ENESCU MATH 8250-LECTURE 1

1. Flatness and completion

Let A be a commutative ring, with identity. A sequence of the form

$$\cdots \to M_i \overset{\phi_i}{\to} M_{i-1} \overset{\phi_{i-1}}{\to} M_{i-2} \to \cdots$$

is called *exact* if ϕ_i is A-linear and $Ker(\phi_{i-1}) = Im(\phi_i)$, for all i.

Let M be an A-module. We say that M is A-*flat* if the functor $- \otimes_A M$ is exact. That is, whenever the sequence of A-modules $E \to F \to G$ is exact, then the sequence $E \otimes_A M \to F \otimes_A M \to G \otimes_A M$ is also exact.

Similarly, we say that M is A-*faithfully flat* if M is A-flat and, whenever $E \otimes_A M \to F \otimes_A M \to G \otimes_A M$ is exact, then A-modules $E \to F \to G$ is exact.

Remark 1.1. *Let S be a mulplicatively closed set in A and ideal I of A. We have the following natural homomorphisms:*

(1)
$$S^{-1}A \otimes_A M \simeq S^{-1}M.$$

(2)
$$A/I \otimes_A M \simeq M/IM.$$

(3)
$$A^{(I)} \otimes_A M \simeq M^{(I)},$$

where $A^{(I)}$ is the free A-module of basis of cardinality I, where I is an arbitrary set.

For an A-algebra B, we say that B is a flat A-algebra if it is flat as an A-module. A ring homomorphism $f : A \to B$ is called a flat homomorphism if B is an A-flat algebra.

We note that $S^{-1}A$, with S a multiplicatively closed set, and $A[x_1, \dots, x_n]$ are A-flat algebras. In fact any nonzero free A-module, including polynomial rings over A, are faithfully flat.

The following fact, called the right exactness of the tensor product, is well known:

Remark 1.2. *If $E \to F \to G \to 0$ is an exact sequence, then $E \otimes_A M \to F \otimes_A M \to G \otimes_A M \to 0$ is also exact.*

Note that this implies that if $E \stackrel{\phi}{\to} F \to 0$ is exact (i.e. ϕ is surjective) then $E \otimes_A M \to F \otimes_A M \to 0$ is exact.

Lemma 1.3. *Let M be an A-module. Then M is A-flat if and only if whenever $0 \to E \to F$ is an exact sequence, then $0 \to E \otimes_A M \to F \otimes_A M$ is exact.*

Proof. Let $E \stackrel{\phi}{\to} F \stackrel{\psi}{\to} G$ be an exact sequence of A-modules. So, $Ker(\psi) = Im(\phi)$ and we have two exact sequence $E \stackrel{\phi}{\to} Im(\phi) \to 0$ and $0 \to Ker(\psi) = Im(\phi) \to F$.

In addition we also have that

$$E \stackrel{\phi}{\to} F \to F/Im(\phi) \to 0$$

is exact and so

$$E \otimes_A M \stackrel{\phi\otimes 1}{\to} F \otimes_A M \to F/Im(\phi) \otimes_A M \to 0,$$

which gives $Im(\phi \otimes 1) = Ker(F \otimes_A M \to F/Im(\phi) \otimes_A M)$ which equals $Im(f) \otimes M$ from the exact sequence $0 \to Im(\phi) \to F$ and the fact that tensoring with M preserves injectivity.

By the right exactness of the tensor product we see that $E \otimes_A M \stackrel{\phi\otimes 1}{\to} Im(\phi) \otimes_A M \to 0$ is exact, and so $Im(\phi \otimes 1) = Im(\phi) \otimes_A M$.

By the hypothesis, $0 \to Ker(\psi) \otimes_A M \stackrel{\psi\otimes 1}{\to} F \otimes_A M$ is also exact and so $Ker(\psi \otimes 1) = Ker(\psi) \otimes_A M$.

Since $Ker(\psi) = Im(\phi)$, it follows $Ker(\psi \otimes 1) = Ker(\psi) \otimes_A M = Im(\phi) \otimes_A M = Im(\phi \otimes 1)$, which shows that $E \otimes_A M \to F \otimes_A M \to G \otimes_A M$ is exact.

□

The following two propositions list some simple consequences of flatness and faithful flatness which can be proven easily by manipulating the definitions.

Proposition 1.4. *Let B be an A-algebra and M a B-module. Then*

(1) *B is A-flat (respectivey A-faithfully flat) and M is B-flat (respectively B-faithfully flat) then M is A-flat (respectively A-faithfully flat);*
(2) *M is B-faithfully flat and M is A-flat (respectively A-faithfully flat) then B is A-flat (respectively A-faithfully flat) .*

Proposition 1.5. *Let B be an A-algebra and M an A-module.*

Then M A-flat (respectively A-faithfully flat) implies $M \otimes_A B$ is B-flat (respectively B-faithfully flat).

The first important result on flatness shows that checking flatness can be performed locally.

Theorem 1.6. *Let $f : A \to B$ be a ring homomorphism and M a B-module. Then M is A-flat if and only if for every prime ideal P of B , M_P is flat over A_p where $p = P \cap A$ (or the same condition for every maximal ideal P of B).*

Proof. Note that if N is an $S^{-1}A$-module, where S is a multiplicatively closed set in A, then $S^{-1}A \otimes_A N \simeq S^{-1}N = N$.

The map $A \to B$ naturally gives the homomorphism $A_p \to B_P$, and M_P is naturally an A_p-module.

Note that $N \otimes_{A_p} M_P = N \otimes_{A_p} A_p \otimes_A M_P = N \otimes_A M_P = N \otimes_A M \otimes_B B_P$. From this we see immediately that M flat over A together with the observation that B_P is B-flat implies that M_P is flat over A_p.

For the converse, let $0 \to E \to F$ be an injective map and consider $0 \to K \to E \otimes_A M \to F \otimes_A M$. The plan is to show that $K = 0$. Localizing at P, we get an exact sequence $0 \to K_P \to E \otimes_A M \otimes_B B_P = E \otimes_A M_P \to F \otimes_A M \otimes_B B_P = F \otimes_A M_P$.

But note that $E \otimes_A M_P = E \otimes_A A_p \otimes_{A_p} M_P = E_p \otimes_{A_p} M_P$ (and similarly for F).

So, $0 \to K_P \to E_p \otimes_{A_p} M_P \to F_p \otimes_{A_p} M_P$ is exact.

But $0 \to E_p \to F_p$ is injective for any prime ideal p in A. When tensoring with M_P over A_p, exactness is preserved. So, $K_P = 0$. This condition for all maximal (respectively prime) ideals in B implies that $K = 0$.

□

Theorem 1.7. *Let A be a ring and M an A-module. the following assertions are equivalent:*

(1) *M is faithfully flat over A;*
(2) *M is flat over A and $N \otimes M \neq 0$ for all nonzero A-modules N.*
(3) *M is flat over A and $M \neq \mathfrak{m}M$ for all maximal ideal $\mathfrak{m}$ of A.*

Proof. (1) implies (3) Let $0 \to A/\mathfrak{m} \to 0$. Assume that $\mathfrak{m}M = M$. This implies that $0 \to A/\mathfrak{m} \otimes M \to 0$ is exact so the original sequence must be exact. Hence $A = \mathfrak{m}$ which is a contradiction.

(3) implies (2)

Let $0 \neq x \in N$ and consider a maximal ideal $\mathfrak{m}$ containing $Ann(x)$. But then $Ax \simeq A/Ann(x)$ and then $Ax \otimes M = A/Ann(x) \otimes M = M/Ann(x)M$ which is nonzero because $Ann(x)M \subset \mathfrak{m}M \neq M$. Now, we have an R-linear map which is injective $0 \to Ax \to N$ and by flatness of M we get that $Ax \otimes M$ injects into $N \otimes M$, so the latter is nonzero as well.

(2) implies (1)

Let f be an A-linear map between two modules $E \to F$. We claim that $Ker(f) \otimes M = Ker(f \otimes 1)$ and $Im(f) \otimes M = Im(f \otimes 1)$. Indeed, $Ker(f) \to E \to F$ and $E \to F \to F/Im(f)$ are exact, so they remain exact after tensoring with M.

Consider a sequence of A-modules $N' \to N \to N''$ such that $N' \otimes M \overset{f\otimes 1}{\to} N \otimes M \overset{g\otimes 1}{\to} N'' \otimes M$ is exact.

So $(g \circ f) \otimes 1 = 0$ hence $g \circ f = 0$. In conclusion $Im(f) \subset Ker(g)$.

Consider now $H = Ker(g)/Im(f)$. Then by flatness $H \otimes M = (Ker(g) \otimes M)/(Im(f) \otimes M) = 0$. Therefore $H = 0$.

□

Corollary 1.8. *Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ a local homomorphism of rings (that is f is a ring homomorphism and $f(\mathfrak{m}) \subset \mathfrak{n}$). Then B is A-flat if and only if B is A-faithfully flat.*

Proof. Since $f(\mathfrak{m}) \subset \mathfrak{n}$ we get that $\mathfrak{m}B \subset \mathfrak{n} \neq B$. □

The following observations are useful in practice which can be obtained straighforwardly from the flatness definition.

Lemma 1.9. *Let M be a flat A-module. Let I be an ideal of A and $E \subset F$ A-modules. Then $I \otimes_A M \simeq IM$ and $(F/E) \otimes_A M \simeq F \otimes_A M/E \otimes_A M$.*

Proof. Use the short exact sequences $0 \to I \to A \to A/I \to 0$ and $0 \to E \to F \to F/E \to 0$. □

Proposition 1.10.

(1) *Let A be a ring and M an A-flat module. Let N_1, N_2 be two submodules of M. Then*

$$(N_1 \cap N_s) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M),$$

where the objects are regarded as submodules of $N \otimes_A M$.

(2) *Therefore, if $A \to B$ is flat then for any ideals I, J of A, we have $(I \cap J)B = IB \cap JB$. If J is finitely generated, then $(I : J)B = (IB : JB)$.*

(3) *If $f : A \to B$ is faithfully flat, then for any A-module M the natural map $M \to M \otimes_A B$ is injective. In particular f is injective. In particular, for any ideal $I \subset A$, $IB \cap A = I$.*

Proof. For (1), consider the exact sequence of A-modules $0 \to N_1 \cap N_2 \to N \to N/N_1 \oplus N/N_2$, and tensor with M. The resulting exact sequence gives the statement after using Lemma 1.9.

For (2), let $N = A$, $N_1 = I, N_2 = J$, and $M = B$ and use Lemma 1.9. For the second part, let $J = (a_1, \dots, a_k)$. But then $I : J = \cap_{i=1}^k (I : Aa_i)$.

Fix i, and let $0 \to (I : Aa_i) \to A \stackrel{\cdot a_i}{\to} A/I$ which is exact. Since B is A-flat we get that the sequence stays exact after tensoring with B. This gives us $0 \to (I : Aa_i)B \to B \stackrel{\cdot a_i}{\to} B/IB$. Therefore, $(I : Aa_i)B = (IB : Ba_i)$ by computing the kernels in two ways.

Therefore, $(I:J)B = (\cap_{i=1}^k (I:Aa_i))B$ which equals $(\cap_{i=1}^k (I:Aa_i)B)$ by the first part of (2). But this last term equals $\cap_{i=1}^k (IB:Ba_i) = IB:JB$.

Finally, let $m \in M$ such that $m \otimes 1 = 0$ in $M \otimes_A B$. We need $m = 0$, so let us assume that $m \neq 0$. But then $0 \neq Am$ and therefore, since B is A-faithfully flat, we get that $0 \neq Am \otimes_A B$. On the hand $m \otimes 1 = 0$ so $Am \otimes_A B = 0$ as well. Contradiction. The final statement is obtained by letting $M = B$.

□

Lemma 1.11. *Let $i : E \to F$ be an injective A-linear map. Let M be an A-module and consider $u \in ker(1_M \otimes i) \subset E \otimes_A M$, where $1_M \otimes i : E \otimes_A M \to F \otimes_A M$. Then there exists N finitely generated submodule of M and $v \in ker(1_N \otimes i)$ such that v maps to u under the canonical map $E \otimes N \to E \otimes M$.*

Proposition 1.12. *A module M is flat over A if all its finitely generated submodules are flat over A.*

Proof. This is a straightforward application of the Lemma. If there exists an R-linear injection $i : E \to F$ and for any element $u \in Ker(i \otimes_A 1_M)$, we can find a finitely generated submodule N of M and $v \in ker(i \otimes_A 1_N)$ such that v maps onto u under the canonical map. But N is flat so $v = 0$ which gives $u = 0$. □

Proposition 1.13. *Let A be a domain. Then every flat A-module is torsion free. The converse holds, if A is a PID.*

Proof. Let $a \neq 0$ in A. Then multiplication by a is injective on A ($0 \to A \overset{a}{\to} A$ is exact), hence it is stays injective after tensoring with M: $0 \to M \overset{a}{\to} M$ is exact, so M is torsion-free.

For the converse, by Proposition 1.12 it is sufficient to show that any finitely generated submodule N of M is flat. But such an N is torsion-free, so by the structure theorem of finitely generated modules over a PID, N must be free hence flat. □

The following theorems also hold, but their proofs are more difficult so we will not include them.

Theorem 1.14. *Let A be a ring and M be an A-module. Then M is A-flat if and only if for all finitely generated modules $E \subset F$ the map $E \otimes M \to F \otimes M$ is injective.*

Theorem 1.15. *Let A be a ring and M a module over A. Then M is A-flat if and only if for any finitely generated ideal I of A we have $I \otimes_A M \to IM$ is injective, and therefore bijective.*

Theorem 1.16. *Let*

$$0 \to N \to M \to P \to 0$$

be a short exact sequence of A-modules. If N, P are A-flat then M is A-flat as well.

Theorem 1.17. *Let M be a finitely generated A-module, where $(A, \mathfrak{m})$ is a local Noetherian ring. Then M is flat if and only if M is projective if and only if M is free.*

2. I-adic completion

Definition 2.1. *Let A be a commutative ring. Let I be a partially ordered set. A pair $((M_i)_i, \{p_{ij}\}_{i \leq j})$ where M_i are A-modules for all $i \in I$, and $p_{ij} : M_j \to M_i$ are A-linear for all $i \leq j \in I$ such that*

1. $p_{ii} = 1_{M_i}$.
2. $p_{ij} \circ p_{jk} = p_{ik}$ *for all* $i \leq j \leq k$.

is called an inverse (or projective) system of A-modules.

Definition 2.2. *Let I be a partially order set and $((M_i)_i, \{p_{ij}\}_{i \leq j})$ an inverse system of A-modules. A module $M = \varprojlim M_i$ together with a family of A-linear maps $q_i : M \to M_i$, $i \in I$, is called the inverse limit of the system if*

1. $p_{ij} q_j = q_i$, *for all* $i \leq j$.
2. *for every A-module X and any A-linear maps $f_i : X \to M_i$, $i \in A$, such that $p_{ij} f_j = f_i$ for all $j \leq i$, there exists a unique A-linear map $F : X \to M$ such that $q_i F = f_i$ for all $i \in I$.*

Theorem 2.3. *Let A be a ring, I a partially ordered set and $((M_i)_i, \{p_{ji}\}_{i \geq j})$ an inverse system of A-modules. Then the inverse limit $\varprojlim M_i$ exists.*

Proof. Consider the A- submodule M of the direct product $\prod_i M_i$ defined by $\{(m_i)_i : p_{ij}(m_j) = m_i, \text{ for all } i \leq j \in I\}$. It is routine to check that $M = \varprojlim M_i$. The maps $q_i : M \to M_i$ are the canonical projections.

□

Let A be a ring and I an ideal of A. We can put a topology on A, where the basis of the topology is given by the sets of the form $x + I^n$, $x \in A, n \in \mathbb{N}$.

Lemma 2.4. *The collection of sets of the form $x + I^n$, $x \in A, n \in \mathbb{N}$ is a topology basis.*

Proof. Clearly, the union of these sets is A. Let $x + I^n$ and $y + I^m$ that intersect non-trivially, and let z an element in the intersection. We need to show that there is a set of form $u + I^k$ containing z and inside $(x + I^n) \cap (y + I^m)$.

Let $n \geq m$ and so $I^n \subseteq I^m$. Now, $z = x + a = y + b$, with $a \in I^n, b \in I^m$. Then $x - y \in I^m$ and so $x \in y + I^m$. Therefore $x + I^n \subseteq y + I^m$ and so $x + I^n \subseteq (x + I^n) \cap (y + I^m)$.

□

Let us endow A with this topology, which will be called the *I-adic topology.*

We say that a sequence of elements $\{x_n\}_n$ is *Cauchy in the I-adic topology* if for all n there exists N such that $x_i - x_j \in I^n$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from A converges to 0 if for all n there exists N such that $x_i \in I^n$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in A$ such that $\{x_n - x\}_n$ converges to zero. We say that A is *complete in the I-adic topology* if every Cauchy sequence in A converges to an element in A.

Note that A/I^n together with the natural projections $p_{mn} : A/I^n \to A/I^m$ for $n \geq m$ form an inverse system. The I-adic completion of A is by definition $\hat{A}^I := \varprojlim A/I^n$ which is a natural A-algebra. We will generally drop the symbol I from our notation when the ideal is understood from the context. Note that we have a natural A-algebra homomorphism $i : A \to \hat{A}$ with kernel equal to $\cap_n I^n$. We say that A is *separated in the I-adic topology* if $\cap_n I^n = 0$.

Proposition 2.5. *The map $i : A \to \hat{A}$ is a ring isomorphism if and only if A is complete in the I-adic topology.*

Proof. Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Since i is an isomorphism we can find an element $x \in A$ such that $x - x_n \in I^n$ for all n which implies easily that $\{x_n\}_n$ converges to x.

Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Therefore for all n, $x_{n+1} - x_n \in I^n$ for all n. Hence $x_i - x_j \in I^n$ for all $i, j \geq n$ which gives that $\{x_n\}$ is Cauchy in A. Hence it is convergent to x an element in A. Therefore, it is enough to show that if $\{x_n\}_n$ is a sequence that converges to zero, then the corresponding element in $\hat{A}$ is zero as well. By definition, $x_j - x_i \in I^j$ for all $j \leq i$. For all n there exists N such that $x_i \in I^n$ for all $i \geq N$. But $x_i - x_n \in I^n$ for $i \geq n$ so $x_n \in I^n$ for all n. □

Remark 2.6. *A Cauchy sequence in A defines a unique canonical element in $\hat{A}$. One can check that the difference between a Cauchy sequence and a subsequence defines a sequence that converges to zero. This can be used to show that that given a Cauchy sequence $\{x_n\}_n$ in A, we can replace it by a sequence $\{y_n\}_n$ that gives the same canonical element in $\hat{A}$ with the additional property that $y_{n+1} - y_n \in I^n$ for all n. This observation is often useful in computations.*