Lecture 12

1. Integral extensions and the Going-Up Theorem

Definition 1.1. *Let $\phi : R \to S$ be a ring homomorphism. Then $\phi^* : Spec(S) \to Spec(R)$ by mapping each Q to $Q \cap R = \phi^{-1}(Q)$. Let $P \in Spec(R)$. Then a prime ideal $Q \in Spec(S)$* lies over *P if $Q \cap R = P$, i.e., $\phi^*(Q) = P$.*

Proposition 1.2. *If $R \subseteq S$ is an integral extension of rings, $I \leq R$, and $u \in IS \cap R$, then $\exists\ n$ such that $u^n \in I$. Moreover, if $I = Rad(I)$ (in particular if I is prime), then $IS \cap R = I$.*

Proof. Let $u \in IS$. Then $u = \sum_{t=1}^{n} j_t\theta_t$, with $\theta_t \in S$ and $j_t \in I$. So we may assume that $S = R[\theta_1, ..., \theta_n]$, (i.e., S is integral and finitely generated over R). It suffices to show that if $u \in IR[\theta_1, ..., \theta_n] \cap R$, then $u \in \mathrm{Rad}(I)$. Since S is module finite over R, we can write $S = Rs_1 + \cdots + Rs_m$ by taking $s_1 = 1$ if $1 \notin \{s_1, ..., s_n\}$. For each k, $us_k = \sum_{j=1}^{m} a_{kj}s_j$ with $a_{kj} \in I$, since $uS \subseteq IS$. So the matrix $[uI_m - a_{kj}]$ times the column vector of s's equals to zero. But $s_1 = 1$ and the determinant of that matrix must be zero. So, by expanding the determinant, $u^m + r_{m-1}u^{m-1} + \cdots + r_1u + r_0 = 0$, with $r_i \in I^{m-i}$, and all but the first term in I. This implies $u^m \in I$.

The second part of the statement follows immediately from the first.

□

Proposition 1.3. *Let $h : R \to S$ be a homomorphism of rings, and let $P \in Spec(R)$. Then the following assertions are equivalent:*

(i) $\exists\ Q \in Spec(S)$ with $h^{-1}(Q) = Q \cap R = P$;

(ii) $Im(R \setminus P) \cap PS = \emptyset$;

(iii) $h^{-1}(PS) = P$.

Proof. The equivalence between (ii) and (iii) is clear by definition of h. For (i) implies (ii), if $\exists\ Q$ with $Q \in \mathrm{Spec}(S)$ and $Q \cap R = P$, then $P \subseteq h^{-1}(PS) \subseteq h^{-1}(Q) = Q \cap R = P$. So $P = h^{-1}(PS)$. For (ii) implies (i), note that $R \setminus P$ is a multiplicative set in R, so

$h(R\setminus P)$ is a multiplicative set in S. But $h(R\setminus P)\cap PS=\emptyset$, which implies $\exists\, Q\in \operatorname{Spec}(S)$ such that $Q\cap h(R\setminus P)=\emptyset$ and $Q\supseteq PS$. But this means that $Q\cap R=P$. □

Proposition 1.4. *Let $R\subseteq S$, with S a domain, and take $0\neq s\in S$, with s integral over R. Then there is a nonzero multiple of s in R. Moreover, if $0\neq J\leq S$, then $J\cap R\neq 0$.*

Proof. Since s is integral over R, $\exists\, n$ such that $s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0=0$ with $a_i\in R$. Let n be minimal among all integral dependence equations. Then $a_0\neq 0$ because if $a_0=0$ then $s(s^{n-1}+\cdots+a_1)=0$ and since $s\neq 0$, it gives that $s^{n-1}+\cdots+a_1=0$. Since S is a domain, but this contradicts n being minimal. So $s(s^{n-1}+\cdots+a_1s)=-a_0\in R\setminus\{0\}$.

For the second part of the statement, let $0\neq s\in J$. Then a power of s is in R, hence in $J\cap R$, and it is of course nonzero since R is a domain.

□

Our next goal is to prove the following theorem, but we will need to develop a few ideas along the way.

Theorem 1.5 (Going-up Theorem)**.** *Let $R\subseteq S$ be an extension of rings, and assume that S is integral over R. Then the following are true:*

(1) *(Lying Over) For every $P\in Spec(R)$, $\exists\, Q\in Spec(S)$ that lies over P, i.e., $Q\cap R=P$.*
(2) *(Incomparability) If $Q\neq Q'$, with $Q,Q'\in Spec(S)$, and $Q\cap R=Q'\cap R=P$, then Q and Q' are incomparable, i.e., if $Q\subset Q'$ then $Q\cap R\subset Q'\cap R$.*
(3) *(Going-Up) If*

$$P_0\subset P_1\subset\cdots\subset P_n$$

is a strictly ascending chain of prime ideals in R, then there is a strictly ascending chain of prime ideals in S,

$$Q_0\subset Q_1\subset\cdots\subset Q_n$$

with $Q_i\cap R=P_i$ for every $0\leq i\leq n$.

Proof. (1) According to Proposition 1.3, we need to show that $PS\cap R=P$. But this follows at once from Proposition 1.2.

(2) Let us assume that there exists a pair of prime ideals $Q \subsetneq Q'$ such that $Q \cap R = Q' \cap R = P$.

Note that $R \hookrightarrow S$ induces a natural map $R/P \hookrightarrow S/Q$ that takes $\overline{a} \mapsto \overline{\overline{a}}$. Now, since $Q \cap R = P$, the above map is injective, that is it gives an extension of rings. But $R \hookrightarrow S$ is integral, so our new map is also integral. Moreover, S/Q is a domain. We can apply Proposition 1.4 to conclude $\frac{Q'}{Q} \cap \frac{R}{P} \neq \overline{0}$. But $Q' \cap R = P$ gives that $\frac{Q'}{Q} \cap \frac{R}{P} = \overline{0}$. So we obtained a contradiction.

(3) It suffices to show the claim for $n = 1$, so given $P \subseteq P'$ and $Q \cap R = P$, find Q'. We have $R/P \hookrightarrow S/Q$ an integral extension of domains, and $P'/P \in \operatorname{Spec}(R/P)$. Applying part (1), we know that there exist a prime ideal in S/Q say Q'/Q such that $\frac{Q'}{Q} \cap \frac{R}{P} = \frac{P'}{P}$, so $Q' \cap R = P'$, with $Q' \in \operatorname{Spec}(S)$.

□

Corollary 1.6. *If $R \hookrightarrow S$ is an integral extension, then* $\dim R = \dim S$.

Proof. Given a chain of primes in S, by intersecting each term with R, we obtain a chain of primes in R. Applying the part (2) of the above theorem we get $\dim S \leq \dim R$. Parts (1) and (3) show $\dim R \leq \dim S$. □

Corollary 1.7. *If $R \hookrightarrow S$ is an integral extension and $u \in R$ has an inverse in S, then u has an inverse in R.*

Proof. We know uR is an ideal in R. If u has no inverse in R, then $uR \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is maximal in R. By the lying over property, $\exists\ Q \in \operatorname{Spec}(S)$ such that $uS \subseteq Q \subset S$, a contradiction, since $uS = S$. □

Corollary 1.8. *Let $R \hookrightarrow S$ be an integral extension. If S is a field then R is a field.*

Proof. Follows at once from the above Corollary. □

Corollary 1.9. *Let $R \hookrightarrow S$ be an integral extension. Then if $M \in Max(S)$, then $M \cap R = \mathfrak{m} \in Max(R)$.*

Proof. Since $M \cap R \in \operatorname{Spec}(R)$, then $R/\mathfrak{m} \hookrightarrow S/M$ is an integral extension of domains, and so S/M is a field. This implies $\mathfrak{m} \in \operatorname{Max}(R)$. □

2. The dimension of a polynomial ring over a field

Definition 2.1. *Let $K \subseteq L$ a field extension. We say that $\alpha_1, \ldots, \alpha_n \in L$ form a transcendence basis for the extension if they algebraically independent over K and $K(\alpha_1, \ldots, \alpha_n) \subseteq L$ is algebraic. The number n is denoted by $trdeg_K(L)$.*

By definition, for a K-algebra domain A, we set $trdeg_K(A) = trdeg_K(L)$, where L is the fraction field of A.

Note that it is know that the notion of transcendence basis is well defined, and that any set of K-algebraically independent elements of L can be completed to a transcendence basis of L over K. Moreover, if $L = K(X)$ for some subset X of L, then a transcendence basis of L over K can be extracted from X.

Theorem 2.2. *Let k be a field. Then* $\dim(k[X_1, ..., X_n]) = n$.

Proof. Let $A = k[X_1, ..., X_n]$. Clearly

$$0 \subset (X_1) \subset \ldots \subset (X_1, \ldots, X_n)$$

is a strict chain of prime ideals so $\dim(A) \geq n$.

Claim: Let $P \subsetneq Q$ prime ideals in A. Then $\operatorname{trdeg}_k(A/P) > \operatorname{trdeg}_k(A/Q)$.

Since A is a domain any maximal chain of prime ideals will have to start with the zero ideal. The above claim implies a maximal chain of length strictly greater than n gives $\operatorname{trdeg}_k(A) > n$. But it is well known that $k(X_1, \ldots, X_n)$ has transcendence degree equal to n. So all maximal chains of prime ideals in A have length at most n, so $\dim(A) \leq n$.

Let us concentrate on proving the Claim.

Since A/P maps onto A/Q we see that $\operatorname{trdeg}_k(A/P) \geq \operatorname{trdeg}_k(A/Q) = r$. Assume that we have equality.

Clearly, if we denote the images of X_i in A/Q by α_i, then $A/Q = k[\alpha_1, \ldots, \alpha_n]$. Because $k(\alpha_1, \ldots, \alpha_n)$ is the fraction field of A/Q, we can extract a transcendence basis for A/Q over k from $\{\alpha_1, \ldots, \alpha_n\}$, say $\alpha_1, \ldots, \alpha_r$.

Let u_i equal the image of X_i in A/P, for $i = 1, \dots, n$. Since u_i maps to α_i, we get that $u_1, \dots, u_r$ are algebraically independent over k as well so they form a transcendence basis of A/P over k.

Let $S = k[X_1, \dots, X_r] \setminus \{0\}$. Since $\alpha_1, \dots, \alpha_r$ and $u_1, \dots, u_r$ are algebraically independent over k we get that $P \cap S = Q \cap S = \emptyset$.

Let us denote $K = k(X_1, \dots, X_r)$. Then $S^{-1}A = K[X_{r+1}, \dots, X_n]$. Moreover

$$\frac{S^{-1}A}{PS^{-1}A} \simeq k(u_1, \dots, u_r)[u_{r+1}, \dots, u_n].$$

But $u_{r+1}, \dots, u_n$ are algebraic over $k(u_1, \dots, u_r)$, so $k(u_1, \dots, u_r)[u_{r+1}, \dots, u_n]$ is in fact a field.

This gives that $PS^{-1}A$ is a maximal ideal in $S^{-1}A$, or, in ther words, P is a prime ideal of A maximal with the property that $P \cap S = \emptyset$. This is clearly not true since $P \subsetneq Q$ and $Q \cap A = \emptyset$.

□