When does the F-signature exist?*

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Abstract. We show that the F-signature of an F-finite local ring R of characteristic $p > 0$ exists when R is either the localization of an $\mathbf{N}$-graded ring at its irrelevant ideal or $\mathbf{Q}$-Gorenstein on its punctured spectrum. This extends results by Huneke, Leuschke, Yao and Singh and proves the existence of the F-signature in the cases where weak F-regularity is known to be equivalent to strong F-regularity.

Résumé. Nous prouvons dans cet article l'existence de la F-signature d'un anneau local F-fini R, de caractéristique positive p, quand R est la localisation à l'unique idéal homogène maximal d'un anneau $\mathbf{N}$-gradué ou quand R est $\mathbf{Q}$-Gorenstein sur son spectre épointé. Ceci généralise les résultats de Huneke, Leuschke, Yao et Singh et prouve l'existence de la F-signature dans les cas où faible et forte F-régularité sont équivalentes.

1 A sufficient condition for the existence of the F-signature

Let $(R, \mathfrak{m}, k)$ be a reduced, local F-finite ring of positive characteristic $p > 0$ and Krull dimension d. Let

$$R^{1/q} = R^{a_q} \oplus M_q$$

be a direct sum decomposition of $R^{1/q}$ such that M_q has no free direct summands. If R is complete, such a decomposition is unique up to isomorphism. Recent research has focused on the asymptotic growth rate of the numbers a_q as $q \to \infty$. In particular, the F-signature (defined below) is studied in [7] and [3], and more generally the Frobenius splitting ratio is studied in [2].

For a local ring $(R, \mathfrak{m}, k)$, we set $\alpha(R) = \log_p[k_R : k_R^p]$. It is easy to see that, for an $\mathfrak{m}$-primary ideal I of R, $\lambda(R^{1/q}/IR^{1/q}) = \lambda(R/I^{[q]})/q^{\alpha(R)}$, where $\lambda(-)$ represents the length function over R.

We would like to first define the notion of F-signature as it appears in [3] and [7].

Definition 1.1. The *F-signature* of R is $s(R) = \lim_{q\to\infty} \dfrac{a_q}{q^{d+\alpha(R)}}$, if it exists.

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The following result, due to Aberbach and Leuschke [3], holds:

Theorem 1.2. *Let* $(R, \mathfrak{m}, k)$ *be a reduced Noetherian ring of positive characteristic* p*. Then* $\liminf_{q\to\infty} a_q/q^{d+\alpha(R)} > 0$ *if and only if* $\limsup_{q\to\infty} a_q/q^{d+\alpha(R)} > 0$ *if and only if* R *is strongly* F*-regular.*

The question of whether or not, in a strongly F-regular ring, $s(R)$ exists, is open. We show in this paper that its existence is closely connected to the question of whether or not weak and strong F-regularity are equivalent.

Smith and Van den Bergh ([10]) have shown that the F-signature of R exists when R has finite Frobenius representation type (FFRT) type, that is, if only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules occur as direct summands of $R^{1/q}$ for any $q = p^e$. Yao has proven that, under mild conditions, tight closure commutes with localization in a ring of FFRT type, [11]. Moreover, Huneke and Leuschke proved that if R is also Gorenstein, then the F-signature exists, [7]. Yao has recently extended this result to rings that are Gorenstein on their punctured spectrum, [12]. Singh has also shown that the F-signature exists for monomial rings, [9].

Let $(R, \mathfrak{m})$ be an approximately Gorenstein ring. This means that R has a sequence of $\mathfrak{m}$-primary irreducible ideals $\{I_t\}_t$ cofinal with the powers of $\mathfrak{m}$. By taking a subsequence, we may assume that $I_t \supset I_{t+1}$. For each t, let u_t be an element of R which represents a socle element modulo I_t. Then there is, for each t, a homomorphism $R/I_t \hookrightarrow R/I_{t+1}$ such that $u_t + I_t \mapsto u_{t+1} + I_{t+1}$. The direct limit of the system will be the injective hull $E = E_R(R/\mathfrak{m})$ and each u_t will map to the socle element of E, which we will denote by u. Hochster has shown that every excellent, reduced local ring is approximately Gorenstein ([5]).

Aberbach and Leuschke have shown that, for every q, there exists $t_0(q)$, such that

$$a_q/(q^{d+\alpha(R)}) = \lambda(R/(I_t^{[q]} : u_t^q))/q^d,$$

for all $t \geq t_0(q)$ (see [3], p. 55).

The situation when $t_0(q)$ can be chosen independently of q is of special interest.

Definition 1.3. We say that R satisfies *Condition* (A), if there exist a sequence of irreducible $\mathfrak{m}$-primary ideals $\{I_t\}$ and a t_0 such that, for all $t \geq t_0$ and all q

$$(I_t^{[q]} : u_t^q) = (I_{t_0}^{[q]} : u_{t_0}^q).$$

Proposition 1.4. *Let* $(R, \mathfrak{m}, k)$ *be a local reduced* F*-finite ring. If* R *satisfies Condition* A*, then the* F*-signature exists.*

Proof. We know that R is approximately Gorenstein and hence we will use the notation fixed in the paragraph above.

As explained above, Condition A implies that there exists t_0, independent of q, such that

$$a_q/(q^{d+\alpha(R)}) = \lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q))/q^d,$$

for all q.

But $\lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q)) = \lambda(R/I_{t_0}^{[q]}) - \lambda(R/(I_{t_0} + u_{t_0}R)^{[q]})$. Dividing by q^d and taking the limit as $q \to \infty$ yields $s(R) = \mathrm{e}_{HK}(I_{t_0}, R) - \mathrm{e}_{HK}(I_{t_0} + u_{t_0}R, R)$. □

Now we would like to concentrate on another condition, Condition (B), that appeared first in the work of Yao. First we need to introduce some notation.

Assume that E is the injective hull of the residue field k. By $R^{(e)}$ we denote the R-bialgebra whose underlying abelian group equals R and the left and right R-multiplication is given by $a \cdot r * b = arb^q$, for $a, b \in R, r \in R^{(e)}$.

Let $k = Ru \to E$ be the natural inclusion and consider the natural induced map $\phi_e : R^{(e)} \otimes_R E \to R^{(e)} \otimes_R (E/k)$. Then $a_q/q^{\alpha(R)} = \lambda(\ker(\phi_e))$ (by Aberbach-Enescu, Corollary 2.8 in [2], see also Yao's work [12]).

One can in fact see that

$$\lambda(\ker(\phi_e)) = \lambda(R/(c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E)) = \lambda(R/ \cup_t (I_t^{[q]} : u_t^q)).$$

Definition 1.5. We say that R satisfies *Condition* (B) if there exists a finite length submodule $E' \subset E$ such that, if $\psi_e : R^{(e)} \otimes_R E' \to R^{(e)} \otimes_R E'/k$, then $\lambda(\ker(\phi_e)) = \lambda(\ker(\psi_e))$, for all e.

Yao [12] has shown that Condition (B) implies that the F-signature of R exists.

Proposition 1.6. *Let $(R, \mathfrak{m}, k)$ be a local reduced F-finite ring. Then Conditions (A) and (B) are equivalent.*

Proof. Assume that Condition (A) holds. Then one can take $E' = R/I_{t_0}$ and then Condition (B) follows.

If Condition (B) holds, then take t_0 large enough such that $E' \subset \mathrm{Im}(R/I_{t_0} \to E)$.

As noted above, one can compute the length of the kernel of ψ_e as the colength of $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E'\}$. Since R/I_{t_0} injects into E we see that $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E'\}$ is a subset of $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R R/I_{t_0}\} = (I_{t_0}^{[q]} : u_{t_0}^q)$.

Since $(I_{t_0}^{[q]} : u_{t_0}^q) \subset (I_t^{[q]} : u_t^q)$ for all $t \geq t_0$, we see that Condition (B) implies that $(I_{t_0}^{[q]} : u_{t_0}^q) = (I_t^{[q]} : u_t^q)$ for all $t \geq t_0$, which is Condition (A).

□

2 N-Graded Rings

Let $(R, \mathfrak{m})$ be a Noetherian $\mathbf{N}$-graded ring $R = \oplus_{n \geq 0} R_n$, where $R_0 = k$ is an F-finite field of characteristic $p > 0$.

For any graded R-module M one can define a natural grading on $R^{(e)} \otimes M$: the degree of any tensor monomial $r \otimes m$ equals $\deg(r) + q \deg(m)$.

In what follows we will need the following important Lemma by Lyubeznik and Smith ([8], Theorem 3.2):

Lemma 2.1. *Let R be an $\mathbf{N}$-graded ring and M, N two graded R-modules. Then there exists an integer t depending only on R such that whenever*

$$M \to N$$

is a degree preserving map which is bijective in degrees greater than s, then the induced map

$$R^{(e)} \otimes M \to R^{(e)} \otimes N$$

is bijective in degrees greater than $p^e(s+t)$.

Let E be the injective hull of $R_{\mathfrak{m}}$. In fact, E is also the injective hull of $R/\mathfrak{m}$ over R and as a result is naturally graded with socle in degree 0. We can write $E = \oplus_{n \leq 0} E_n$.

Let t be as in the Lemma 2.1, and let $s \leq -t-1$. Obviously the map $E' = \oplus_{s \leq n \leq 0} E_n \to E = \oplus_{n \leq 0} E_n$ is bijective in degrees greater than s. So by Lemma 2.1, the map $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $p^e(s+t) \leq -p^e$.

Theorem 2.2. *Let R be an $\mathbf{N}$-graded reduced ring over an F-finite field k of positive characteristic. Then Condition (B) is satisfied by R and hence the F-signature of R exists.*

Proof. Let E be the injective hull of $k = R/\mathfrak{m}$ over $R_{\mathfrak{m}}$. As above, $E = \oplus_{n \leq 0} E_n$, where 0 is the degree of the socle generator u of E.

Using the notation introduced above, we will let $s = -t-1$ and $E' = \oplus_{s \leq n \leq n_0} E_n \to E$. So, $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $-p^e$. In particular it is bijective in degrees greater or equal to 0.

We have the following exact sequences:

$$0 \to k = Ru \to E \to E/k \to 0$$

and

$$0 \to k = Ru \to E' \to E'/k \to 0.$$

After tensoring with $R^{(e)}$, we get the exact sequences

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \to R^{(e)} \otimes E \xrightarrow{\phi_e} R^{(e)} \otimes E/k \to 0$$

and

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \to R^{(e)} \otimes E' \xrightarrow{\psi_e} R^{(e)} \otimes E'/k \to 0.$$

One can easily see that $\ker(\phi_e)$ and $\ker(\psi_e)$ are the submodules generated by $1 \otimes u$ in $R^{(e)} \otimes E$ and $R^{(e)} \otimes E'$, respectively.

The degree of $1 \otimes u$ is $q \cdot 0 = 0$ and we have noted that the natural map $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $-p^e$. This shows that $\ker(\phi_e) \simeq \ker(\psi_e)$ and hence Condition (B) is satisfied.

□

3 $\mathbb{Q}$-Gorenstein Rings

We turn now to showing that Condition (A) holds in strongly F-regular local rings which are $\mathbb{Q}$-Gorenstein on the punctured spectrum. Let $(R, \mathfrak{m}, k)$ be such a ring of dimension

d, and assume that R has a canonical module (e.g. R is complete). In this case R has an unmixed ideal of height 1, say $J \subseteq R$, which is a canonical ideal. We may pick an element $a \in J$ which generates J at all minimal primes of J, and then an element $x_2 \in \mathfrak{m}$ which is a parameter on R/J such that $x_2 J \subseteq aR$. It is easy to see that then $x^n J^{(n)} \subseteq a^n R$ for all $n \geq 1$ (where $J^{(n)}$ is the height one component of J^n). The condition that R is $\mathbb{Q}$-Gorenstein on the punctured spectrum implies that there is an integer h and two sequences of elements $x_3, \ldots, x_d \in \mathfrak{m}$ and $a_3, \ldots, a_d \in J^{(h)}$ such that $x_i J^{(h)} \subseteq a_i R$ for $3 \leq i \leq d$, and $x_2, \ldots, x_d$ is a s.o.p. on R/J. We may then pick $x_1 \in J$ such that $x_1, \ldots, x_d$ is an s.o.p. for R. See [1], section 2.2 for more detail. Then by [1], Lemma 2.2.3 we have that for any $N \geq 0$ and any $n \geq 0$,

$$(J^{(nh)}, x_2^N, \ldots, \widehat{x_i^N}, \ldots, x_d^N) : x_i^\infty = (J^{(nh)}, x_2^N, \ldots, \widehat{x_i^N}, \ldots, x_d^N) : x_i^n. \tag{3.1}$$

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be an F-finite strongly F-regular ring which is $\mathbb{Q}$-Gorenstein on the punctured spectrum. Then R satisfies Condition* (A). *In particular the F-signature of R exists.*

Proof. If R is not complete, we observe that, since R is excellent, $\widehat{R}$ is strongly F-regular and $\mathbb{Q}$-Gorenstein on the punctured spectrum. If $\{I_t\}$ is a sequence of ideal in $\widehat{R}$ showing condition (A) in $\widehat{R}$, then $\{I_t \cap R\}$ does so for R. Thus we will assume that R is complete.

Let J, h, and $x_1, \ldots, x_d$ be as discussed above. Let $I_t = (x_1^{t-1}J, x_2^t, \ldots, x_d^t)$. Since $x_1^n J \cong J$ as R-modules, the quotient $R/x_1^n J$ is Gorenstein. The hypothesis that $x_2, \ldots, x_d$ are parameters on R/J and $R/x_1 R$ (hence on $R/x_1^n J$) then shows that I_t is irreducible (see [4], Proposition 3.3.18). The sequence $\{I_t\}$ is then a sequence of $\mathfrak{m}$-primary irreducible ideals cofinal with the powers of $\mathfrak{m}$. If u_1 represents the socle element of I_1, then we may take $u_t = (x_1 \cdots x_d)^{t-1} u_1$ to represent the socle element of I_t. We will show that t_0 may be taken to be 3.

Suppose that $c \in I_t^{[q]} : u_t^q$ for some q. We will show that $c \in I_3^{[q]} : u_3^q$. Raising to the q'th power we have $c^{q'} u_t^{qq'} = c^{q'}((x_1 \cdots x_d)^{t-1} u_1)^{qq'} \in I_t^{[qq']} = (x_1^{t-1}J, x_2^t, \ldots, x_d^t)^{[qq']}$. Hence $c^{q'}((x_2 \cdots x_d)^{t-1} u_1)^{qq'} \in (x_2^t, \ldots, x_d^t)^{[qq']} : x_1^{(t-1)qq'} + (J, x_2^t, \ldots, x_d^t)^{[qq']} = (J, x_2^t, \ldots, x_d^t)^{[qq']}$.

Write $qq' = n_{q'} h + r_{q'}$ with $0 \leq r_{q'} < h$. Repeated application of equation 3.1 (using 1 rather than h for x_2) gives

$$c^{q'}((x_2 \cdots x_d) u_1)^{qq'} \in (J^{(n_{q'} h)}, x_2^{2qq'}, \ldots, x_d^{2qq'}). \tag{3.2}$$

Let $d \in J^{(h)} \subseteq J^{(r_{q'})}$. Multiplying by $x_2^{qq'}$ and using that $x_2^{qq'} J^{(qq')} \subseteq a^{qq'} R \subseteq J^{[qq']}$ we have $dc^{q'}((x_2 \cdots x_d)^2 u_1)^{qq'} \in (J, x_2^3, \ldots, x_d^3)^{[qq']}$. Multiplying by $x_1^{2qq'}$ shows that $dc^{q'} u_3^{qq'} = d(cu_3^q)^{q'} \in (I_3^{[q]})^{[q']}$. Thus $cu_3^q \in (I_3^{[q]})^* = I_3^{[q]}$, as desired. ☐

References

[1] I. M. Aberbach *Some conditions for the equivalence of weak and strong F-regularity*, Comm. Alg. 30 (2002), 1635–1651.

[2] I. M. Aberbach, F. Enescu, *The structure of F-pure rings*, Math. Zeit., to appear.

[3] I. M. Aberbach, G. Leuschke, *The F-signature and strong F-regularity*, Math. Res. Lett. 10 (2003), 51–56.

[4] W. Bruns, J. Herzog *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[5] M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. 231 (1977), no. 2, 463–488.

[6] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Meḿoire no. 38, Soc. Math. France, 1989, 119–133,

[7] C. Huneke, G. Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. 324 (2002), 391–404.

[8] G. Lyubeznik, K.E. Smith, *Strong and weak F-regularity are equivalent for graded rings*, Amer. J. Math. 121 (1999), 1279–1290.

[9] A. K. Singh, *The F-signature of an affine semigroup ring*, J. Pure Appl. Algebra, 196 (2005) 313–321.

[10] K. E. Smith, M. Van den Bergh, *Simplicity of rings of differential operators in prime characteristic*, Proc. London. Math. Soc. (3) 75, 1997, no. 1, 32–62.

[11] Y. Yao, *Modules with finite F-representation type*, preprint 2002.

[12] —, *Observations on the F-signature of local rings of characteristic* $p > 0$, preprint 2003.