LECTURE 8: THE HILBERT-SAMUEL FUNCTION

This next result clarifies the degree of the polynomial describing the Hilbert function of a graded module.

Theorem 0.1. *(Hilbert polynomial)*

Let M be a finitely generated graded module over $R = R_0[R_1]$ where R is Noetherian, R_0 is Artinian, and $\dim M = d$. Then there exists a polynomial HP of degree $d-1$, called the Hilbert polynomial of M, such that for all $n \gg 0$ $H_M(n) = HP(n)$ (the zero polynomial will have degree 1 by convention here).

Proof. We will use induction on d.

The case $d = 0$ is immediate since in this case M is Artinian and so $M_n = 0$ for all $n \gg 0$, hence $H_M(n) = 0$ for $n \gg 0$. First let us prove the statement for the case $M = R/p$ where p is a graded prime ideal of R.

Assume $d > 0$ and so let x be a homogeneous element of R/P of degree 1 (this is possible because R is standard graded). We have the following short exact sequence:

$$0 \to (R/P)(-1) \xrightarrow{x} R/P \to R/(xR+P) \to 0.$$

Apply the additivity of the length function in each degree n and get

$$H_M(n) - H_M(n-1) = H_N(n),$$

where $N = R/(xR+P)$. Since the dimension of N is $d-1$ and H_N is a function of polynomial type of degree $d-2$ it follows from the previous equality that H_M is of polynomial type of degree $(d-2)+1 = d-1$.

For the general case, note that, similar to the ungraded case, there is prime filtration for M: $0 = M_0 \subset M_1 \subset M_i \subset M_n = M$, where $M_{i+1}/M_i \simeq (R/p_i)(a_i)$ where p_i are graded prime ideals in R and $a_i \in \mathbb{Z}$. The additivity of the length function gives that $H_M(n) = \sum_{i=i}^{n} H_{(R/p_i)(a_i)}$. Note that d is indeed the maximum of all dimensions of R/p_i

and the Hilbert polynomials have nonnegative leading coefficients since they define a nonnegative function. In conclusion, the case R/p implies the general case. □

Definition 0.2. *Let M be a finitely generated graded over $R = R_0[R_1]$ where R is Noetherian, R_0 is Artinian, and $\dim M = d$. The Hilbert iterated functions are defined as $H_{M,0} : I\!N \to \mathbb{Z}$, where $H_{M,0} = H_M$, and $H_{M,i} = \sum_{j \le n} H_{M,i-1}(j)$. We take it as fact that the latter is a function of polynomial type with degree $d + i - 1$ (the reader can try it as an exercise based upon Theorem 0.1).*

We write $HP(n) = \frac{e}{(d-1)!} n^{d-1} + \text{lower degree terms}$. By definition the multiplicity of M, $e(M)$, is e if $d > 0$ and $l(M)$ if $d = 0$.

Note that as a consequence $H_{M,1}(n) = \sum_{i=0}^{d} a_i \binom{n+i}{i}$, for $n >> 0$, with $a_d = e(M)$, and the sum is of polynomial type with degree d.

Example 0.3. The curve $x^2 - y^2 - y^3 = 0$ defines a one-dimensional ring $\dfrac{k[x,y]}{(x^2 - y^2 - y^3)}$. The ring is not graded, however we still want to find a natural way to define the multiplicity of the curve at the origin.

Now we will lead up to the Hilbert Samuel function of a finitely generated module M over a semi-local (has finitely many maximal ideals), Noetherian ring R. First, we need some definitions.

Definition 0.4. *Let $\mathfrak{m} = Jac(R) = \bigcap_{m \in Max(R)} m$. Remember that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Let $I \le R$. Then I is said to be an* **ideal of definition** *for R if $\exists\, \theta > 0$, integer, such that $\mathfrak{m}^\theta \subset I \subset \mathfrak{m}$.*

Definition 0.5. *Let $gr_I(M) = \bigoplus_{n=0}^{\infty} \dfrac{I^n M}{I^{n+1} M}$ over $gr_I(R) = \bigoplus_{n=0}^{\infty} \dfrac{I^n}{I^{n+1}}$ where $I^n = R$ for $n = 0$. Let M be finitely generated over R. The* **Hilbert-Samuel function** *is $HS_M(n) = \ell\left(\dfrac{M}{I^{n+1}M}\right)$.*

Consider the exact sequence

$$0 \to \frac{I^n M}{I^{n+1} M} \to \frac{M}{I^{n+1}} \to \frac{M}{I^n M} \to 0.$$

The exactness implies $\ell\left(\frac{I^nM}{I^{n+1}M}\right) = \ell\left(\frac{M}{I^nM}\right) - \ell\left(\frac{M}{I^{n+1}}\right)$, which says $\mathrm{HS}_M(n) = \mathrm{Hgr}_{I(M),1}(n)$. This can be shown through the following string:

$$\mathrm{Hgr}_{I(M),1}(n) = \sum_{j\le n} \mathrm{Hgr}_{I(M)}(j) = \sum_{j\le n} \ell([\mathrm{gr}_I(M)]_j) = \sum_{j\le n} \ell\left(\frac{I^jM}{I^{j+1}M}\right) = \ell\left(\frac{M}{I^{n+1}M}\right) = \mathrm{HS}_M(n).$$

Hence by Theorem 0.1 and above considerations, the Hilbert-Samuel function of M is of polynomial type with degree d, where $d = \dim(\mathrm{gr}_I(M))$.

Let R be semilocal and Noetherian, $\mathfrak{m} = \mathrm{Jac}(R)$. Take $I \le R$ such that $\exists\, \theta > 0$ with $\mathfrak{m}^\theta \subseteq I \subseteq \mathfrak{m}$. Let M be a finitely generated R-module. We want to show $\deg HSP_{M,I} = \dim M$, where $HPS_{M,I}$ is the *Hilbert-Samuel polynomial* of M with respect to I. We know $HP_{\mathrm{gr}_I(M)}$ has degree equal to $\dim \mathrm{gr}_I(M) - 1$, and hence $HSP_{M,I}$ has degree equal to $\dim \mathrm{gr}_I(M)$. So it suffices to show $\dim M = \dim \mathrm{gr}_I(M)$, which will take some effort and is one of the main results of the lecture.

Denote the degree of $HSP_{M,I}$ by the number $\tilde{d}_M$. This does not depend on I, our ideal of definition. In fact, $m^{n\theta+1} \subseteq I^{n+1} \subseteq \mathfrak{m}^{n+1}$ for all n. So $HS_{M,\mathfrak{m}}(n) \le HS_{M,I}(n) \le HS_{M,\mathfrak{m}}(n\theta)$. This comes from the fact that the maps in

$$\frac{M}{\mathfrak{m}^{n\theta+1}} \to \frac{M}{I^{n+1}} \to \frac{M}{\mathfrak{m}^{n+1}}$$

are surjective. Thus the degree of $HSP_{M,\mathfrak{m}}$ equals that of $HSP_{M,I}$.

Proposition 0.6. *Let R, I be as above, and M', M, M'' be finitely generated R-modules such that $0 \to M' \to M \to M'' \to 0$ is exact. Then $\tilde{d}_M = \max\{\tilde{d}_{M'}, \tilde{d}_{M''}\}$, and the polynomials $HSP_{M'',I}$, $HSP_{M,I} - HSP_{M',I}$ have the same leading coefficient.*

Proof. We can think of $M'' = M/M'$. Consider $\frac{M''}{I^nM''} = \frac{M/M'}{I^n\, M/M'} = \frac{M}{I^nM + M'}$. This implies

$$0 \to \frac{I^nM + M'}{I^nM} \to \frac{M}{I^nM} \to \frac{M}{I^nM + M'} \to 0$$

is exact. Thus

$$\ell\left(\frac{M}{I^nM}\right) = \ell\left(\frac{M''}{I^nM''}\right) + \ell\left(\frac{M' + I^nM}{I^nM}\right) = \ell\left(\frac{M''}{I^nM''}\right) + \ell\left(\frac{M'}{M' \cap I^nM}\right).$$

We write $\phi(n) = \ell\left(\frac{M'}{M' \cap I^{n+1}M}\right)$, so $\phi(n)$ is of polynomial type, and $\phi(n) \ge 0$ for all $n \ge 0$. So $\tilde{d}_M = \max\{\tilde{d}_{M''}, \deg\phi\}$. It now suffices to show $\deg\phi = \tilde{d}_{M'}$. The Artin-Rees lemma implies $I^i(I^kM \cap M') = I^{k+i}M \cap M'$ for some k, and all $i \ge 0$. We will now show the claim that $I^{n+1}M' \subseteq M' \cap I^{n+1}M \subseteq I^{n-k+1}M'$ for some large n and some fixed k. From the lemma result, we get $I^{k+1}M \cap M' \subseteq I(IM \cap M') \subseteq IM'$, and $I^{k+2}M \cap M' \subseteq I^2(I^kM \cap M') \subseteq I^2M'$. This implies $I^{m+k}M \cap M' \subseteq I^mM'$ for all m. Thus $I^{n+1}M \cap M' \subseteq I^{n-k+1}M'$ for every $n \geq k-1$, and the claim is established. Therefore all the maps in

$$\frac{M'}{I^{n+1}M'} \to \frac{M'}{M' \cap I^{n+1}M} \to \frac{M'}{I^{n-k+1}M'},$$

are surjective. So $HS_{M',I}(n) \geq \phi(n) \geq HS_{M',I}(n-k)$ which implies $\tilde{d}_{M'} = \deg HSP_{M',I} = \deg \phi$, so the first part of the proposition holds. For the leading coefficients, $LC(\phi) = LC(HSP_{M',I})$, but $HSP_{M,I} - HSP_{M',I} = HSP_{M'',I}$, so the second part holds. □

Proposition 0.7. *Let $(R, \mathfrak{m})$ be local and Noetherian, and let ${}_RM$ be finitely generated. Then M is of finite length over R if and only if* $\dim M = 0$.

Proof. Note that $\dim M = 0$ if and only if $\dim R/\mathrm{Ann}_R M = 0$ if and only if $Rad(\mathrm{Ann}_R M) = \mathfrak{m}$. This is equivalent to the existence of a positive integer k such that $\mathfrak{m}^k \cdot M = 0$.

Assume that $\ell(M) = n$ and so there exists a composition series $0 = M_0 \subset M_1 \cdots \subset M_n = M$ where M_{i+1}/M_i are simple R-modules, and hence isomorphic to $R/\mathfrak{m}$. Let us show that $\mathfrak{m}^n \cdot M = 0$. We will do induction on $n \geq 1$. Clearly this is true for $n = 1$ since then M is isomorphic to $R/\mathfrak{m}$.

For $n > 1$, consider the short exact sequence

$$0 \to M_{n-1} \to M \to M/M_{n-1} \to 0,$$

where $\mathfrak{m}$ annihilates $M/M_{n-1} \simeq R/\mathfrak{m}$ and $\mathfrak{m}^{n-1} \cdot M_{n-1} = 0$ due to the induction hypothesis. Let $x \in M$, then $\mathfrak{m} \cdot x \in M_{n-1}$ and multiplying both sides by $\mathfrak{m}^{n-1}$ we get that $\mathfrak{m}^n \cdot x = 0$ and so $\mathfrak{m}^n \cdot M = 0$. Theorefore, $\dim M = 0$.

Conversely, assume $\dim M = 0$ and hence there exists $n \geq 1$ such that $\mathfrak{m}^n \cdot M = 0$. We will use again induction on n to show that M is finite length. If $n = 1$, then M is a vector space over $R/\mathfrak{m}$ which together with the hypothesis that M is Noetherian implies that M is Artinian as well.

For $n > 1$ let $N = \mathfrak{m} \cdot M$. By induction both $\ell(N), \ell(M/N) < \infty$. So the additivity of the length function on the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

implies that $\ell(M) < \infty$ and we are done. □

Theorem 0.8. *(Krull-Chevalley) Let $(R, \mathfrak{m})$ be local and Noetherian, and let ${}_RM$ be finitely generated. Let $s(M) := \min\{k \mid \exists\, x_1, ..., x_k \in \mathfrak{m}$ with $\ell(M/(x_1, ..., x_k)M) < \infty\}$. Then $s = \dim M$.*

Before we prove this result, let us make the following non-trivial observation.

Proposition 0.9. *Let $(R, \mathfrak{m})$ be local and Noetherian, and let ${}_RM$ be finitely generated. Then*

$$s(M) = s(R/\mathrm{Ann}_R M).$$

Proof. Let $s = s(M)$ and $t = s(R/\mathrm{Ann}_R M)$.

Let $x_1, \ldots, x_t$ such that $\frac{R}{(x_1,\ldots,x_t)+\mathrm{Ann_R}(M)}$ is of finite length, hence Artinian. So, there exists k such that $\mathfrak{m}^k \subseteq (x_1, \ldots, x_t) + \mathrm{Ann_R}(M) \subseteq \mathrm{Ann}_R(M/(x_1, \ldots, x_t)M)$. This shows that $\dim(M/(x_1, \ldots, x_t)M) = 0$.

So, by the above Proposition, $\ell(M/(x_1, \ldots, x_t)M < \infty$ and this gives that $t \geq s$.

Conversely, let $x_1, \ldots, x_s$ such that $\ell(M/(x_1, \ldots, x_s)M) < \infty$ or, in other words, $\dim(M/(x_1, \ldots, x_s)M = 0$. This gives that $\mathfrak{m}^k \subset \mathrm{Ann}_R(M/(x_1, \ldots, x_s)M)$, or equivalently $\mathfrak{m}^k M \subseteq (x_1, \ldots, x_s)M$.

Let $z \in \mathfrak{m}^k$. Then $zM \subseteq (x_1, \ldots, x_s)M$. The determinantal trick (as in Lemma 2.2 in Lecture 3) says that there exists $b \in (x_1, \ldots, x_s)$ such that $(z^n + b)M = 0$, where n is the number of generators of M as an R-module. This gives in particular that $z^n \in (x_1, \ldots, x_s) + \mathrm{Ann}_R(M)$. So, $\mathfrak{m}^{kn} \subseteq (x_1, \ldots, x_s) + \mathrm{Ann}_R(M)$ and then $\dim(\frac{R}{(x_1,\ldots,x_t)+\mathrm{Ann_R}(M)}) = 0$ or $\ell(\frac{R}{(x_1,\ldots,x_t)+\mathrm{Ann_R}(M)}) < \infty$

In conclusion, it follows that $s \geq t$. Putting the two parts together, $s = t$.

□

Proof. (of Krull-Chevalley) By the Proposition above combined with the definition of the Krull dimension of a module, we see that we can replace M by $R/\mathrm{Ann}_R M$.

Let $I = \mathrm{Ann}_R M$. The structure of the R-module R/I is the same as that of the R/I-module R/I. So, we can regard R/I as an R/I-module. In conclusion, the theorem reduces to the ring case. The required statement was proved for rings in earlier lectures.

□

Theorem 0.10. *Let $(R, \mathfrak{m})$ be local and Noetherian, ${}_RM$ be finitely generated, and I an $\mathfrak{m}$-primary ideal of R. Let $\tilde{d}_M = \dim gr_I(M)$. Then $\dim M = \tilde{d}_M$.*

Proof. Using the interpretation of the dimension as the degree of the Hilbert-Samuel polynomial we can replace I by $\mathfrak{m}$.

First we show $\tilde{d_M} \geq \dim M$. We proceed by induction on $\tilde{d}_M$. Assume first that $M = R$. If $\tilde{d}_R = 0$, then $gr_{\mathfrak{m}}(R)$ is Artinian which implies $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Now, by NAK, we have $\mathfrak{m}^n = 0$, which gives $\dim R = 0$. Next, let $\tilde{d_R} > 0$, and take $P_0 \subsetneq \cdots \subsetneq P_n$ a chain of prime ideals in R. Let $x \in P_1 \setminus P_0$, and let $B = \dfrac{R}{P_0 + xR}$. Note that $\dim(B) \geq n - 1$. Then the sequence

$$0 \to R/P_0 \to R/P_0 \to B \to 0$$

is exact with the first nontrivial map being multiplication by x. This gives $\tilde{d_{R/P_0}} = \max(\tilde{d_{R/P_0}}, \tilde{d_B})$, so $\tilde{d_B} \leq \tilde{d_{R/P_0}}$, but in fact we have a strict inequality by a leading coefficient argument using the previous theorem. Since R surjects onto R/P_0, we have $R/\mathfrak{m}^n$ surjecting onto $\dfrac{R}{\mathfrak{m}^n + P_0} = \dfrac{R/P_0}{\mathfrak{m}^n(R/P_0)}$, so $\tilde{d}_B < \tilde{d}_{R/P_0} \leq \tilde{d}_R$. By the induction hypothesis, we have $n - 1 \leq \dim B \leq \tilde{d}_B \leq \tilde{d}_R - 1$. This implies $\tilde{d}_R \geq n$.

For a general M the proof goes through with the help of prime filtrations and Proposition 1.6 as in the proof of the existence of the Hilbert polynomial.

Next we show $\dim M \geq \tilde{d}_M$ and proceed by induction on $\dim M$. If $\dim M = 0$, then M is Artinian which implies $\ell(M) < \infty$. Then $\ell\left(\dfrac{M}{\mathfrak{m}^n M}\right)$ are bounded by $\ell(M)$ for all n since M surjects onto $\dfrac{M}{\mathfrak{m}^n M}$, so $\tilde{d}_M = 0$. Now let $\dim M = s > 0$. By the remark $\exists\ x_1, ..., x_s \in \mathfrak{m}$ such that $\ell\left(\dfrac{M}{(x_1, ..., x_s)M}\right) < \infty$. Let $M_i = \dfrac{M}{(x_1, ..., x_i)M}$ for each $1 \leq i \leq s$. Note that $\dim M_i = \dim M - i = s - i$. By the second isomorphism theorem,

$\ell\left(\frac{M_1}{\mathfrak{m}^n M_1}\right) = \ell\left(\frac{M}{x_1 M + \mathfrak{m}^n M}\right)$, and since the sequence

$$0 \to \frac{xM + \mathfrak{m}^n M}{\mathfrak{m}^n M} \to \frac{M}{\mathfrak{m}^n M} \to \frac{M}{x_1 M + \mathfrak{m}^n M} \to 0$$

is exact, the length equation is equal to $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) - \ell\left(\frac{x_1 M}{x_1 M \cap \mathfrak{m}^n M}\right)$, and considering $(\mathfrak{m}^n M : x_1) = \{m \in M \mid x_1 m \in \mathfrak{m}^n M\}$, the equation now becomes $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) - \ell\left(\frac{M}{(\mathfrak{m}^n M : x_1)}\right)$. Consider the map $\varphi : M \to \frac{x_1 M}{x_1 M \cap \mathfrak{m}^n M}$ which acts by multiplication by x_1, and then by the appropriate modulo. We have $\ker \varphi = \{m \in M \mid \overline{x_1 m} = 0\} = \{m \mid x_1 m \in \mathfrak{m}^n M\} = (\mathfrak{m}^n M : x_1)$. But $\mathfrak{m}^{n-1} M \subseteq (\mathfrak{m}^n M : x_1)$ since $x_1 \mathfrak{m}^{n-1} M \subseteq \mathfrak{m}^n M$ because $x_1 \in \mathfrak{m}$. Thus $\frac{M}{\mathfrak{m}^{n-1} M}$ surjects onto $\frac{M}{(\mathfrak{m}^n M : x_1)}$ which implies that the length equation above is greater than or equal to $\ell\left(\frac{M}{\mathfrak{m}^n M}\right) - \ell\left(\frac{M}{\mathfrak{m}^{n-1} M}\right)$. Therefore $\mathrm{HSP}_{M_1}(n) \geq \mathrm{HSP}_M(n) = \mathrm{HSP}_M(n-1)$ (the leading terms on the right hand side cancel out), so $\tilde{d}_{M_1} \geq \tilde{d}_M - 1$. But $\dim M_1 = \dim M - 1$. Applying the induction hypothesis, $\dim M - 1 = \dim M_1 \geq \tilde{d}_{M_1} \geq \tilde{d}_M - 1$, so $\dim M \geq \tilde{d}_M$. Thus we have equality, and the theorem is established. □

Remark 0.11. *The same statement is true for R a semilocal Noetherian ring with $\mathfrak{m} = Jac(R)$, I an ideal of definition for R, and M a finitely generated R-module.*

Note that $d = \dim M$ if $I \subseteq \mathfrak{m} = \mathrm{Jac}(R)$ an ideal of definition for M. The leading coefficient of the Hilbert-Samuel polynomial is $\frac{e(\mathrm{gr}_I(M))}{d!}$.

Definition 0.12. We let $e(I, M) := e(gr_I(M))$ denote the *multiplicity of M with respect to I. The multiplicity of M* is the multiplicity of M with respect to $\mathfrak{m} = \mathrm{Jac}(R)$, but most often this is used for the case when R is local.

This Theorem allows us to work with the multiplicity of an ideal I primary to the maximal ideal in a local ring R, without necessarily mentioning the associated graded ring of R with respect to I. One can do this by thinking of the multiplicity as a certain limit, as described in the previous lecture.

Corollary 0.13. $\lim_{n\to\infty} d! \cdot \frac{\ell(M/I^n M)}{n^d}$ *exists and equals $e(I, M)$.*

Proof. Consider $\ell(\frac{M}{I^{n+1}M}) = \frac{e(\mathrm{gr}_I(M))}{d!}n^d +$ lower degree terms. Then, dividing by $(n+1)^d$ and letting $n \to \infty$, the limit exists. □

Corollary 0.14. $e(I^r, M) = e(I, M)r^d$.

Proof. Left as a simple exercise, using the Corollary above. □