LECTURE 7: GRADED RINGS AND MODULES; THE HILBERT FUNCTION

Definition 0.1. *Let R be a ring, G an abelian group (or an Abelian cancellative semi-group), and $R = \bigoplus_{i \in G} R_i$ a direct sum decomposition of abelian groups. R is* graded *(G-graded) if $R_iR_j \subseteq R_{i+j}$ for all $i, j \in G$. The easiest example is that of polynomial rings where R_i consists of all degree polynomials of degree i. Similarly, let $M = \bigoplus_{i \in G} M_i$ be an R-module. If $R_iM_j \subseteq M_{i+j}$ for all $i, j \in G$ then M is a* graded R-module. *M_i is called the i^{th} graded homogeneous component of M, and elements of M_i are called i^{th} forms.*

Example 0.2.
(1) Consider $k[x] = \bigoplus\limits_{n \in \mathbb{Z}} kx^n$ where $kx^n = 0$ if $n < 0$. Then $k[x] = \cdots 0 \oplus \cdots \oplus 0 \oplus k \oplus kx \oplus kx^2 \oplus \cdots$.
(2) Consider $k[x, y] = \bigoplus\limits_{(i,j) \in \mathbb{Z}^2} kx^iy^j$ where $kx^iy^j = 0$ if $i, j < 0$.

Remark 0.3. R_0 is a subring of R, and $R_0 \hookrightarrow R$ as a direct summand. Also, each R_i is a R_0-module because $R_0R_i \subseteq R_i$. The same is true for M_i, since $R_0M_i \subseteq M_i$.

Definition 0.4. *Let M, N be graded R-modules, and $\phi : M \to N$ where ϕ is R-linear. Then ϕ is* graded of degree d *(sometimes called* homogeneous *if $d = 0$) if $\phi(M_i) \subseteq N_{i+d}$ for all $i \in G$. Now we have a category of R-graded modules.*

For each $x \in M$, a graded modules, we can write $x = \sum x_i$, where each $x_i \neq 0$, and $x_i \in M_i$. This is a unique representation and each x_i has degree i. By convention, 0 has arbitrary degree.

There is great importance to graded modules. The grading helps to prove statements that otherwise might seem intractable. The added structure of grading is what is so powerful.

Definition 0.5. *Let $R = \bigoplus\limits_{i \in G} R_i$ be a graded ring. If $G = \mathbb{N}$ and R is generated by 1-forms (elements of degree 1) over R_0, we say R is* homogeneous or standard graded *($R = R_0[R_1]$).*

Definition 0.6. *Let R be as above. If $G = \mathbb{N}$ and the generators have positive degree, then R is called* **positively graded** *over R_0.*

Example 0.7. $R = R_0[x_1, \ldots, x_d]$ *is standard graded.*

Example 0.8. $R = \dfrac{k[x,y]}{(x^2+y^2)}$ *is standard graded.*

Example 0.9. $R = \dfrac{k[x,y]}{(x^2+y^3)}$ *with respect to the grading setting degree $x = 3$, and degree $y = 2$ is positively graded over k, but it is not homogeneous.*

Definition 0.10. *Let $M = \bigoplus_{i \in G} M_i$ be graded over $R = \bigoplus_{i \in G} R_i$. Let $\alpha \in G$. Then $M(\alpha)$ is the graded R-module with the property that $M(\alpha)_i = M_{i+\alpha}$ for all i.*

Example 0.11. *Let $R = k[x,y]$, and $\phi_x : R \to R$ acting by multiplication by x. Then ϕ_x is not a homogeneous map of R-modules because it does not preserve degrees, but the same map considered on $R(-1)$ is homogeneous.*

Proposition 0.12. *Let R be a positively graded R_0-algebra. Let $x_1, ..., x_n$ be elements of positive degree. Then $(x_1, ..., x_n) = \bigoplus_{i=1}^{\infty} R_i$ if and only if $\{x_1, ..., x_n\}$ generates R as an R_0-algebra. In particular, R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.*

Proof. For the reverse direction, let r be homogeneous in R, with $\deg r > 0$, and $r = f(x_1, ..., x_n)$, for f a polynomial. By degree reasons, $f(x_1, \ldots, x_n)$ contains only terms of degree equal to the degree of r and this shows that $r \in (x_1, \ldots, x_n)$.

For the forward direction, we proceed by induction on the degree of y, to show that every y homogeneous can be written as $y = f(x_1, \ldots, x_n)$, with f polynomial. If $\deg y = 0$, the proof is clear, so assume a positive degree. We have $y \in (x_1, ..., x_n)$, so $y = \alpha_1 x_1 + \cdots + \alpha_n x_n$, with $\alpha_i \in R$, and if $\deg x_i = d_i$, then $\alpha_i \in R_{\deg y - d_i}$. To finish, apply the induction hypothesis to the α_i's.

For the last part, the reverse implication is clear. For the forward direction, part one of the first equivalence holds, so this implies R is a finitely generated R_0-algebra. Also, $R \cong \dfrac{R}{\bigoplus_{i=1}^{\infty} R_i}$ is Noetherian since quotients of Noetherian rings are Noetherian. □

Similarly one has

Theorem 0.13. *Let R be a $\mathbb{Z}$-graded ring. Then the following are equivalent:*

(1) *Every graded ideal $I \leq R$ is finitely generated.,*
(2) *R is Noetherian,*
(3) *R_0 is Noetherian and R is finitely generated over R_0,*
(4) *R_0 is Noetherian and both $\bigoplus_{i=0}^{\infty} R_i$ and $\bigoplus_{i=0}^{\infty} R_{-i}$ are finitely generated R_0-algebras.*

1. Prime Ideals of Graded Rings

Definition 1.1. *Let R be $\mathbb{Z}$-graded and $I \leq R$. Let I^h be the ideal generated by all homogeneous elements of I. Then I^h is homogeneous and $I^h \subseteq I$.*

Proposition 1.2. *Let R be $\mathbb{Z}$-graded and M be R-graded. Then the following are true:*

(1) *For all $\mathfrak{p} \in \operatorname{Spec}(R)$, $\mathfrak{p}^h \in \operatorname{Spec}(R)$,*
(2) *If $\mathfrak{p} \in \operatorname{Supp}(M)$, then $\mathfrak{p}^h \in \operatorname{Supp}(M)$,*
(3) *If $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p}$ is graded,*
(4) *If $\operatorname{Ann}(x) = \mathfrak{p}$, then x can be taken homogeneous.*

Proof. For (1), let $ab \in \mathfrak{p}^h$, and assume $a, b \notin \mathfrak{p}^h$. Then $a = \sum a_i$, and $b = \sum b_i$. Choose $m, n \in \mathbb{Z}$ such that $a_m \notin \mathfrak{p}^h$, but $a_i \in \mathfrak{p}^h$ for every $i < m$, and $b_n \notin \mathfrak{p}^h$, but $b_j \in \mathfrak{p}^h$ for every $j < n$. Then the $(m+n)^{\text{th}}$ homogeneous component of ab is $\sum_{i+j=m+n} a_i b_j$. It is in $\mathfrak{p}^h$. Since all terms except $a_m b_n$ are in $\mathfrak{p}^h$, it implies $a_m b_n \in \mathfrak{p}^h$, so $a_m b_n \in \mathfrak{p}$, a prime ideal, so either a_m or b_n is in $\mathfrak{p}$. This says either a_m or b_n is in $\mathfrak{p}^h$, a contradiction. Before continuing with the other items, we state the following immediate consequence: a minimal prime $\mathfrak{p}$ must be graded, because $\mathfrak{p}^h \subseteq \mathfrak{p}$ and hence $\mathfrak{p}^h = \mathfrak{p}$, if $\mathfrak{p}$ is minimal.

For (2), if $\mathfrak{p} \in \operatorname{Supp}(M)$, then $M_{\mathfrak{p}} \neq 0$. Assume $M_{\mathfrak{p}^h} = 0$, so if x is a homogeneous elements, $\frac{x}{1} = 0$ in $M_{\mathfrak{p}^h}$. This implies $\exists\, a \in R \setminus \mathfrak{p}^h$ such that $ax = 0$. If $a = \sum a_i$ then $a_i x_i = 0$ for every i. But $a \notin \mathfrak{p}^h$, so $\exists\, i$ such that $a_i \notin \mathfrak{p}$. So $\frac{x}{1} = 0$ in $M_{\mathfrak{p}}$ which implies $M_{\mathfrak{p}} = 0$, a contradiction, so $M_{\mathfrak{p}^h} \neq 0$.

For (3), let $\mathfrak{p} = \operatorname{Ann}(x)$, and let $a \in \mathfrak{p}$, so $ax = 0$. Let $x = x_m + \cdots + x_n$, and $a = a_s + \cdots + a_t$, so $\sum_{i+j=r} a_i x_j = 0$ for $r = m+s, ..., n+t$. Note that $a_s x_m = 0$, and $a_s x_{m+1} + a_{s+1} x_m = 0$, so $a_s^2 x_{m+1} = 0$. Repeat this process to show that $a_s^i x_{m+i-1} = 0$ for all i. This implies that if $m+i-1=n$, then $i=n-m+1$, so $a_s^{n-m+1}x_n=0 \Rightarrow a_s^{n-m+1}x=0 \Rightarrow a_s^{n-m+1}\in\mathfrak{p}$, and thus $a_s\in\mathfrak{p}$, which gives $(a_{s+1}+\cdots+a_t)x=0$. Repeat the whole argument to show that all homogeneous components of a are in $\mathfrak{p}$, which implies $\mathfrak{p}=\mathfrak{p}^h$, and so $\mathfrak{p}$ is graded.

For (4), let $\mathfrak{p}=\mathrm{Ann}(x)$, and let $x=\sum x_i$. Since $\mathfrak{p}x=0$, $\mathfrak{p}$ is graded, so $\mathfrak{p}x_i=0$ for all i. Take $A_i=\mathrm{Ann}(x_i)\supseteq\mathfrak{p}$. But $\bigcap_i A_i\subseteq\mathfrak{p}$, so $\prod_{\text{finite}} A_i\subseteq\mathfrak{p}$. This implies $\exists\, i$ such that $A_i\subseteq\mathfrak{p}$ since $\mathfrak{p}$ is prime. Therefore $\mathfrak{p}=A_i=\mathrm{Ann}(x_i)$. □

Corollary 1.3. *All minimal primes are graded.*

Definition 1.4. *Let $\mathfrak{p}\in\mathrm{Spec}(R)$. Let S be the set of homogeneous elements not in $\mathfrak{p}$. Then S is a multiplicative set. Let M be a graded R-module. By definition, $M_{(\mathfrak{p})}=S^{-1}M$ which is the homogeneous localization of M at $\mathfrak{p}$. If x is homogeneous and $\frac{x}{a}\in M_{(\mathfrak{p})}$ then we define $\deg(\frac{x}{a})=\deg x-\deg a$. The ith component of $M_{(\mathfrak{p})}$ is $(M_{\mathfrak{p}})_i=\{\frac{x}{a}\in M_{(\mathfrak{p})}\mid a\notin\mathfrak{p},\ x,a$ homogeneous, $\deg(\frac{x}{a})=i\}$. Then $M_{(\mathfrak{p})}$ is a graded $R_{(\mathfrak{p})}$-module.*

Now we will list some facts without proof which is left as an exercise.

(1) $\mathfrak{p}^hR_{(\mathfrak{p})}$ is a graded prime ideal of $R_{(\mathfrak{p})}$.
(2) $R_{(\mathfrak{p})}/\mathfrak{p}^hR_{(\mathfrak{p})}$ has the following property: every nonzero homogeneous element is invertible.

Let R be $\mathbb{Z}$-graded. Then $R=\bigoplus_{n\in\mathbb{N}}R_n$. Assume R_0 is Artinian and $M=\bigoplus_{n\in\mathbb{N}}M_n$, a graded R-module with the property that M_n is a finitely generated R_0-module for every n.

Definition 1.5. *We define $H_M(-):\mathbb{Z}\to\mathbb{Z}$ to act as $H_M(n)=\ell_{R_0}(M_n)<\infty$, and it is called the Hilbert function of M. The Hilbert series is $\sum_{n\in\mathbb{Z}}H_M(n)t^n=HS(t)$.*

Lemma 1.6. *Let R_0 be Artinian, and $R=R_0[x_1,...,x_d]$. Then $H_R(n)=\binom{n+d-1}{d-1}\ell(R_0)$.*

Proof. We know $H_R(n)=\ell_{R_0}(R_n)$ and R_n is R_0-free, generated by all monomials of degree n. Now, proceed by induction on d. If $d=1$, then $\binom{n}{0}=1=d$. For $d>1$, consider the exact sequence

$$0\to R_0[x_1,...,x_d](-1)\to R_0[x_1,...,x_d]\to\frac{R_0[x_1,...,x_d]}{(x_d)}\cong R_0[x_1,...,x_{d-1}]\to 0,$$

where the first nontrivial map is multiplication by x_d. So $H_R(n) = H_{R(-1)}(n)+H_{R/x_dR}(n)$. Fix d and do induction on n (the $n = 1$ case is clear), to obtain $H_R(n) = H_R(n-1) + H_{R/x_dR}(n) = \binom{n+d-1}{d-1}\ell(R_0)$. □

Proposition 1.7. *Let $F(x_1, ..., x_r)$ be a homogeneous polynomial of degree s over a field k with $r \geq 2$. Let $R' = \dfrac{k[x_1, ..., x_r]}{(F(x))}$, and $R = k[x_1, ..., x_n]$. Then $\ell(R'_n) = \binom{n+r-1}{r-1} - \binom{n-s+r-1}{r-1}$. Therefore, $HP_{R'}$ is a polynomial of degree $r-2$ and leading coefficient $\frac{s}{(r-2)!}$.*

Proof. For the proof, consider the exact sequence

$$0 \to R(-s) \to R \to R/(F) \to 0,$$

where the first nontrivial map is the multiplication by F and apply the preceding result. □

The Proposition shows that the Hilbert polynomial of the graded R-module R' is

$$P_{R'} = \frac{s}{(r-2)!}x^{r-2} + \text{lower degree terms}$$

so the leading coefficient is equal to the degree of $F(x)$ divided by $(r-2)!$. In the next lecture, we will see that this implies that the multiplicity of F is $deg(F)$.

Theorem 1.8. *(Hilbert series) Let R be a Noetherian positively graded ring over an Artinian ring R_0. Let M be a finitely generated graded module over R. Suppose that $R = R_0[x_1, \ldots, x_r]$ with x_i homogeneous of degree d_i. Then there exists a integer polynomial $f(t)$ such that*

$$HS_M(t) = \frac{f(t)}{\prod_{i=1}^{n}(1-t^{d_i})}.$$

Proof. We will prove this by induction on r.

Let $r = 0$, then R is Artinian and hence M is finite length, therefore $l(M_n) = 0$ for $n \gg 0$, which means that $HS(t)$ is a polynomial.

Let us assume that $r > 0$. Consider the following exact sequence:

$$0 \to K_n \to M_n \xrightarrow{\cdot x_r} M_{n+d_r} \to L_n \to 0,$$

where K_n is the kernel of the map defined by multiplication by x_r, while $L_{n+d_r} = \frac{M_{n+d_r}}{x_rM_n}$.

The modules $K = \oplus_n K_n$ and $L = \oplus_n L_n$ are graded R-modules, both killed by x_r and so the induction hypothesis appplies to each of them, because they are modules over $R/x_rR = R_0[x_1, \ldots, x_{r-1}]$.

Applying the length function to the exact sequence above we get

$$l(K_n) + l(M_{n+d_r}) = l(M_n) + l(L_{n+d_r}).$$

Multiply both sides by t^{n+d_r} and add to get

$$HS_K(t)t^{d_r} + HS_M(t) - t^{d_r}HS_M(t) - HS_L(t) = P(t),$$

where $P(t)$ is a polynomial with integer coefficients.

Applying the induction hypothesis and solving for $HS_M(t)$ leads to the desired statement.

□

Corollary 1.9. *Assume further that R is standard graded (i.e. $d_i = 1$, for all $i = 1, \ldots, n$). Then there exists a rational polynomial ϕ such that $H_M(n) = \phi(n)$ for all $n \gg 0$.*

Proof. We know that

$$HS_M(t) = \frac{f(t)}{\prod_{i=1}^n (1 - t^{d_i})},$$

and since $d_i = 1$ we can simplify the rational function such that $HS_M(t) = g(t)/(1-t)^d$, with $d \neq 0$ and such that $g(1) \neq 0$ if $d > 0$.

Write

$$(1-t)^d = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n.$$

Therefore, after using $g(t) = a_0 + a_1 t + \ldots + a_s t^s$, we can get a expanded presentation for HS_M which leads to

$$H_M(n) = a_0 \binom{d+n-1}{d-1} + \ldots + a_s \binom{d+n-s-1}{d-1},$$

where we make the convention that $\binom{m}{d-1} = 0$ if $m < d-1$.

□

Definition 1.10. *Let R be a standard graded ring. For a finitely generated R-module M, we let HP_M denote the polynomial such that $HP_M(n) = l(M_n)$ for $n \gg 0$. This polynomial exists by the previous Corollary. We will revisit this notion.*

Remark 1.11. From the previous proof we can identify the value of n_0 such that $H_m(n) = \phi(n)$ for all $n \geq n_0$. This value is $n \geq s+1-d$, where we keep the notations from the proof. Moreover the leading coefficient of ϕ is $g(1)/(d-1)!$ where $d-1$ is the degree of ϕ. In the next lecture we will give a precise description for d.