LECTURE 10

1. Affine Space & the Zariski Topology

Definition 1.1. *Let k a field. Take S a set of polynomials in $k[T_1, ..., T_n]$. Then $Z(S) = \{x \in k^n \mid f(x) = 0, \ \forall f \in S\}$.*

It is easy to check that $Z(S) = Z((S))$ with (S) denoting the ideal generated by elements of S.

Definition 1.2. *$Y \subseteq k^n$ is an (affine) algebraic set if $\exists\ S \subseteq k[T_1, ..., T_n] = A$ such that $Z(S) = Y$.*

Example 1.3.
(1) *Consider the ideal $I = (xy) \subset k[x, y]$. Then $Z(I) = \{(x, y) \in k^2 \mid xy = 0\} = \{x = 0\} \cup \{y = 0\}$.*
(2) *Consider the ideal $I = (x^2 - y^3) \subset k[x, y]$. Then $Z(I) = \{(x, y) \in k^2 \mid x^2 - y^3 = 0\}$.*

Proposition 1.4. *The following are true:*

(1) *The union of a finite collection of algebraic sets is algebraic.*
(2) *Arbitrary intersections of algebraic sets are algebraic.*
(3) *$\emptyset$ and $\mathbb{A}^n$ are algebraic.*

Proof. (1) It suffices to show this for the union of two sets. The general case can then be established by induction. Let $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$. We claim that $Y_1 \cup Y_2 = Z((T_1)(T_2))$. For the forward inclusion, let $x \in Y_1 \cup Y_2$, so without loss of generality, assume $x \in Y_1$. This implies $f(x) = 0$ for all $f \in (T_1)$. Since $(T_1)(T_2) \subseteq (T_1)$, we have $f(x) = 0$ for every $f \in (T_1)(T_2)$ so $x \in Z((T_1)(T_2))$.

For the reverse inclusion, let $x \in Z((T_1)(T_2))$ so $h(x) = 0$ for all $h \in (T_1)(T_2)$. Assume $x \notin Z((T_2))$ so $\exists\ f \in (T_2)$ such that $f(x) \neq 0$. Take $g \in (T_1)$. Then $gf \in (T_1)(T_2)$ which implies $(gf)(x) = 0 \Rightarrow g(x)f(x) = 0$. But $f \neq 0$ so $g(x) = 0$ for every g. Thus $x \in Z((T_1)) = Y$, and we have equality of sets.

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(2) Let $\{Z(I_\lambda)\}_{\lambda\in\Lambda}$. We claim that $\bigcap_{\lambda\in\Lambda} Z(I_\lambda) = Z(\sum_\lambda I_\lambda)$ where the latter expression is a sum over a finite subset of Λ. Beginning with reverse containment, let $x \in Z(\sum I_\lambda) \Rightarrow f(x) = 0$ for all $f \in \sum I_\lambda$. In particular, $f(x) = 0$ for $f \in I_\lambda$, so $x \in Z(I_\lambda)$ for all λ.

For forward containment, let $x \in Z(I_\lambda)$ for every λ. Then $f(x) = 0$ for every $f \in I_\lambda$ for all λ. Then for $g \in \sum I_\lambda$, we have $g = h_1 + \cdots + h_n$ with $h_i \in I_{\lambda_i}$. This implies that $g(x) = 0$ because $h_i(x) = 0$ for all i. Hence $x \in Z(\sum I_\lambda)$, and we have equality of sets.

(3) $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$.

$\square$

Remark 1.5. The above Proposition shows that the collection of algebraic sets defines a topology on k^n where the closed sets are the algebraic sets. This topology will be called Zariski topology and k^n endowed with this topology will be denoted $\mathbb{A}^n$.

Remark 1.6. Is the Zariski topology Haussdorf? In general no. For an example, in the special case $\mathbb{A}^1$ with $k = \overline{k}$, every $f \in k[T]$ has finitely many zeroes, so closed sets in the Zariski topology have a finite number of points. We may ask if $\mathbb{A}^1$ is Hausdorff by taking $x, y \in \mathbb{A}^1$, and assuming that $\exists\ U_x, U_y$ open disjoint neighborhoods of x and y, respectively. But this would be equivalent to having closed sets that cover $\mathbb{A}^1$, a contradiction since $\mathbb{A}^1$ is infinite.

1.1. The Ideal-Variety Correspondence.

Definition 1.7. *Let $Y \subset \mathbb{A}^n$, and $A = k[T_1, ..., T_n]$. Then $I(Y) = \{f \in A \mid f(P) = 0 \quad \forall\ P \subseteq Y \subseteq k^n\}$. Also, $I(Y)$ is an ideal of A. This ideal is in fact radical.*

Proposition 1.8. *Let $A = k[T_1, ..., T_n]$. Then the following are true:*

(1) *If $A_1 \subseteq A_2 \subseteq A$, then $Z(A_1) \supseteq Z(A_2)$ in $\mathbb{A}^n$.*
(2) *If $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$ in A.*
(3) *If $Y_1, Y_2 \subseteq \mathbb{A}^n$,$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
(4) *If I, J are ideals in A, then $Z(I) \cup Z(J) = Z(IJ)$. Also, $Z(\cup S_j) = \cap Z(S_j)$, for any family of subsets $\{S_j\}$ of $\mathbb{A}^n$.*
(5) *If $S \subseteq \mathbb{A}^n$ and J ideal of A, then $S \subseteq Z(I(S))$ and $J \subseteq I(Z(J))$.*
(6) *If V is an algebraic set then $V = Z(I(V))$. If J is an ideal of A of the form $J = I(S)$, then $I(Z(J)) = J$.*

Corollary 1.9. *The functions $Z(-)$ defined on the family of ideals of the form $I(S)$ for some $S \subseteq \mathbb{A}^n$ and $I(-)$ defined on algebraic sets in $\mathbb{A}^n$ are inverses to each other.*

Theorem 1.10. (1) *If $k = \overline{k}$, then $I(Z(J)) = Rad(J)$ for all $J \leq A = k[T_1, \ldots T_n]$. This is known as the Hilbert Nullstellensatz.*

(2) $Z(I(Y)) = \overline{Y}$ *for all* $Y \subseteq \mathbb{A}^n$.

Proof. The statement of (1) can be restated as $f \in I(Z(J) \Leftrightarrow f(P) = 0$ for all $P \in Z(J)$, where $Z(J) = \{x \in \mathbb{A}^n \mid g(x) = 0 \quad \forall\, g \in J\}$. The statement implies that if f vanishes where J vanishes then $\exists\, h$ such that $f^h \in J$. This only follows when $k = \overline{k}$.

(2) We start with forward inclusion. Let $Y \subseteq Z(I(Y))$ which implies $\overline{Y} \subseteq Z(I(Y))$. For the reverse, let W be a closed superset of Y. Then $W = Z(J)$, which gives $Z(J) \supseteq Y$. Examine the ideals corresponding to these sets, and we get $I(Y) \supseteq I(Z(J))$, so $J \subseteq I(Y)$. Now go to the sets corresponding to the ideals, and we get $Z(I(Y)) \subseteq Z(J) = W$, so any closed set containing Y contains $Z(I(Y))$. This statement applied to $W = \overline{Y}$ gives that $\overline{Y} \supseteq Z(I(Y))$. □

Corollary 1.11. *Assume that k is an algebraically closed field. The maps $Z(-)$ and $I(-)$ are inverses to each other and establish a one-to-one correspondence between the family of algebraic sets in $\mathbb{A}^n$ and radical ideals of A.*

Corollary 1.12. *In this correspondence, a point $(a_1, \ldots, a_n) \in \mathbb{A}^n$ corresponds to the maximal ideal $(T_1 - a_1, \ldots, T_n - a_n)$ of A.*

Proof. Let $I = (T_1 - a_1, \ldots, T_n - a_n)$ which is a maximal (hence radical) ideal.

The Corollary follows at once since $Z(I) = (a_1, \ldots, a_n)$. The correspondence implies that $I(a_1, \ldots, a_n) = (T_1 - a_1, \ldots, T_n - a_n)$ which is a non-trivial statement (and which may not be true if k is not algebraically closed).

□

Definition 1.13. *Let $\emptyset \neq Y \subseteq X$, with X a topological space. Then Y is irreducible if Y is not a union of two proper closed subsets of Y.*

An example of a reducible set in $\mathbb{A}^2$ is the set of points satisfying $xy = 0$ which is the union of the two axis of coordinates.

Definition 1.14. *We call Y an affine algebraic variety if Y is an irreducible algebraic set.*

Corollary 1.15. *Let Y be algebraic variety. Then $I(Y)$ is prime. Conversely, $I(Y)$ is prime implies that Y is an algebraic variety. Therefore, in our 1-1 correspondence, varieties (irreducible algebraic sets) correspond to prime ideals.*

Proof. Take $Y = Z(I)$ irreducible. Let $fg \in I(Y)$, so $(fg)(y) = 0$ for all $y \in Y$. This implies $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Then $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$. Note that both sets in our union are closed in the subspace topology. But Y is irreducible, so the sets $Y \cap Z(f)$ and $Y \cap Z(g)$ are not simultaneously proper. Assume, without loss of generality, that $Y \cap Z(f) = Y$ which implies that $Y \subseteq Z(f)$, so $f \in I(Y)$ by definition. This gives $f \in I(Y)$ and hence $I(Y)$ is prime.

Conversely, let $Y = Y_1 \cup Y_2$ with $Y_i = Z(I_i), I_i$ ideals in A, for $i = 1, 2$. Assume, for a contradiction, that Y_1, Y_2 are strictly contained in Y. First note that $I(Y) \subset I(Y_i)$ since equality would give $Z(I(Y_i) = Z(I(Y))$, for $i = 1, 2$. But $Z(-), I(-)$ are inverses to each other when restricted to the set of ideals of algebraic sets and, respectively, algebraic sets, hence $Y_i = Y$, false.

Now, let $f_i \in I(Y_i) \setminus I(Y)$. Then $f_1 f_2 \in I(Y_1)I(Y_2) \subset I(Y_1) \cap I(Y_2) \subset I(Y_1 \cup Y_2) = I(Y)$.

Bt $I(Y)$ is prime. Therefore, either $f_1 \in I(Y)$ or $f_2 \in I(Y_2)$. This is a contradiction.

$\square$

Definition 1.16. *If $A = k[T_1, ..., T_n]$, and $Y \subseteq \mathbb{A}^n$ is an algebraic set, then $k[Y] = A/I(Y)$ is called the coordinate ring of functions of Y.*

Definition 1.17. *A map between two algebraic sets $\phi : V \subseteq \mathbb{A}^n \to W \subseteq \mathbb{A}^m$ is called a morphism (or regular map) if there are polymomials $F_1, \dots F_m$ such that*

$$\phi(a_1, \dots, a_n) = (F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)),$$

for all $(a_1, \dots, a_n) \in V$.

A morphism ϕ is called isomorphism between V and W if there exists a morphism $\psi : W \to V$ inverse to ϕ.

Let $\phi : V \subseteq \mathbb{A}^n \to W \subseteq \mathbb{A}^m$ be a morphism. This morphism induces a natural map $\phi_* : k[W] \to k[V]$ by $\phi_*(\hat{f}) = f \hat{\circ} \phi$.

Indeed, if $f - g \in I(W)$ then $f(w) = g(w)$ for all $w \in W$, so $f(\phi(v)) = g(\phi(v)$ for all $v \in V$, since $\phi(v) \in W$. This means that $f \circ \phi - g \circ \phi \in I(V)$ and hence ϕ_* is well defined. It is a routine check that Φ_* is in fact a k-algebra homomorphism.

Moreover, every k-algebra homomorphism $\Phi : k[W] \to k[V]$ is induced by a unique ϕ, that is $\Phi = \phi_*$. The morphism ϕ is an isomorphism if and only if ϕ_* is an isomorphism of k-algebras.

Given $\Phi : k[W] \to k[V]$, k-algebra homomorphism, let us construct ϕ:

Let $\Phi(\hat{T}_i) = \hat{F}_i$, for all $i = 1, \ldots, m$ and $F_i \in k[T_1, \ldots, T_n]$. Then $\phi = (F_1, \ldots, F_m)$ defines a morphism between $\mathbb{A}^n$ and $\mathbb{A}^m$. Let us show that it maps V to W.

Let $g \in I(W)$ so $g(w) = 0$ for all $w \in W$. Moreover $g(\hat{T}_1, \ldots, \hat{T}_m) = g(\widehat{T_1, \ldots, T_m}) = \hat{0}$ in $k[W]$, since $g \in I(W)$.

Therefore $\Phi(g(\hat{T}_1, \ldots, \hat{T}_m) = 0$ in $k[V]$ since a homomorphims maps 0 to 0.

But Φ is a k-algebra homomorphism, so $g(\Phi(\hat{T}_1), \ldots, \Phi(\hat{T}_m)) = 0$ which is equivalent to $g(\hat{F}_1, \ldots, \hat{F}_m) = 0$ in $k[V]$, or $g(F_1, \hat{\ldots}, F_m) = 0$ in $k[V]$. This gives $g(F_1, \ldots, F_m) \in I(V)$ or in other words, $g(F_1, \ldots, F_m)(v) = 0$ for all $v \in V$, i.e. $g(F_1(v), \ldots, F_m(v)) = 0$ for all $v \in V$.

Since $\phi(v) = (F_1(v), \ldots F_m(v))$ we see that $g(\phi(v)) = 0$ for all $v \in V$ and so $g \in I(W)$ implies that $\phi(v) \in Z(g)$. In other words, $\phi(v) \in Z(I(W))$. But $Z(I(W)) = W$, since W is an algebraic set, and so $\phi(V) \in W$.

Note that $\phi_*(\hat{T}_i) = \widehat{T_i \circ \phi} = \hat{F}_i = \Phi(\hat{T}_i)$ for all $i = 1, \ldots, m$. Since $\hat{T}_i$ are k-algebra generators for $k[W]$ we get $\phi_* = \Phi$.

Proposition 1.18. *Every nonempty affine algebraic set V may be uniquely written in the form*

$$V = V_1 \cup \cdots \cup V_n$$

where each V_i is an algebraic variety and $V_i \not\subseteq V_j$ for all $j \neq i$. (These V_i's are called irreducible components on V).

2. Dimension

Definition 2.1. *Let $V \subseteq \mathbb{A}^n$ be an algebraic set. The supremum over all n such that there exists a chain $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$ of distinct irreducible algebraic sets in V is called the* dimension *of V.*

Definition 2.2. *Let P be a prime ideal in a ring A. The supremum over all n such that there exists a chain of distinct prime ideals*

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$$

contained in P is called the height *of P and it is denoted by $ht(P)$.*

The Krull dimension *of A, $dim(A)$, is the supremum of all $ht(P)$ over all $P \in Spec(A)$.*

If I is an arbitrary ideal of A, we let the height *of I, $ht(I)$, equal the infimum of all $ht(P)$ over all prime ideals P containing I.*

Theorem 2.3. *Let V be an algebraic set in $\mathbb{A}_k^n$ where k is algebraically closed. Then*

$$dim(V) = dim(k[V]).$$

Proof. This follows at once since irreducible algebraic sets in V correspond to prime ideals containing $I(V)$ in our correspondence. $\square$