Lecture 2

1. Noetherian and Artinian rings and modules

Let A be a commutative ring with identity, ${}_AM$ a module, and $\phi : M \to N$ an A-linear map. Then $\ker\phi = \{m \in M : \phi(m) = 0\}$ is a submodule of M and $\operatorname{im}\phi$ is a submodule of N.

We sometimes consider $\mathcal{L}_A(M) = \{M' \mid M' \leq_A M\}$, the lattice of all A-submodules of M. Note that $(\mathcal{L}_A(M), \subseteq)$ is, in particular, a partially ordered set under inclusion. If $M = A$, then $\mathcal{L}_A(A)$ is the set of ideals of A.

Definition 1.1. *If every chain of A-submodules of M, $\cdots \subseteq N_m \subseteq N_{m+1} \subseteq \cdots$, stabilizes, i.e., $\exists\, n$ such that $N_n = N_{n+1} = \cdots$, then we say that M satisfies the Ascending Chain Condition (ACC) on submodules.*

Definition 1.2. *Similarly, if every descending chain of A-submodules of M stabilizes, then M satisfies the Descending Chain Condition (DCC) on submodules.*

Proposition 1.3. *Let A be a commutative ring, ${}_AM$ a module. Then TFAE:*

(1) *Every $\emptyset \neq P \subseteq \mathcal{L}_A(M)$ has a maximal element,*
(2) *Every ascending chain of submodules stabilizes (ACC).*

Proof. For (1) implies (2), take an ascending chain $N_1 \subseteq N_2 \subseteq \cdots$. Then take $P = \{N_1, ..., N_m, ...\}$, which has a maximal element, N_n. Hence $N_n = N_{n+1} = \cdots$.

For (2) implies (1), take the same subset P as above, and $N_1 \in P$. If N_1 is not maximal, then there is a $N_2 \in P$ such that $N_1 \subsetneq N_2$. Repeat as necessary and remark that the procedure stops since M has (ACC) on submodules. So, there is a maximal element of P. $\square$

Definition 1.4. *${}_AM$ is* **Noetherian** *if M has (ACC) on submodules, and is* **Artinian** *if it has (DCC) on submodules. A ring A is Noetherian (Artinian) if ${}_AA$ is Noetherian (Artinian).*

Proposition 1.5. *Let A, ${}_AM$ be as above. Then TFAE:*

(1) *M has (ACC) on submodules,*
(2) *Every submodule of M is finitely generated.*

Proof. For (1) implies (2), let N be a submodule of M and take $P = \{L \mid L$ is finitely generated and $L \subseteq N\}$. Then P is not empty since if $0 \neq n \in N$ implies $An \in P$. Then P admits a maximal element, L_0. We claim that $L_0 = N$. If not, then $\exists\ x \in N$ such that $x \notin L_0$. Then $L_0 \nsubseteq L_0 + Ax$. This contradicts the maximality of L_0.

For (2) implies (1), let $N = \bigcup_m N_m$. Let $x, y \in N$ and $r, s \in A$. Then there exist i, j with say $i < j$ such that $x \in N_i, y \in N_j$. Since $i < j$ we note that $N_i \subseteq N_j$ which gives $x, y \in N_j$. But N_j is an A-submodule of M so $rx + sy \in N_j$ which implies that $rx+sy \in N$. This shows that N is an A-submodule of M, so it is finitely generated. Then $N = (x_1, ..., x_k)$, for some $x_1, \dots, x_k \in M$. Take n large enough such that $x_1, ..., x_k \in N_n$, so $N_n = N_{n+1} = \cdots = N$. □

Proposition 1.6. *Let A be a ring, and ${}_AM$ a module. Then TFAE:*

(1) *M has (DCC) on submodules,*
(2) *For every $\{N_i\}_{i\in I}$ family of A-submodules, $\exists\ J \subseteq I$ with $|J| < \infty$ such that $\bigcap_{i\in I} N_i = \bigcap_{j\in J} N_j$.*

Proof. For (1) implies (2), take $\{N_i\}_{i\in I}$, with $N_i \leq M$ for every i. Define $P = \{N_F \leq M \mid N_F = \bigcap_{k\in K} N_k,\ |K| < \infty,\ K \subseteq I\}$. Then M has (DCC) so P has a minimal element, $N_{F_0} = \bigcap_{j\in J} N_j$ and $|J| < \infty$. Take $i \in I$. Then $N_i \cap N_{F_0} \in P$ and $N_i \cap N_{F_0} \subseteq N_{F_0}$. The minimality of N_{F_0} says that $N_i \cap N_{F_0} = N_{F_0}$ for all i. We claim that $\bigcap_{i\in I} N_i = \bigcap_{j\in J} N_j = N_{F_0}$ since any $x \in N_{F_0}$ is in N_i because $N_i \cap N_{F_0} = N_{F_0}$ implies $x \in \bigcap_{i\in I} N_i$.

For (2) implies (1), take a descending chain $N_m \supseteq N_{m+1} \supseteq \cdots$. Then $\bigcap_m N_m = \bigcap_{j\in J} N_j$ (with $|J| < \infty$). Take n the maximal integer of J. Then $\bigcap_m N_m = N_n$ so (DCC) implies $N_n = N_{n+1} = \cdots$. □

Example 1.7. (1) $\mathbb{Z}$ is Noetherian but not Artinian; Indeed, all ideal of $\mathbb{Z}$ are principal, so finitely generated. The (DCC) condition is not satisfied due to the existence of the chain

$$\cdots \subseteq (2^n) \subseteq (2^{n-1}) \subseteq \cdots \subseteq (2) \subseteq \mathbb{Z}.$$

$\mathbb{Z}[x_1, x_2, ...]$ is neither Noetherian nor Artinian; The (ACC) condition is not satisfied due to the chain

$$\cdots I_n \subseteq I_{n+1} \subseteq \cdots \subseteq \mathbb{Z}[x_1, x_2, ...],$$

where $I_n = (x_1, \ldots, x_n)$, for all $n \geq 1$.

The (DCC) condition is not satisfied due to the chain

$$\cdots J_n \subseteq J_{n+1} \subseteq \cdots \subseteq \mathbb{Z}[x_1, x_2, ...],$$

where $J_n = (x_i : i \geq n)$, for all $n \geq 1$.

There are examples of Artinian modules that are not Noetherian, however any Artinian ring is Noetherian, as we will see later.

Fix p a prime integer. Let $\mathbb{C}_{p^\infty} = \{z \in \mathbb{C} : \text{there exists } n \text{ such that } z^{p^n} = 1\}$. This is an Abelian group under multiplication and so, it is module over $\mathbb{Z}$. In fact, it is an Artinian $\mathbb{Z}$-module which is not Noetherian. Prove this by showing that any submodule of $\mathbb{C}_{p^\infty}$ is of the form $\mathbb{C}_{p^n} = \{z \in \mathbb{C} : z^{p^n} = 1\}$.

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2. Serre Class

Definition 2.1. *A* **short exact sequence (ses)** *of A-modules*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a triple of A-modules (M', M, M'') together with a pair (f, g) of A-linear maps such that f is injective, g is surjective, and $Im\, f = \ker g$.

Definition 2.2. *Take $\mathcal{A}$ a family of modules over A. Assume that $\mathcal{A}$ is not empty. Then $\mathcal{A}$ is a* **Serre class** *if and only if for every short exact sequence*

$$0 \to M' \to M \to M'' \to 0,$$

$M \in \mathcal{A}$ if and only if $M', M'' \in \mathcal{A}$.

Proposition 2.3. *Let $\mathcal{A}$ be a Serre class. Then the following are true:*

1. *$0 \in \mathcal{A}$;*
2. *$M \in \mathcal{A}$ and $N \cong M$ implies $N \in \mathcal{A}$;*
3. *$M \in \mathcal{A}$ and $L \leq M$ implies $L,\ M/L \in \mathcal{A}$;*
4. *$M_1, ..., M_n \in \mathcal{A}$ implies $\bigoplus_{i=1}^{n} M_i \in \mathcal{A}$.*

The proof is left as an easy exercise.

Proposition 2.4. *The class of Noetherian (Artinian) A-modules forms a Serre class.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. First we will show that if M' and M'' are Noetherian then M is Noetherian. First, M' is isomorphic to its image in M and M'' is isomorphic to a quotient of M. The Fundamental Isomorphism Theorem for modules together with the definition of a short exact sequence allows us allows us to regard $L = M' \leq M$ and $M'' = M/L$. Take $N_1 \subseteq N_2 \subseteq \cdots$ a chain of A-submodules of M. Then $\{N_i \cap L\}_{i \in \mathbb{N}}$ is an ascending chain in L. Similarly, $\left\{\frac{N_i + L}{L}\right\}_{i \in \mathbb{N}}$ is an ascending chain in M/L. Both chains have to stabilize due to our Noetherian assumption. Take n to be the maximum of the stabilizing index of the two chains, so that $N_n \cap L = N_{n+1} \cap L = \cdots$ and $\frac{N_n + L}{L} = \frac{N_{n+1} + L}{L} = \cdots$. Let $x \in N_{n+1}$. We have $x \in N_{n+1} \subseteq N_{n+1} + L$, so $x + L \subseteq N_{n+1} + L$, and thus $\overline{x} \in \frac{N_{n+1} + L}{L} = \frac{N_n + L}{L}$. And $\overline{x} \in \frac{N_n + L}{L}$ implies $\exists\ z \in N_n + L$ such that $x - z \in L$. So $x = z + \ell$, where $z \in N_n + L$, and $\ell \in L$. Thus $x = y + \ell_1$, with $y \in N_n$, and $\ell_1 \in L$. Then $\ell_1 = x - y$, $x \in N_{n+1}$, $y \in N_n \subseteq N_{n+1}$, so $\ell_1 \in N_{n+1} \cap L = N_n \cap L$. Thus $x = y + \ell_1$ with $y \in N_n$, and $\ell_1 \in N_n$ which implies $x \in N_n$. Therefore $N_n = N_{n+1} = \cdots$. The discussion for the Artinian case can be done similarly.

For the reverse implication we again consider a short exact sequence of the form

$$0 \to L \to M \to M/L \to 0.$$

Any submodule of L is a submodule of M so the (ACC) (or (DCC)) condition on submodules is satisfied for L. Similarly, the lattice of submodules of M/L is naturally identified with the lattice of submodules of M containg L, so the (ACC) (or (DCC)) condition for submodules of M/L follows naturally from the (ACC) condition (or (DCC)) for M.

□

Corollary 2.5. *Let R be a Noetherian (respectively Artinian) ring and M a finitely generated R-module. Then M is a Noetherian (respectively Artinian) module.*

Proof. Since M is finitely generated, then there exists a R-linear surjection $R^n \to M$, for some $n \in \mathbb{N}$. But the Serre class property implies that R^n is Noetherian (respectively

Artinian) if R is. Applying again the Serre class property we obtain that M is Noetherian (respectively Artinian).

□

Proposition 2.6. *A quotient of a Noetherian (respectively Artinian) ring is Noetherian (respectively Artinian).*

Proof. This is immediate because the lattice of ideals of a quotient ring R/I naturally embeds in the lattice of ideals of R via an inclusion preserving map.

□

Theorem 2.7. *(Hilbert Basis Theorem- general case) Let R be a Noetherian ring. Then $R[X]$ is Noetherian.*

Proof. Let I be an ideal of $R[X]$. For a polynomial $f(X)$, let us denote by $lc(f)$ its leading coefficient.

Consider $I_n = \{a \in R : a = lc(f), \text{for some polynomial } f \in I \text{ of degree } n\} \cup \{0\}$. Note that $I_0 = I \cap R$.

It is easy to see that I_n is an ideal in R, and since $f \in I$ implies that $Xf \in I$, then we have an ascending chain $\{I_n\}$ of ideals in R which must stabilize. Say that we have fixed m such that $I_n = I_m$ for all $n \geq m$.

For all $i \leq m$ let $a_{1,i}, \ldots, a_{n_i,i}$ be a finite set of generators for I_i. Now consider $a_{r,s} = lc(f_{r,s})$ for all $0 \leq r \leq n_s$, $1 \leq s \leq m$ with $f_{r,s} \in I$. We claim that the set of all such $f_{r,s}$ generate I.

Let $f \in I$. The plan is to prove that $f \in J$. We will show by induction of the degree of f that every $f \in I$ belongs to $J := (f_{r,s} : 1 \leq r \leq n_s, 0 \leq s \leq m)$.

Let $deg(f) = t$.

If $t = 0$, then $f \in I_0$ and so f is generated by $f_{j,0}$ $1 \leq j \leq n_0$.

Let a be the leading coefficient of f and assume $t > 0$.

If $t \leq m$ then

$$a = \sum_{i=1}^{n_t} r_i a_{i,t},$$

with $r_i \in R$. But then the polynomial $g = f - \sum r_i f_{i,t}$ has degree less than t so by induction is in J. But all $f_{i,t}$ are in J as well, so $f \in J$.

If $t > m$, then $f \in I$ implies that since $a \in I_n = I_m$ and we get that

$$a = \sum_{i=1}^{n_m} r_i a_{i,m},$$

with $r_i \in R$. But then the degree of the polynomial

$$g - \sum r_i X^{t-m} f_{i,m}$$

is strictly less than t. Again, by the induction hypothesis, we obtain that $g \in J$, but since all $f_{i,m}$ are in J as well we get $f \in J$. □

Definition 2.8. *Let R, S be commutative rings. We say that S is an R-algebra if there exists a ring homomorphism $\phi : R \to S$. Note that S becomes an R-module via $r \cdot s := \phi(r)s$, for all $r \in R, s \in S$. We say that S is an R-algebra generated by $\Gamma \subset S$ if every element in S is of form $P(z_1, \ldots, z_m)$ for some $z_1, \ldots, z_m \in \Gamma$ and $P(T_1, \ldots, T_m)$ polynomial in $R[T_1, \ldots, T_m]$ and $m \in \mathbb{N}$. If Γ is a finite set, then we say that S is a finitely generated algebra over R*

Corollary 2.9. *Let R be a Noetherian ring. Then any finitely generated R-algebra is Noetherian.*

Proof. Let $S = R[z_1, \ldots, z_m]$ be a finitely generated R-algebra. Then $S = R[z_1, \ldots, z_m]$ for some $z_1, \ldots, z_m \in S$. Therefore we have a natural ring homomorphism $R[T_1, \ldots, T_m] \to S$ by sending P to $P(z_1, \ldots, z_m)$. Then S is quotient of $R[T_1, \ldots, T_m]$, which is Noetherian by Theorem 2.7, because R is Noetherian. Then S is Noetherian by Proposition 2.6.

□