

- (1) Let \mathbb{Q} be the field of all rational numbers, R the ring of all 4×4 matrices with entries in \mathbb{Q} , and J a two-sided ideal of R such that $A \in J$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

- (a) Determine whether the 4×4 identity matrix I_4 is contained in J . Justify your claim fully.
 - (b) Determine whether J coincides with R . Explain why.
- (2) Let R be a commutative ring (with identity). Assume that R has only three distinct ideals: 0 , I , R . Prove that
- (a) If $a \in I$ then $1 - a$ is invertible in R .
 - (b) If a, b are nonzero elements in I then $ab = 0$.
- (3) Show there is no simple group of order 72.
- (4) Let $R = \mathbb{Q}[x]$ and M be the quotient of R^3 modulo the R -submodule generated by the columns of the 3×3 matrix $xI - A$ where I is the 3×3 identity matrix and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}.$$

- (a) Find the Smith normal form of $xI - A$.
 - (b) Up to isomorphism, express M as a direct sum of cyclic R -modules.
 - (c) What are the free rank, the invariant factors, and the elementary divisors of M ?
 - (d) Determine the rational canonical form and the Jordan canonical form of A .
- (5) Compute the Galois group of $\mathbb{Q}(\sqrt{5}, \sqrt{3})$ over \mathbb{Q} .
- (6) Prove that $\mathbf{F}_p(x^p, y^p) \subseteq \mathbf{F}_p(x, y)$ is not a simple extension. (Here p is a prime number.)