1. The Spectrum of a Ring & the Zariski Topology

Definition 1.1. *Let A be a ring. For I an ideal of A, define $V(I) = \{P \in Spec(A) \mid I \subseteq P\}$.*

Proposition 1.2. *Let A be a ring. Let Λ be a set of indices and let I_l denote ideals of A. Then*

(1) $V(0) = Spec(R), V(R) = \emptyset$;
(2) $\cap_{l\in\Lambda} V(I_l) = V(\sum_{l\in\Lambda} I_l)$;
(3) $\cup_{l=1}^{k} V(I_l) = V(\cap_{l=1}^{k} I_l)$;

Then the family of all sets of the form $V(I)$ with I ideal in A defines a topology on $\text{Spec}(A)$ where, by definition, each $V(I)$ is a closed set. We will call this topology *the Zariski topology* on $\text{Spec}(A)$.

Let $\text{Max}(A)$ be the set of maximal ideals in A. Since $\text{Max}(A) \subseteq \text{Spec}(A)$ we see that $\text{Max}(A)$ inherits the Zariski topology.

Now, let k denote an algebraically closed fied. Let Y be an algebraic set in $\mathbb{A}^n_k$. If we consider the Zariski topology on $Y \subseteq \mathbb{A}^n_k$, then points of Y correspond to maximal ideals in A that contain $I(Y)$. That is, there is a natural homeomorphism between Y and $\text{Max}(k[Y])$.

Example 1.3. Let $Y = \{(x, y) \mid x^2 = y^3\} \subset k^2$ with k algebraically closed. Then points in Y correspond to $\text{Max}(\frac{k[x,y]}{(x^2 - y^3)}) \subseteq \text{Spec}(\frac{k[x,y]}{(x^2 - y^3)})$. The latter set has a Zariski topology, and the restriction of the Zariski topology to the set of maximal ideals gives a topological space homeomorphic to Y (with the Zariski topology).

Let $(A, \mathfrak{m}_A)$ be a local ring. Then $V(\mathfrak{m}_A)$ is the set of all prime ideals containing $\mathfrak{m}_A$, that is ideals equal to $\mathfrak{m}_A$. This implies that $\overline{\{\mathfrak{m}_A\}} = \{\mathfrak{m}_A\}$. Moreover, this shows that that a closed point P in $\text{Spec}(A)$ (i.e., $\overline{\{P\}} = \{P\}$) is a maximal ideal of A.

Take $P_0 \leq A$, with P_0 a minimal prime ideal. Then $V(P_0) = \{P \in \text{Spec}(A) \mid P_0 \subseteq P\}$ which is naturally identified with $\text{Spec}(A/P_0)$.

If $P_0 = 0$ with A a domain, then $V(0) = \text{Spec}(A)$, and $\overline{(0)} = \text{Spec}(A)$. We call such a prime ideal the generic point of $\text{Spec}(A)$: $P \in \text{Spec}(A)$ such that $\text{Spec}(A) = \overline{\{P\}}$.

Example 1.4. Let $k = \overline{k}$, $A = k[x, y]$, and consider $\text{Spec}(A) \supseteq \text{Max}(A) = \{(x-\alpha, y-\beta) \mid \alpha, \beta \in k\}$. This latter set corresponds to k^2.

Definition 1.5. *Principal open sets are open sets of the form* $D(f) = \{P \in Spec(A) \mid f \notin P\}$*, for* $f \in A$*. Note that this requirement is equivalent to* $(f) \nsubseteq P$ *or* $f \notin P$*.*

For $f \in A$, $D(f) \subseteq \mathrm{Spec}(A)$. This implies that for $P \in D(f)$ one has that $P \cap \{1, f, f^2, ..., f^n, ...\} = \emptyset$, which implies that $D(f)$ can be naturally identified with $\mathrm{Spec}(A_f)$.

If $Y \subseteq \mathbb{A}^n$ is an algebraic set, and $Y = Z(f)$ with $f \in k[T_1, ..., T_n]$ then $\mathbb{A}^n \setminus Y = \{x \mid f(x) \neq 0\}$. $\mathrm{Spec}(A_f)$ is the natural ring that corresponds to the principal open set $D(f)$, since having $f(x) \neq 0$ 'implies' that we can write f as a denominator.

Let $f : A \to B$ be a ring homomorphism, and let $\mathfrak{p} \in \mathrm{Spec}(A)$, so $f^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A \in \mathrm{Spec}(A)$. We can consider

$$\phi = f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A),$$

defined by $\phi(\mathfrak{p}) = \mathfrak{p} \cap A$. If $I \leq A$, then $\phi^{-1}(V(I)) = V(IB)$. Note that ϕ is therefore continuous, because a preimage under ϕ of a closed set is a closed set.

However, if $M \in \mathrm{Max}(B)$, this does not imply that $M \cap A = f^{-1}(M) = f^*(M) = \phi(M)$ belongs to $\mathrm{Max}(A)$.

Consider π the projection map from a surface onto a line. Take t a point on our line. Then $\pi^{-1}(t)$ is a curve (not a point) on the surface so it is natural to study $\mathrm{Spec}(A)$, instead $\mathrm{Max}(A)$. ($\mathrm{Max}(A)$ would not allow us to pullback all the points, as $\pi^{-1}(t)$ is not a point in general.) The curve does define a prime ideal.

Definition 1.6. *Let* ϕ *be defined as above, and take* $\mathfrak{p} \in Spec(A)$*. The fiber over* $\mathfrak{p}$ *is defined to be* $\phi^{-1}(\mathfrak{p}) = \{Q \in Spec(B) \mid Q \cap A = \mathfrak{p}\}$*.*

It can be shown that $\phi^{-1}(\mathfrak{p})$ is homeomorphic to $\mathrm{Spec}(B \otimes_A \dfrac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}})$. The ring $\frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}}$ is a field since it is a quotient of a local ring by its maximal ideal.

Definition 1.7. *If* $\mathfrak{p} \in Spec(A)$*, then we call* $\dfrac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}}$ *the residue field at* $\mathfrak{p}$*, denoted* $k(\mathfrak{p})$*. One can see that this field is also the fraction field of* $A/\mathfrak{p}$

We can now see the following string of equalities:

$$B \otimes_A k_\mathfrak{p} = B \otimes_A \frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}} = \frac{B \otimes_A A_\mathfrak{p}}{\mathfrak{p}(B \otimes_A A_\mathfrak{p})} = \frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}} = (\frac{B}{\mathfrak{p}B})_{(A \setminus \mathfrak{p})}.$$

Let us compute $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$. We know that $S = A \setminus P \subseteq A \subseteq B \to B/\mathfrak{p}B$, so $\dfrac{Q}{\mathfrak{p}B} \cap S = \emptyset$. The first term in our intersection is in $\mathrm{Spec}(B/\mathfrak{p}B)$, and hence $\mathrm{Spec}(B \otimes_A k(\mathfrak{p})) =$

$\{Q \in \mathrm{Spec}(B) \mid Q \cap (A \setminus P) = \emptyset, Q \supseteq \mathfrak{p}B\} = \{\mathfrak{p} \in \mathrm{Spec}(B) \mid P \cap A = \mathfrak{p}, P \supseteq \mathfrak{p}B\} = \{\mathfrak{p} \in \mathrm{Spec}(B) \mid P \cap A = \mathfrak{p}\} = \phi^{-1}(\mathfrak{p})$. So it is natural to call $B \otimes_A k(\mathfrak{p})$ *the fiber ring at* $\mathfrak{p}$.

1.1. Exercises.

(1) Let M be an A-module, and let $x \in M$. If $x = 0$ in $M_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(A)$, then $x = 0$.

Proof. Assume $x \neq 0$, which implies $\mathrm{Ann}_A(x) \nsubseteq A$. Then $\mathrm{Ann}_A(x) \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Max}(A)$. But $x = 0$ in $M_\mathfrak{m}$ so $\exists\ u \in A \setminus \mathfrak{m}$ such that $ux = 0$, which implies $u \in \mathrm{Ann}_A(x) \subseteq \mathfrak{m}$, a contradiction. Hence $x = 0$. Note that $M_\mathfrak{m}$ is a module over the local rin $A_\mathfrak{m}$. □

(2) Let M be a finitely generated A-module. If $M \otimes_A k(\mathfrak{m}) = 0$ for all $\mathfrak{m} \in \mathrm{Max}(A)$ then $M = 0$.

Proof. (Sketch) We know $M \otimes_A k(\mathfrak{m}) \cong \frac{M_\mathfrak{m}}{\mathfrak{m}M_\mathfrak{m}}$, so $M_\mathfrak{m} = \mathfrak{m}M_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$. Now apply NAK and exercise 1. □

2. Krull Dimension

Definition 2.1. *Let $P \in Spec(A)$. A chain of prime ideals descending from P of length m is:*

$$P_0 \subset P_1 \subset \cdots \subset P_m = P$$

with each $P_i \in Spec(A)$.

Definition 2.2. *We define the height of P, denoted $ht(P)$ to be the supremum of the lengths of chains of prime ideals descending from P.*

Definition 2.3. *The Krull dimension of a ring A, denoted* $\dim(A)$*, is defined to be the supremum of $ht(P)$, where P runs over all prime ideals in $Spec(A)$.*

We can have $\dim(A) = \infty$, even if A is Noetherian.

2.1. Some examples.

If k is a field, then $\dim(k) = 0$.

If A is a PID, then $\dim(A) = 1$. Some examples of PID's and chains of prime ideals of length one are: $0 \subseteq (x) \subseteq k[x]$, and $0 \subseteq (p) \subseteq \mathbb{Z}$.

To a prime ideal P, we associate an algebraic set that is defined by taking common the zeroes of all polynomials in P. Each ideal in the chain descending from P correspond to subsets of the original algebraic set. We now have the following chains,

$$\text{point} \subseteq \text{some curve} \subseteq \text{ some plane} \subseteq \cdots \subseteq \text{whole variety}$$

$$\text{Maximal ideal} \supseteq \cdots \supseteq (0),$$

where single points are the zero sets for maximal ideals, and the whole space is the zero set for the zero polynomial.

Remark 2.4. (1) If Y is algebraic variety in $\mathbb{A}^n$, one can define the *codimension* of Y as the supremum of all n such that $Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$ of strict chains of irreducible algebraic varieties in $\mathbb{A}^n$. This is deonoted by $\text{codim}(Y)$.

If Y is an algebraic set,then $\text{codim}(Y)$ is by definition the infimum of all $\text{codim}(Z)$ where Z is an irreducible component of Y.

(2) Let $x \in Y$, where Y is an algebraic set in $\mathbb{A}^n$. The dimension of Y at x is by definition the infimum of $\dim(U)$ where U runs over all open sets in Y containing x.

3. Integral Extensions

Definition 3.1. *If we have a map $R \hookrightarrow S$, then $s \in S$ is integral over R if $\exists$ n and $r_0, ..., r_{n-1} \in R$ such that $x^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0$.*

Some Facts:

1. An algebra over a field k is module finite over k if and only if it is a finite dimensional vector space over k. Also, $R \subseteq S$ is module finite if $\exists$ $s_1, ..., s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$.

2. $K \subseteq L$ a field extension is module finite if and only if it is a finite algebraic extension.

Example 3.2. $\mathbb{Z}[\frac{1}{2}] = \{\frac{n}{2^k} \mid n \in \mathbb{Z},\ k \in \mathbb{W}\}$ *is not module finite over* $\mathbb{Z}$ *since* $\mathbb{Z}[\frac{1}{2}] \neq \mathbb{Z}\frac{1}{2} + \mathbb{Z}\frac{1}{2^2} + \cdots + \mathbb{Z}\frac{1}{2^k}$ *because we cannot get higher powers of 2 in the denominator.*

Example 3.3. $\mathbb{Z}[\sqrt{2}]$ *is module finite over* $\mathbb{Z}$*.*

In general, R/I is module finite over R for all $I \leq R$: the projection map can be written as $\pi : R \to R/I = R\overline{1} = R(\pi(1))$ when we consider $\pi(a) = \overline{a} = a + I$ and so R/I si R-spanned by $\pi(1)$.

Remark 3.4. *One can talk of $\phi : R \to S$ as an integral map by regarding S as an R-algebra via ϕ, that is by reducing to an extension $R/\ker(\phi) = Im(\phi) \hookrightarrow S$.*

We will show:

1. $R \hookrightarrow S$ integral implies $\dim R = \dim S$.

2. Every finitely generated k-algebra is module finite over a polynomial ringover k. This is known as the Noether Normalization Theorem.

Theorem 3.5. *If $R \hookrightarrow S$, then the following are equivalent:*

(1) *S is a finitely generated R-algebra and every $s \in S$ is integral over R.*
(2) *$S = R[v_1, ..., v_n]$ where v_i are integral over R.*
(3) *S is module finite over R.*

Proof. (1) implies (2) is trivial.

For (2) implies (3), we proceed by induction on n. If $n = 1$, then $S = R[v_1]$ with v_1 integral over R so $\exists$ m such that $v_1^m + r_{m-1}v^{m-1} + \cdots + r_1 v + r_0 = 0$ with $r_i \in R$. This implies that $v^m \in \langle 1, v, ..., v^{m-1} \rangle_R$. We also have that $v^{m+1} + r_{m-1}v^m + \cdots + r_1 v^2 + r_0 v = 0$ which implies $v^{m+1} \in \langle 1, v, ..., v^{m-1} \rangle_R$, and by induction, $v^k \in \langle 1, v, ..., v^{m-1} \rangle_R$ for all $k \geq m$, so $S = R + Rv + \cdots + Rv^{m-1}$. For $n > 1$, we have $T = R[v_1, ..., v_{n-1}]$ is module finite over R. Let $S = T[v_n]$. We can say that $R \subseteq T$ is module finite with $t_1, ..., t_k$ as generators of R over T. The fact that $T \subseteq S$ is also module finite implies that $R \subseteq S$ is module finite by transitivity: If $s_1, ..., s_h$ generate S over T, then $s_i t_j$ generate S over R.

For (3) implies (1), let $S = Rs_1 + \cdots + Rs_n$. Clearly $S = R[s_1, ..., s_n]$. Need $s \in S$ integral over R. If $s_1 \neq 1$, then put 1 in the list of generators and rename $s_1 = 1$. For all i, we have $ss_i = \sum_{j=1}^{n} v_{ij} s_j$, with $v_{ij} \in R$. Then $A = sI_n - (r_{ij}) \in M_n(S)$, so the product of A with the column vector consisting of all the s's must be 0. Multiply by $\mathrm{Adj}(A)$, to obtain that $\det(A)$ times this column vector equals 0. Since $s_1 = 1$, we have $\det(A)s_1 = \det(A) = 0$. The form of $A = sI_n - (r_{ij})$ shows that $\det(A)$ is a monic polynomial expression in s with coefficients in R, and thus s is integral over R. □

Proposition 3.6. *1. If $R \subseteq S$, then the integral elements of S form a subring called the integral closure of R in S.*

Proof. Let $t, s \in S$ with t and s integral in S. Then $R[s, t]$ is integral over R by previous theorem, and $s + t, st \in R[s, t]$ so $s + t$ and st are integral. □

Definition 3.7. *We call $R \hookrightarrow S$ an integral extension if every $s \in S$ is integral over R.*

Definition 3.8. *Let R be a domain, and $R \subseteq L$ where L is a field. The integral closure of R in L is denoted by R'_L (or $\overline{R_L}$). If $L = Q(R)$, the field of fractions of R, then $R'_{Q(R)}$, called the integral closure of R, is denoted by R' or $\overline{R}$.*