Lecture 3: Krull's Theorems

1. Krull's Principal Ideal Theorem

Lemma 1.1. *Let R be a Noetherian ring and P a prime ideal. For $n \in \mathbb{N}$, let $P^{(n)} = P^n R_P \cap R$. Then*

$$P^{(n)} R_P = (PR_P)^n.$$

Proof. Clearly, $P^{(n)} \subseteq P^n R_P$, so $P^{(n)} R_P \subseteq P^n R_P = (PR_P)^n$.

Conversely, let $a/s \in P^n R_P = (PR_P)^n$, with $a \in P^n$, $s \notin P$. But then $a \in P^{(n)}$ and so $a/s = a \cdot 1/s$ belongs to $P^{(n)} R_P$.

□

Theorem 1.2 (Krull's PIT)**.** *A minimal prime Q over a principal ideal xR in a Noetherian ring R has height less or equal to* 1*.*

Proof. Assume there is a chain $P_0 \subset P \subset Q$ a chain of prime ideals with $xR \subseteq Q$ minimal. Consider R_Q, a local ring, and QR_Q its maximal ideal, and all relevant data is preserved. We can assume $(R, \mathfrak{m})$ is local, Noetherian, $xR \subseteq \mathfrak{m}$ is minimal, and $P_0 \subset P \subset \mathfrak{m}$. Replace R by R/P_0. Then R/P_0 is local, and since $P_0 \subset P \subset \mathfrak{m}$, we have $0 \subset P/P_0 \subset \mathfrak{m}/P_0$, and $\overline{x} \subseteq \mathfrak{m}/P_0$. Also, $(R, \mathfrak{m})$ is a domain, with $xR \subseteq \mathfrak{m}$, minimal, and $0 \subset P \subset \mathfrak{m}$. If we can show $P = 0$, a contradiction, we are done. We know $P^{(t)} = P^t R_P \cap R \leq R$, and $P^{(t)}$ are P-primary for every t. Now look at R/xR. Since $\mathfrak{m}/xR$ is the only prime ideal in R/xR we conclude that R/xR is Artinian.

Next consider $\overline{P^{(t)}}$ in R/xR, a descending chain of ideals, so $\exists\, n$ such that $\overline{P^{(t)}} = \overline{P^{(t+1)}}$ for all $t \geq n$, because R/xR is Artinian. Then $P^{(t)} + xR = P^{(t+1)} + xR$. For all $v \in P^{(t)}$, there is $w \in P^{(t+1)}$ such that $v = w + xr$. So $v - ww = xr \in P^{(t)}$. But $x \notin P = \mathrm{Rad}(P^{(t)})$, so $r \in P^{(t)}$ by the definition of a primary ideal, which implies $P^{(t)} \subseteq P^{(t+1)} + xP^{(t)}$. Therefore,

$$\frac{P^{(t)}}{P^{(t+1)}} = x\frac{P^{(t)}}{P^{(t+1)}}.$$

By NAK, we have $P^{(t)} = P^{(t+1)} = P^{(n)}$ for all $t \geq n$. Let $J = \bigcap P^{(t)} = P^{(n)}$. Remember that R is a domain, so $R \to R_P$ is injective. Hence $J = \bigcap P^{(t)} \subseteq \bigcap P^{(t)} R_P$. Then in R_P,

$P^{(t)}R_P = (PR_P)^t$ (by the above lemma), and so $J \subseteq \bigcap (PR_P)^t = 0$, so $0 = J = P^{(n)}$, which implies $P^n \subseteq P^{(n)} = 0$, and thus $P = 0$. □

2. Krull's height theorem

Theorem 2.1 (HT)**.** *Let R be a noetherian ring. Let I be an ideal with n generators. Let P be a prime ideal of R such that P is minimal among the prime ideals containg I. Then* $\mathrm{ht}(P) \leq n$*.*

Proof. Let $I = (x_1, \ldots, x_n)$. Localize at P and note that therefore one can assume that R is local with maximal ideal equal to P.Since P is a minimal prime over I and also a maximal ideal, we see that $P = Rad(I)$.

Let $P_0 \subset P_1 \subset \ldots P_k = Q \subset P$ a chain of distinct prime ideals. We can refine this chain and due to the Noetherian hypothesis on R, we can assume that Q is such that the chain $Q \subsetneq P$ is saturated. Let us show that any such ideal Q is a prime minimal over an ideal generated by $n-1$ elements. By inductive hypothesis, we get that $\mathrm{ht}(Q) \leq n-1$ and so $\mathrm{ht}(P) \leq n$.

By hypothesis Q cannot contain all x_i and so let us say $x_1 \notin Q$. Then P is minimal over $Q + x_1R$, so $Rad(Q + x_1R) = P$. Hence for all $i \geq 2$, there exist $n_i \in \mathbb{N}$, $r_i \in R$ and $y_i \in Q$ such that

$$x_i^{n_i} = r_i x_1 + y_i.$$

We can note that every prime contains $(x_1, y_2, \ldots, y_n)$ if and only if it contains $Rad(x_1, x_2, \ldots, x_n)$, and so $Rad(x_1, y_2, \ldots, y_n) = Rad(x_1, x_2, \ldots, x_n) = P$. Therefore the image of P in $R/(y_2, \ldots, y_n)$ is minimal over the class of x_1 and therefor by PIT it has height at most 1. So, the image of Q must be a minimal prime. Lifting back to R we get that Q is minimal over $(y_2, ..., y_n)$, an $n-1$ generated ideal.

□

Since any chain of prime ideals in a Noetherian ring descending from a prime P is bounded by above by its minimal number of generators we conclude the following consequence.

Corollary 2.2. *In a Noetherian ring the prime ideals satisfy the (DCC) condition.*

The following remark follows easily from HT as well.

Remark 2.3. *The ideal* $(x_1, \ldots, x_n)$ *has height equal to* n *in* $R = k[x_1, \ldots, x_n]$, k *field.*

We have the following converse of Krull's height theorem:

Theorem 2.4. *Let* R *be a Noetherian ring. Any prime ideal of height* n *is minimal over an ideal generated by* n *elements.*

Proof. Consider a chain of length n of prime ideals descending from P:

$$P = P_n \supset P_{n-1} \supset \cdots \supset P_1 \supset P_0.$$

We need to find $x_1, \ldots, x_n$ such that P is minimal over $(x_, \ldots, x_n)$.

We claim that for all $1 \leq i \leq n$ we can find $x_1, \ldots, x_i$ in P_i such that $\mathrm{ht}(x_1, \ldots, x_i) = i$ and P_i is minimal over $(x_1, \ldots, x_i)$.

First note that $\mathrm{ht}(P_i) = i$. Let us consider the case $i = 1$.

Let $x \in P_1$ but not in any of the minimal primes of R (note that P_0 is one of them and that P_1 is not contained in the union of the minimal primes of R by the Prime Avoidance Lemma). Then P_1 is minimal over xR, otherwise there exists another prime, say Q, minimal over x and strictly contained in P_1. But Q cannot be minimal in R because it contains x, therefore there exists a minimal prime, Q', strictly contained in Q, which gives $\mathrm{ht}(P_1) \geq 2$, false.

Assume that we have chose $x_1, \ldots, x_i$ in P_i such that $\mathrm{ht}(x_1, \ldots, x_i) = i$ and P_i is minimal over $(x_1, \ldots, x_i)$.

By the Prime Avoidance Lemma we know that $P_{i+1} \not\subseteq \cup Q$, where Q runs over the finite set of minimal primes of $(x_1, \ldots, x_i)$, otherwise by Krull's height theorem, $\mathrm{ht}(P_{i+1}) \leq i$. We can choose an element $x_{i+1} \in P_{i+1}$ but not in any minimal prime of $x_1, \ldots, x_i$.

Let us consider P' a minimal prime of R containing $x_1, \ldots, x_{i+1}$. Clearly, $\mathrm{ht}(P') \leq i+1$ by Krull's height theorem. But P' contains $x_1, \ldots, x_i$ and so it contains one of the minimal primes of $(x_1, \ldots, x_i)$, which by hypothesis has height i. Since P' is not a minimal prime of $(x_1, \ldots, x_i)$ because it contains x_{i+1} we conclude that $\mathrm{ht}(P') > i$. In conclusion,

$\mathrm{ht}(P') = i+1$. So, we get that $\mathrm{ht}(x_1, \ldots, x_{i+1}) = i+1$, and since $(x_1, \ldots, x_{i+1}) \subset P_{i+1})$ with $\mathrm{ht}(P_{i+1}) = i+1$, it follows that P_{i+1} is minimal over $(x_1, \ldots, x_{i+1})$ and therefore it completes our induction step. □

Let $(R, \mathfrak{m})$ be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Since $\dim(R) = \mathrm{ht}(\mathfrak{m})$, we know that $\dim(R)$ is finite. Say $\dim(R) = n$. Then there exists a sequence of elements $x_1, \ldots, x_n$ such that $\mathfrak{m}$ is minimal over $(x_1, \ldots, x_n)$. This condition is equivalent to $\mathfrak{m} = Rad(x_1, \ldots, x_n)$. We will call $x_1, \ldots, x_n$ *a sequence of parameters* for R.

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and I an ideal of R.*

Then $Rad(I) = \mathfrak{m}$ if and only if $\mathfrak{m}$ is minimal over I if and only if I is $\mathfrak{m}$-primary.

Proof. Recall that $\mathfrak{m}$ is the only maximal ideal of R.

Since the radical of an ideal equals the intersection of all minimal primes containing it, we see that the first equivalence is trivial.

For the second one, note that I $\mathfrak{m}$-primary means that $Rad(I) = \mathfrak{m}$ since any ideal that has its radical equal to a maximal ideal is primary.

□

Proposition 2.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Then*

$$\dim(R) = min\{n : \text{there exists } x_1, \ldots, x_n \in R \text{ with } \mathfrak{m} \text{ minimal over } (x_1, \ldots, x_n)\}$$

Proof. Let n_0 equal the right hand side of the equality to be proven.

We have seen that $\dim(R) \geq n_0$.

Assume that the inequality is strict. Then there exists a sequence of elements $x_1, \ldots, x_n$ such that $\mathfrak{m}$ is minimal over $(x_1, \ldots, x_n)$ and $n < \dim(R)$. But $\mathfrak{m}$ minimal over $(x_1, \ldots, x_n)$ implies, by the Krull Height Theorem, that $\mathrm{ht}(\mathfrak{m}) \leq n < \dim(R) = \mathrm{ht}(\mathfrak{m})$, which is impossible.

□

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, Then* $\dim(R)$ *equals the smallest number n such that there exist $x_1, \ldots, x_n \in \mathfrak{m}$ such that* $\dim(R/(x_1, \ldots, x_n)) = 0$.

Proposition 2.8. *Let P be a prime ideal of height n in a Notherian ring R. Then*

(1) *If $x_1, \dots, x_k$ in P, then* $\mathrm{ht}(P/(x_1, \dots, x_k)) \geq n - k$.
(2) *Assume that P is a minimal prime over $(x_1, \dots, x_n)$. Then*

$$\mathrm{ht}(P/(x_1, \dots, x_i) = n - i,$$

for all $i = 1, \dots, n$.

Proof. (1) Let $h = \mathrm{ht}(P/(x_1, \dots, x_k))$. Then we can find $a_1, \dots, a_h$ such that P is minimal over $(x_1, \dots, x_k, a_1, \dots, a_k$. But then $\mathrm{ht}(P) \leq k + h$. So, $n \leq k + h$.
(2) The ideal $P/(x_1, \dots, x_i)$ is minimal prime over the classes of $x_{i+1}, \dots, x_n$ in $R/(x_1, \dots, x_i)$. So, $\mathrm{ht}(P/(x_1, \dots, x_i)) \leq n - i$. The reverse inequality follows from the first part.

□

Proposition 2.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, with $x_1, \dots, x_d$ system of parameters. Then*

(1)
$$\dim(R/(x_1, \dots, x_i)) = d - i,$$
for all $1 \leq i \leq d$.
(2) *There exists a system of parameters in R such that for every subset of indices $I \subseteq \{1, \dots, d\}$, the ideal $(x_i : i \in I)$ has height equal to the cardinality of I.*

Proof. The first part follows from Proposition 2.8. For $d = 0, 1$ the second assertion is trivial. Assume that $d > 1$. Choose $x_1 \in \mathfrak{m}$ not in any minimal prime of R. Choose $x_2 \in \mathfrak{m}$ not in any minimal prime of x_1R and not in any minimal prime of R. Clearly, $\mathrm{ht}(x_2) = 1, \mathrm{ht}(x_1, x_2) = 2$. If $d > 2$, Continue by choosing $x_3 \in \mathfrak{m}$, not in any minimal prime of $R, (x_1), (x_2), (x_1, x_2)$. Proceed like this and the result follows. □