implies that M is Noetherian over A. □

The following is an interesting result, provided here without proof.

Theorem 2.5. *(Eakin-Nagata) Let $A \subseteq B$ be a subring where B is finitely generated as an A-module. Then B Noetherian implies A Noetherian.*

(1) If $f : A \to B$ is a ring homomorphism, then B is an A-algebra by $ab = f(a)b$. In fact, by abuse of notation, ab is written and understood as $f(a)b$.

(2) If $f : A \to B$ is a ring homomorphism, and ${}_BM$ is a module, then M is also an A-module by $am = f(a)m$. One says that ${}_AM$ is obtained from ${}_BM$ by *restriction of scalars.*

Whenever we have a ring homomorphism $f : A \to B$, then B is naturally an A-module. We call B an A-algebra (note that B is an A-module with a ring structure that is compatible with the multiplication with scalars from A).

A homomorphism $\phi : B \to C$ of A-algebras $f : A \to B, g : A \to C$ is a ring homomorphism such that $\phi \circ f = g$.

Example 2.6. $k[x] \hookrightarrow \dfrac{k[x,y]}{(y^3)}$, *and the right hand side is a ring, so it is also a* $k[x]$-*algebra.*

Let A be a commutative ring. The following alternate definition of the polynomial ring with coefficients in A is going to be useful.

Definition 2.7. *Let* $(A[x_1, ..., x_n], \{x_1, \dots, x_n\})$ *be a pair consisting of an* A-*algebra* $A[x_1, \dots, x_n]$ *and a string of elements* $x_1, ..., x_n$ *in* $A[x_1, \dots, x_n]$. *Such a pair with the property that for every* A-*algebra* B *and string of elements* $b_1, b_2, \dots, b_n \in B$, *there exists a unique* A-*algebra homomorphism*

$$\phi : A[x_1, \dots, x_n] \to B$$

such that $\phi(x_i) = b_i$ *for all* $i = 1, \dots, n$ *is called the polynomial ring in indeterminates* $x_1, \dots, x_n$ *and coefficients in* A.

The reader should check as an exercise that a polynomial ring over A in finitely many variables (under the old definition) satisfies the alternate definition provided above.

(1) We say that B is finitely generated as an A-algebra, if $\exists\, b_1, ..., b_n \in B$ such that $B = A[b_1, ..., b_n]$. Here $A[b_1, ..., b_n]$ is the image in B of the natural homomorphism of A-algebras

$$A[x_1, ..., x_n] \to B$$

which sends x_i to b_i, $i = 1, \dots, n$. The existence of this homomorphism is guaranteed by the universal property of polynomial ring $A[x_1, \dots, x_n]$.

(2) If B is finitely generated as an A-module, then $\exists\, b_1, ..., b_n \in B$ such that

$$B = Ab_1 + \cdots + Ab_n.$$

Note that this implies that B is finitely generated as an A-algebra by $b_1, \ldots, b_n$ but it is not equivalent to it. For example, $\mathbb{Z}\sqrt{2} \subsetneq \mathbb{Z}[\sqrt{2}]$.