LECTURE 2

1. I-adic completion

Definition 1.1. *Let A be a commutative ring. Let I be a partially ordered set. A pair $((M_i)_i, \{p_{ij}\}_{i\leq j})$ where M_i are A-modules for all $i \in I$, and $p_{ij} : M_j \to M_i$ are A-linear for all $i \leq j \in I$ such that*

(1) $p_{ii} = 1_{M_i}$.
(2) $p_{ij} \circ p_{jk} = p_{ik}$ *for all* $i \leq j \leq k$.

is called an inverse (or projective) system of A-modules.

Definition 1.2. *Let I be a partially order set and $((M_i)_i, \{p_{ij}\}_{i\leq j})$ an inverse system of A-modules. A module $M = \varprojlim M_i$ together with a family of A-linear maps $q_i : M \to M_i$, $i \in I$, is called the inverse limit of the system if*

(1) $p_{ij}q_j = q_i$, *for all* $i \leq j$.
(2) *for every A-module X and any A-linear maps $f_i : X \to M_i$, $i \in A$, such that $p_{ij}f_j = f_i$ for all $j \leq i$, there exists a unique A-linear map $F : X \to M$ such that $q_iF = f_i$ for all $i \in I$.*

Theorem 1.3. *Let A be a ring, I a partially ordered set and $((M_i)_i, \{p_{ji}\}_{i\geq j})$ an inverse system of A-modules. Then the inverse limit $\varprojlim M_i$ exists.*

Proof. Consider the A- submodule M of the direct product $\prod_i M_i$ defined by $\{(m_i)_i : p_{ij}(m_j) = m_i$, for all $i \leq j \in I\}$. It is routine to check that $M = \varprojlim M_i$. The maps $q_i : M \to M_i$ are the canonical projections.

□

Let A be a ring and I an ideal of A. We can put a topology on A, where the basis of the topology is given by the sets of the form $x + I^n$, $x \in A, n \in \mathbb{N}$.

Lemma 1.4. *The collection of sets of the form $x + I^n$, $x \in A, n \in \mathbb{N}$ is a topology basis.*

Proof. Clearly, the union of these sets is A. Let $x + I^n$ and $y + I^m$ that intersect non-trivially, and let z an element in the intersection. We need to show that there is a set of form $u + I^k$ containing z and inside $(x + I^n) \cap (y + I^m)$.

Let $n \geq m$ and so $I^n \subseteq I^m$. Now, $z = x + a = y + b$, with $a \in I^n, b \in I^m$. Then $x - y \in I^m$ and so $x \in y + I^m$. Therefore $x + I^n \subseteq y + I^m$ and so $x + I^n \subseteq (x + I^n) \cap (y + I^m)$.

□

Let us endow A with this topology, which will be called the *I-adic topology.*

We say that a sequence of elements $\{x_n\}_n$ is *Cauchy in the I-adic topology* if for all n there exists N such that $x_i - x_j \in I^n$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from A converges to 0 if for all n there exists N such that $x_i \in I^n$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in A$ such that $\{x_n - x\}_n$ converges to zero. We say that A is *complete in the I-adic topology* if every Cauchy sequence in A converges to an element in A.

Note that A/I^n together with the natural projections $p_{mn} : A/I^n \to A/I^m$ for $n \geq m$ form an inverse system. The I-adic completion of A is by definition $\hat{A}^I := \varprojlim A/I^n$ which is a natural A-algebra. We will generally drop the symbol I from our notation when the ideal is understood from the context. Note that we have a natural A-algebra homomorphism $i : A \to \hat{A}$ with kernel equal to $\cap_n I^n$. We say that A is *separated in the I-adic topology* if $\cap_n I^n = 0$.

Proposition 1.5. *The map $i : A \to \hat{A}$ is a ring isomorphism if and only if A is complete in the I-adic topology.*

Proof. Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Since i is an isomorphism we can find an element $x \in A$ such that $x - x_n \in I^n$ for all n which implies easily that $\{x_n\}_n$ converges to x.

Let $\{x_n\}_n$ be a sequence that gives an element of $\hat{A}$. Therefore for all n, $x_{n+1} - x_n \in I^n$ for all n. Hence $x_i - x_j \in I^n$ for all $i, j \geq n$ which gives that $\{x_n\}$ is Cauchy in A. Hence it is convergent to x an element in A. Therefore, it is enough to show that if $\{x_n\}_n$ is a sequence that converges to zero, then the corresponding element in $\hat{A}$ is zero as well. By definition, $x_j - x_i \in I^j$ for all $j \leq i$. For all n there exists N such that $x_i \in I^n$ for all $i \geq N$. But $x_i - x_n \in I^n$ for $i \geq n$ so $x_n \in I^n$ for all n. □

Remark 1.6. (1) A Cauchy sequence in A defines a unique canonical element in $\hat{A}$. One can check that the difference between a Cauchy sequence and a subsequence defines a sequence that converges to zero. This can be used to show that that given a Cauchy sequence $\{x_n\}_n$ in A, we can replace it by a sequence $\{y_n\}_n$ that gives the same canonical element in $\hat{A}$ with the additional property that $y_{n+1} - y_n \in I^n$ for all n. This observation is often useful in computations.

(2) Let $\mathcal{C}_I(R)$ be the collection of Cauchy sequences on R with the I-adic topology. Let $\mathcal{C}_0(R)$ be the collection of Cauchy sequences in R that converge to 0 in the I-adic topology.

Prove that there is a canonical isomorphism:

$$\frac{\mathcal{C}_I(R)}{\mathcal{C}_0(R)} \simeq \hat{R}^I.$$

2. Properties of completion; Artin-Rees lemma

It is helpful to note that any element of $\hat{A}^I$ is given by a sequence $\{\overline{x}_n\}$ such that $x_{n+1} - x_n \in I^n$. Hence we can find $a_{n+1} \in I^n$ for all $n \geq 0$ such that $x_n = a_1 + \cdots + a_n$ for all $n \geq 1$.

Example 2.1. *It can easily be checked that if $A = R[x_1, \ldots, X_n]$ is a polynomial ring over a ring R, and $I = (X_1, \ldots, X_n)$, then $\hat{A}^I = R[[X_1, \ldots, X_n]]$.*

Proposition 2.2. *Let A, B be two rings and let I be an ideal of A, respectively J be an ideal of B. Consider $f : A \to B$ be a ring homomorphism such that $f(I) \subset J$. Then there is a canonical ring homomorphism $\hat{f} : \hat{A}^I \to \hat{B}^J$.*

Moreover, if f is surjective such that $f(I) = J$, then $\hat{f}$ is surjective.

Proof. We have natural maps $A/I^n \to B/J^n$, so $\varprojlim A/I^n \to B/J^n$ for all n which implies, by applying the universal property of the inverse limit, the first part.

For the second part, consider a sequence of elements in B, say $\{b_n\}$ such that $b_{n+1} \in J^n$, $y_n = b_1 + \cdots + b_n$ and let $\{\overline{y}_n\}_n$ give an element in $\hat{B}^J$. But I^n maps onto J^n via f, so we can find a sequence of elements $a_{n+1} \in I^n$ mappint onto b_{n+1}. Set $x_n = a_1 + \cdots + a_n$, for $n \geq 1$. Then $\{\overline{x}_n\}_n$ gives an element in $\hat{A}^I$ that maps onto the element corresponding to $\{\overline{y}_n\}_n$ in $\hat{B}^J$. □

Corollary 2.3. *If A is Noetherian and $I = (r_1, \ldots, r_n) \subset A$, then $\hat{A}^I$ is Noetherian.*

Remark 2.4. *In fact, we have $\hat{A}^I = \frac{A[[X_1,\ldots,X_n]]}{(X_i - r_i, \ldots, X_n - r_n)}$. This fact is left as an exercise.*

Proof. Indeed, let $I = (r_1, \ldots, r_n)$. Map $R[[X_1, \ldots, X_n]]$ onto A by sending $X_i \to r_i$. This maps $(X_1, \ldots, X_n)$ onto I and hence we obtain $\hat{A}^I$ as a quotient of the Noetherian ring $A[[X_1, \ldots, X_n]]$. □

Theorem 2.5. *Let A be a ring and I and ideal of A. Let $\pi : \hat{A}^I \to A/I$ the natural projection. Then $I\hat{A}^I \subseteq Ker(\pi) \subseteq Jac(\hat{A}^I)$. This implies that there is a one-to-one correspondence between the maximal ideals in $\hat{A}^I$ and the maximal ideal of A/I. In particular, the completion of a local ring $(A, \mathfrak{m})$ at its maximal ideal is a local ring as well.*

Proof. Let $\{\overline{x}_n\}$ an element x of $\hat{A}^I$ that belongs to $Ker(\pi)$: $x_n \in I$ for all n. We will show that $1+x$ is invertible in $\hat{A}^I$. Consider $y_n = \sum_{i=0}^{n+1} (-1)^i x_n^i$. It is clear that y_n define a Cauchy sequence in A which therefore gives an element y of the completion. But then $z_n = 1 - (1 + x_n)y_n = x_n^{n+2} \in I^{n+2}$. This implies that $z = \{\overline{z}_n\}_n$ is 0 in $\hat{A}^I$ and then $1 = (1 + x)y$ in $\hat{A}^I$. □

Definition 2.6. Let A be a ring, I an ideal of A and M an A-module. We say that a sequence of elements $\{x_n\}_n$ in M is *Cauchy in the I-adic topology* if for all n there exists N such that $x_i - x_j \in I^n M$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from M converges to 0 if for all n there exists N such that $x_i \in I^n M$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in M$ such that $\{x_n - x\}_n$ converges to zero in M. We say that M is *complete in the I-adic topology* if every Cauchy sequence in M converges to an element in M. We say that M is *I-adically separated* if $\cap_{n=1}^{\infty} I^n M = 0$.

Definition 2.7. *The I-adic completion of M is $\hat{M}^I : \varprojlim M/I^n M$. It can be checked that $\hat{M}^I$ is a $\hat{A}^I$-module and there exists a natural A-module homomorphism $M \to \hat{M}^I$ with kernel $\cap_n I^n M$.*

We say that a filtration of submodules of M say $\{N_n\}$ is cofinal with the filtration $\{I^nM\}$ if for all n there exists m such that $N_m \subseteq I^nM$ and for all t there exists s such that $I^sM \subset N_t$. It can be checked that $\varprojlim M/N_n \simeq \varprojlim M/I^nM$ (in fact, the filtrations define the same linear topology on M).

Moreover, we can see that a Cauchy sequence and a subsequence of it define the same element in $\hat{M}^I$, so we assume that every element $m \in \hat{M}$ is defined by a sequence $\{m_n\}$ such that $m_{n+1} - m_n \in I^n$. Therefore there exists $z_{n+1} \in I^n$ such that for $y_n = z_0 + \ldots + z_n$ we have that $\{y_n\}_n$ gives m.

Proposition 2.8. *Let A be a ring, I an ideal of A. Then*

(1) *Any A-linear map $f : M \to N$ of A-modules induces an $\hat{A}^I$-linear map $\hat{f} : \hat{M}^I \to \hat{N}^I$. Moreover, f surjective implies that $\hat{f}$ is surjective.*
(2) *There exists a natural isomorphism of $\hat{A}^I$-modules $\hat{M}^I \oplus \hat{N}^I \simeq \hat{M}^I \oplus \hat{N}^I$, for any two A-modules M, N.*
(3) *The multiplication by an element $M \xrightarrow{a} N$ defines a natural map $\hat{A}^I$-linear map $\hat{M}^I \to \hat{N}^I$ given by the multiplication by the image of $a \in \hat{A}^I$.*

Proof. The proof of the first part follows the ring case mutatis mutandis. The last two parts are straightforward □

Let $N \subset M$ be a pair of A-modules. In what follows we need to compare the I-adic topology on N with the topology induced by the I-adic topology on M restricted to N. In essence we will show that $\varprojlim N/I^nM \cap N = \hat{N}^I$. To prove this we need to develop some considerations on filtrations of modules and in fact we will be proving a statement that is more general.

Definition 2.9. Let M be an A-module and I an ideal of A. Let $\mathcal{M} = \{M_n\}_n$ be a filtration of submodules of M, i.e. $M_{n+1} \subset M_n$ and $M_0 = M$. We say that $\mathcal{M}$ is an *I-filtration* if $IM_n \subset M_{n+1}$ for all $n \geq 0$. The filtration $\mathcal{M}$ is called *I-stable* if $I^nM_m = M_{n+1}$ for $n \gg 0$.

An example of an I-stable filtration is the one given by $\{I^nM\}_n$. The case of $M = A$ is particularly important because we can associate the following object to the filtration $\{I^n\}_n$: $gr_I(A) := I^n/I^{n+1}$ which is an A-module naturally. In fact this object, which is called the *associated graded ring* with respect to the ideal I is a ring with multiplication defined as follows: $\overline{a}\overline{b} = \overline{ab}$ for any two elements $a \in I^n, b \in I^m$. It can be checked that this is well-defined and that is extends via distributivity to a multiplication on $gr_I(A)$.

Now consider an I- filtration $\mathcal{M}$. We can define the following A-module $gr_{\mathcal{M}}(M)$: $\oplus_{n\geq 0} M_n/M_{n+1}$. An important feature of it is that this object is in fact an $gr_I(A)$-module. For $\overline{a} \in I^n/I^{n+1}$ and $\overline{m} \in M_k/M_{k+1}$ we let $\overline{am} : \overline{am} \in M_{n+k}/M_{n+k+1}$. It can be checked that definition is well-defined. By distributivity, we can extend this to a scalar multiplication on $gr_{\mathcal{M}}(M)$ with elements from $gr_I(A)$ and we call it *the associated graded module* of M with respect to $\mathcal{M}$.

Proposition 2.10. *Let A be a ring, I be an ideal of A, M be an A-module, and $\mathcal{M}$ be an I-filtration on M. Then*

(1) *If $A[It]$ is a finitely generated A-algebra, if I is a finitely generated ideal.*
(2) *If $A[It]/IA[It] \simeq gr_I(A)$ as A-algebras.*

Proof. For (1), let $I = (a_1, \ldots, a_r)$. Then $A[X_1, \ldots, x_n]$ maps onto $A[It]$ under $X_i \to a_i$. For (2), let $A[It] \to gr_I(A)$ that sends at to $\overline{a} \in I/I^2$, for any $a \in I$. It can be easily check that this is an well-defined A-algebra homomorphism with kernel equal to $IA[It]$.

□

Proposition 2.11. *Let A be a ring, I be an ideal of A, M be a finitely generated module over A, and $\mathcal{M}$ be an I-stable filtration on M composed of finitely generated submodules. Then $gr_{\mathcal{M}}(M)$ is a finitely generated module over $gr_I(A)$.*

Proof. Since the filtration is I-stable so $I^k M_N = M_{N+k}$ for some $N \geq 0$ and all $k \geq 0$. Therefore $\frac{I}{I^2} \cdot \frac{M_n}{M_{n+1}} = \frac{M_{n+1}}{M_{n+2}}$, for all $n \geq N$.

This shows that $gr_{\mathcal{M}}(M)$ is generated by the union of all the generators of M_n/M_{n+1} for $n \leq N$. This is a finite set which proves the claim. □

Definition 2.12. Let I be an ideal in A. The A-algebra $R_I(A) = A[It] \subset A[t]$ is called the *Rees algebra, or the blowup algebra,* of A with respect to I. Note that $A[It] = \oplus_{n\geq 0} I^n$.

Similarly, for an I-filtration $\mathcal{M}$ on an A-module M, we can define the *Rees module* of M with respect to $\mathcal{M}$ by $R_{\mathcal{M}}(M) := \oplus_{n\geq 0} M_n t^n = \oplus_{n\geq 0} M_n$. Note that $R_{\mathcal{M}}(M)$ is a module over $A[It]$ in a natural way.

Theorem 2.13. *Let A be a ring, I be an ideal of A, M be an A-module with I-filtration $\mathcal{M}$ consisting of finitely generated A-submodules of M. Then the filtration $\mathcal{M}$ is I-stable if and only if $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$-module.*

Proof. If $\mathcal{M}$ is I-stable, then $M_{N+k} = I^k M_N$ for some $N \geq 0$ and for all $k \geq 0$. Then $R_{\mathcal{M}}(M)$ is finitely generated by the union of the generators of M_i, with $i \leq N$.

If $R_{\mathcal{M}}(M)$ is finitely generated over $A[It]$, there exists $N \geq 0$ such that all generators belong to the union of M_i, $i \leq N$. But $\oplus_{k>0} M_{N+k}$ is finitely generated as an $A[It]$-module (since it is a homomorphic image of $R_{\mathcal{M}}(M)$). Using this and the fact that $\mathcal{M}$ is an I-filtration we derive that $I^k M_N = M_{N+k}$, for all $k \geq 0$.

□

Corollary 2.14 (Artin-Rees Lemma)**.** *Let M be a finitely generated A-module, where A is Noetherian. Assume that I is an ideal of A and let N an A-submodule of M.*

1. *Let $\mathcal{M}$ be an I-stable filtration on M. Then $\{M_n \cap N\}_n$ is an I-stable filtration on N.*
2. *The filtration $\{I^n M \cap N\}_n$ is I-stable that is there exists $c > 0$ such that*

$$I^n M \cap N = I^{n-c}(I^c M \cap N),$$

for all $n \geq c$.

Proof. It suffices to prove (1). Let $\mathcal{M}'$ the filtration with terms $M_n \cap N$. Clearly $R_{\mathcal{M}'}(N)$ is an $A[It]$-submodule of $R_{\mathcal{M}}(M)$.

Note that $A[It]$ is a Noetherian A-algebra and $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$-module by Theorem 2.13. So, $R_{\mathcal{M}'}(N)$ is a finitely generated $A[It]$-module, hence by Theorem 2.13 we get that $\mathcal{M}'$ is I-stable on N. □

A simple consequence of the Artin-Rees lemma is the Krull Intersection Theorem

Theorem 2.15. *Let A be a Noetherian ring and I an idea and M a finitey generated A-module. Let $N = \cap_n I^n M$. Then exists an element $a \in A$ with $1 + a \in I$ that kills N. In particular, if A is local then $N = 0$.*

Proof. Show that $IN = N$ and then use NAK. □

Theorem 2.16. *Let A be a Noetherian ring and I an ideal of A.*

(1) *If $0 \to N \to M \to P \to 0$ is a short exact sequence of finitely generated A-modules then*
$$0 \to \hat{N}^I \to \hat{M}^I \to \hat{P}^I \to 0$$
is a short exact sequence of $\hat{A}^I$-modules.

(2) *The universal property of the tensor product implies that there is a natural $\hat{A}^I$-map $\hat{A}^I \otimes_A M \to \hat{M}^I$ for any A-module M. For every finitely generated A-module M, $\hat{A}^I \otimes_A M \simeq \hat{M}^I$ is an isomorphism.*

(3) *$\hat{A}^I$ is a flat A-algebras which is faithfully flat if $(A, \mathfrak{m})$ is a local ring.*

Proof. For (1), consider the short exact sequence
$$0 \to N \to M \to P \to 0$$
and tensor with A/I^n. We get
$$N/I^n N \to M/I^n M \to P/I^n P \to 0,$$
and further
$$0 \to N/(I^n M \cap N) \to M/I^n M \to P/I^n P \to 0.$$

This gives a short exact sequence of inverse limits

$$0 \to \varprojlim N/(I^n M \cap N) \to \hat{M}^I \to \hat{N}^I \to 0,$$
because the maps in the inverse system defined by $N/(I^{n+1} M \cap N) \to N/(I^n M \cap N)$ are surjective. But by the Artin-Rees Lemma the filtrations $\{I^n M \cap N\}_n$ and $I^n N$ are cofinal and hence
$$\varprojlim N/(I^n M \cap N) \simeq \varprojlim N/I^n N = \hat{N}^I.$$

For (2), note that we have the isomorphism

$$M \otimes_A \hat{A}^I \simeq \hat{M}^I,$$

if M is finitely generated and free. Since A is Noetherian, we can find two finitely generated free A-modules such that

$$E = A^m \to F = A^n \to M \to 0.$$

to obtain this presentation of M first map a finitely generated free A-module onto M and then map another free finitely generated A onto the kernel of this first map.

We obtain the following to exact sequences

$$(\hat{A}^I)^m \to (\hat{A}^I)^n \to \hat{M}^I \to 0$$

and

$$A^m \otimes_A \hat{A}^I \to A^n \otimes_A \hat{A}^I \to M \otimes_A \hat{A}^I \to 0.$$

An application of the Five Lemma (or Snake Lemma) gives the result.

Finally for (3) it is enough to show that tensoring with $\hat{A}^I$ preserves exactness of $0 \to N \to M$, where N, M are finitely generated. But this follows directly from (2).

$\square$