LOWER BOUNDS FOR HILBERT-KUNZ MULTIPLICITIES IN LOCAL RINGS OF FIXED DIMENSION

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Abstract. Let $(R, \mathfrak{m})$ be an unmixed local ring of positive prime characteristic and dimension d. Assume that R is either excellent or a homomorphic image of a Cohen-Macaulay ring. We examine the implications of having small Hilbert-Kunz multiplicity (i.e., close to 1). In particular, we show that if R is not regular, there exists a lower bound, strictly greater than one, depending only on d, for its Hilbert-Kunz multiplicity.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a local ring of positive characteristic p. For an $\mathfrak{m}$-primary ideal I, one can consider the Hilbert-Samuel multiplicity and the Hilbert-Kunz multiplicity of I with respect to R.

Definition 1.1. Let I be an $\mathfrak{m}$-primary ideal in $(R, \mathfrak{m})$. Let $\lambda(-)$ denote the usual length function.

1. *The Hilbert-Samuel multiplicity of R at I* is defined by $\mathrm{e}(I) = \mathrm{e}(I, R) := \lim_{n\to\infty} d!\,\dfrac{\lambda(R/I^n)}{n^d}$. The limit exists and it is a positive integer.

2. *The Hilbert-Kunz multiplicity of R at I* is defined by $\mathrm{e}_{HK}(I) = \mathrm{e}_{HK}(I, R) := \lim_{q\to\infty} \dfrac{\lambda(R/I^{[q]})}{q^d}$. Monsky has shown that the latter limit exists and is positive.

The Hilbert-Samuel multiplicty of R, denoted $\mathrm{e}(R)$, is by definition $\mathrm{e}(\mathfrak{m})$. Similarly, the Hilbert-Kunz multiplicity of R, denoted $\mathrm{e}_{HK}(R)$, is $\mathrm{e}_{HK}(\mathfrak{m})$.

It is known that for parameter ideals I, one has $\mathrm{e}(I) = \mathrm{e}_{HK}(I)$. The following sequence of inequalities is also known to hold whenever I is $\mathfrak{m}$-primary:

$$\max\{1, \frac{\mathrm{e}(I)}{d!}\} \le \mathrm{e}_{HK}(I) \le \mathrm{e}(I).$$

We call a local ring R *unmixed* if R is equidimensional and $\mathrm{Min}(R) = \mathrm{Ass}(R)$, that is, $\dim(R/P) = \dim(R)$ for all minimal primes P, and all associated primes of R are minimal.

In this paper we investigate rings that have small Hilbert-Kunz multiplicity. It is known that an unmixed local ring of characteristic p is regular if and only if $\mathrm{e}_{HK}(R) = 1$. In fact, similar statements hold true for the Hilbert-Samuel multiplicity and they are considered classical. (The unmixedness assumption is essential as there are examples of nonregular rings that are not unmixed with $\mathrm{e}_{HK}(R) = 1$. The reason is that neither Hilbert-Samuel multiplicity nor Hilbert-Kunz multiplicity can pick up lower dimensional components of R). Since $\mathrm{e}(R)$ is always a positive integer we have that $\mathrm{e}(R) \ge 2$ if R is unmixed but not regular. The situation is much more subtle in the case of the Hilbert-Kunz multiplicity because it

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often takes on non-integer values (although it is conjectured to always be rational). So, the question becomes: If one fixes the dimension d, how close to 1 can $\mathrm{e}_{HK}(R)$ be (when R is unmixed, but not regular)? What can be said about the structure of rings of small Hilbert-Kunz multiplicity? This problem has been intensively studied in recent years (with success mostly for rings of small dimension) by Blickle-Enescu [3], Watanabe-Yoshida [13], [14], [15], and Enescu-Shimomoto [5]. In the current paper, we will develop techniques that shed light on this problem independent of dimension.

The goal is at least twofold: find the following constants (as introduced in [3]),

$$\epsilon_{HK}(d,p) = \inf\{\mathrm{e}_{HK}(R) - 1 : R \text{ non-regular, unmixed, } \dim R = d, \operatorname{char} R = p\}$$

and

$$\epsilon_{HK}(d) = \inf\{\epsilon_{HK}(d,p) : p > 0\}$$

and describe the structure of the rings with small Hilbert-Kunz multiplicity from both an algebraic and geometric point of view.

It is known that $\epsilon_{HK}(d,p) > \dfrac{1}{d!p^d}$ by results in [3]. Clearly, however, as $p \to \infty$, the right hand side tends toward 0, so this does not give a positive lower bound for $\epsilon_{HK}(d)$. A byproduct of our work is that it leads us to a proof of the fact that $\epsilon_{HK}(d) > 0$, answering positively a problem raised in [3], Section 3. We should mention that a conjecture of Watanabe and Yoshida [15] asserts that if $(R, \mathfrak{m}, k)$ has residue field equal to $\overline{\mathbf{F}_p}$, $p > 2$, then $\mathrm{e}_{HK}(R) \geq \mathrm{e}_{HK}(R_{p,d})$, where $R_{p,d} = \overline{\mathbf{F}_p}[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)$. This conjecture has been answered positively for dimensions $d = 2, 3, 4, 5$ (the difficult cases of dimension $4, 5$ are due to Watanabe and Yoshida) and in the case of complete intersections by Enescu and Shimomoto ([5]).

The starting point of our investigation is the following:

Theorem 1.2 (Blickle-Enescu)**.** *Let R be an unmixed ring that is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$. If*

$$\mathrm{e}_{HK}(R) \leq 1 + \max\{1/d!, 1/\mathrm{e}(R)\},$$

then R is Cohen-Macaulay and F-rational.

Remark 1.3. The proof of the above result shows that, in fact, the inequality $\mathrm{e}_{HK}(R) < \dfrac{\mathrm{e}(R)}{\mathrm{e}(R) - 1}$ forces R to be Cohen-Macaulay and F-rational.

In fact, the hypotheses of Theorem 1.2 suffice to show that R must be (strongly) F-regular. This is the content of Corollary 3.6 which states:

Corollary. *Let $(R, \mathfrak{m}, k)$ be an unmixed ring of characteristic p and $\dim(R) = d \geq 2$. Assume either that R is the image of a Cohen-Macualay ring, or is excellent. If $\mathrm{e}_{HK}(R) \leq 1 + \max\{1/d!, 1/\mathrm{e}(R)\}$, then R is weakly F-regular. If R is excellent, then R is strongly F-regular.*

Theorem 4.12 gives a positive lower bound for $\epsilon(d)$ which does not depend on p:

Theorem. *Let $(R, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic p and dimension d that is either a homomorphic image of a Cohen-Macaulay ring or excellent. If R is not regular then*

$$\mathrm{e}_{HK}(R) \geq 1 + \frac{1}{(d-1)\cdot(d!(d/2-1)+1)^{d-1}}.$$

While this result shows that $\epsilon(d) > 0$, our techniques can be refined to give sharper estimates. In a future paper, we will give results that are considerably better, but the cost is that the arguments are very much more technical, so we have opted to give a more accessible proof of the fact that such an $\epsilon(d)$ exists. Although the above mentioned conjecture of Watanabe and Yoshida is still open, we have developed techniques that, for the first time, work regardless of dimension or additional hypotheses on the rings.

In dealing with Hilbert-Kunz multiplicities it often useful to assume that the rings that are studied are either formally unmixed or unmixed and homomorphic images of Cohen-Macaulay rings. This will also be the case in our paper.

2. Definitions and known results

First we would like to review some definitions and results that will be useful later. Throughout the paper R will a Noetherian ring containing a field of characteristic p, where p is prime. Also, q will denote p^e, a varying power of p.

If I is an ideal in R, then $I^{[q]} = (i^q : i \in I)$, where $q = p^e$ is a power of the characteristic. Let $R^\circ = R \setminus \cup P$, where P runs over the set of all minimal primes of R. An element x is said to belong to the *tight closure* of the ideal I if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all sufficiently large $q = p^e$. The tight closure of I is denoted by I^*. By a *parameter ideal* we mean here an ideal generated by a full system of parameters in a local ring R.

Let $F : R \to R$ be the Frobenius homomorphism $F(r) = r^p$. We denote by F^e the eth iteration of F, that is $F^e(r) = r^q$, $F^e : R \to R$. One can regard R as an R-algebra via the homomorphism F^e. Although as an abelian group it equals R, it has a different scalar multiplication. We will denote this new algebra by $R^{(e)}$. For an R-module M we let $F^e(M) = R^{(e)} \otimes_R M$, where we consider this an R-module via $R^{(e)}$, i.e., $a(r \otimes m) = (ar) \otimes m$, but $r \otimes (am) = a^q r \otimes m$. For an element $m \in M$, let $m^q = 1 \otimes m \in F^e(M)$. If $N \subseteq M$ then we denote the image of $F^e(N)$ in $F^e(M)$ by $N^{[q]}$, and this is the same as the submodule of $F^e(M)$ generated by the elements n^q for $n \in N$. We then say that $x \in M$ is in the tight closure of N in M, denoted N^*_M, if there exists $c \in R^0$ such that $cx^q \in N^{[q]}$ for all $q \gg 0$.

Definition 2.1. R is *F-finite* if $R^{(1)}$ is module finite over R, or, equivalently (in the case that R is reduced), $R^{1/p}$ is module finite over R. R is called *F-pure* if the Frobenius homomorphism is a pure map, i.e, $F \otimes_R M$ is injective for every R-module M.

If R is F-finite, then $R^{1/q}$ is module finite over R, for every q. Moreover, any quotient and localization of an F-finite ring is F-finite. Any finitely generated algebra over a perfect field is F-finite. An F-finite ring is excellent.

Definition 2.2. A reduced Noetherian F-finite ring R is *strongly F-regular* if for every $c \in R^0$ there exists q such that the R-linear map $R \to R^{1/q}$ that sends 1 to $c^{1/q}$ splits over R, or equivalently $Rc^{1/q} \subset R^{1/q}$ splits over R.

The notion of strong F-regularity localizes well, and all ideals are tightly closed in strongly F-regular rings. Regular rings are strongly F-regular and strongly F-regular rings are Cohen-Macaulay and normal.

Let $E_R(k)$ be the injective hull of the residue field of R. Then an F-finite ring R is strongly F-regular if and only if $0^*_{E_R} = 0$. More generally, when $(R, \mathfrak{m})$ is excellent (but not necessarily F-finite) we will say that R is strongly F-regular if $0^*_{E_R} = 0$.

Definition 2.3. A ring R is called F-rational if all parameter ideals are tightly closed. A ring R is called weakly F-regular if all ideals are tightly closed. The ring R is F-regular if and only if $S^{-1}R$ is weakly F-regular for all multiplicative sets $S \subset R$.

Regular rings are (strongly) F-regular. For Gorenstein rings, the notions of F-rationality and F-regularity coincide (and if in addition the ring is excellent, these coincide with strong F-regularity).

Proposition 2.4 (Associativity formula)**.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Denote* $\mathrm{Assh}(R) = \{P \in \mathrm{Ass}(R) : \dim(R/P) = \dim(R)\}$*. Then*

$$\mathrm{e}_{HK}(R) = \sum_{P \in \mathrm{Assh}(R)} \lambda\left(R_P\right) \cdot \mathrm{e}_{HK}(R/P).$$

Remark 2.5*.* The associativity formula immediately implies that if $\mathrm{e}_{HK}(R) < 2$ then $\mathrm{Assh}(R)$ contains one element, and if this is the prime P then the P-primary component of 0 is P. Thus, if R is unmixed and $\mathrm{e}_{HK}(R) < 2$ then R is a domain.

We will also need the following technical notion:

Definition 2.6. Let $(R, \mathfrak{m}, k)$ be a local ring of positive characteristic p and let $J \subset I$ be $\mathfrak{m}$-primary ideals. Define the *star length* of J in I, $\lambda^*(I/J)$, to be the minimum length n of a sequence of ideals

$$J^* = I_0 \subset I_1 \subset ... \subset I_n = I^*$$

such that, for each k, $I_{k+1} = (I_k, x_k)^*$ for some element x_k with $\mathfrak{m}x \subset I_k$.

The definition of star length was introduced by Hanes [6], who also noted some of the basic properties of the star length function:

Proposition 2.7. *Let* $J \subset I$ *be any* $\mathfrak{m}$*-primary ideals of a local ring* $(R, \mathfrak{m}, k)$ *of prime characteristic* $p > 0$*. Then*

a) $\lambda^*(I/J) \leq \lambda(I/J)$ *and* $\lambda^*(I/J) = \lambda^*(I^*/J^*)$*;*

b) $\mathrm{e}_{HK}(J) \leq \mathrm{e}_{HK}(I) + \lambda^*(I/J)\, \mathrm{e}_{HK}(R)$*. Moreover,* $\mathrm{e}_{HK}(J) \leq \lambda^*(R/J)\, \mathrm{e}_{HK}(\mathfrak{m})$*.*

Definition 2.8. Let $I \subseteq J$ be two $\mathfrak{m}$-primary ideals in $(R, \mathfrak{m}, k)$ and M a finitely generated R-module. The Hilbert-Kunz multiplicity of I on M is $\mathrm{e}_{HK}(I; M) = \lim_{q \to \infty} \dfrac{1}{q^d}\, \lambda(M/I^{[q]}M)$. The relative Hilbert-Kunz multiplicity of I and J on M is $\mathrm{e}_{HK}(I, J; M) = \mathrm{e}_{HK}(I; M) - \mathrm{e}_{HK}(J; M)$.

When $M = R$, we simply drop it from the notation.

The following Proposition offers a natural characterization of strong F-regularity in terms of the relative Hilbert-Kunz multiplicity.

Proposition 2.9. *Let* $(R, \mathfrak{m}, k)$ *be an excellent local ring. Then the following are equivalent:*

1) *R is strongly F-regular.*
2) $\inf\{\mathrm{e}_{HK}(I,J) | I \subsetneq J\} > 0$.
3) $\inf\{\mathrm{e}_{HK}(I,(I,x)) | I$ *is $\mathfrak{m}$-primary, irreducible and x is a socle element modulo I*$\} > 0$

Proof. By [2], Theorem 0.2, R is strongly F-regular if and only if $\liminf \lambda(R/0 :_{F^e(E)} u^q)/q^d > 0$ (the theorem is stated there for F-finite rings, but the proof works in the excellent case too).

We first show that (1) implies (3). Let $I \subseteq R$ be irreducible and $\mathfrak{m}$-primary. Say x is a socle element modulo I. There is then an injection $R/I \hookrightarrow E$ sending x to u. Applying Frobenius gives a map $R/I^{[q]} \to F^e(E)$ sending x^q to u^q, from which it is clear that $I^{[q]} : x^q \subseteq 0 :_{F^e(E)} u^q$. Hence $\mathrm{e}_{HK}(I,(I,x)) \geq \liminf \lambda(R/0 :_{F^e(E)} u^q)/q^d > 0$.

To see that (3) implies (2) we note that it suffices to take $J = (I, y)$ for a socle element y modulo I. In this case we can embed $R/I \hookrightarrow R/I_1 \oplus \cdots R/I_t$ where each I_n is irreducible, and $y \mapsto (x, 0, \ldots, 0)$ where x is the socle element modulo I_1. It is then clear, after applying Frobenius, that $\mathrm{e}_{HK}(I,J) \geq \mathrm{e}_{HK}(I_1,(I_1,y))$.

Clearly (2) implies (3).

Suppose that (3) holds, but R, of dimension d, is not strongly F-regular. Choose $c \in R^0$ such that $cu^q = 0$ in $F^e(E)$ for all q. Then $\dim R/cR = d-1$. Let $\mathrm{e}_1 = \mathrm{e}_{HK}(R)$ and $\mathrm{e}_2 = \mathrm{e}_{HK}(R/cR)$. Fix q_0 such that $\lambda(R/(c, \mathfrak{m}^{[q_0]})) \leq (\mathrm{e}_2 + 1)q_0^{d-1}$. Since $cu^{q_0} = 0$, we can choose an irreducible ideal I with socle representative x such that $cx^{q_0} \in I^{[q_0]}$. Since $\mathfrak{m}x \subseteq I$ we see that for all q, $(\mathfrak{m}^{[q_0]}, c)^{[q]} x^{q_0 q} \subseteq I^{[q_0 q]}$. Hence

$$\lambda\left(\frac{R}{I^{[q_0 q]} : x^{q_0 q}}\right) \leq \lambda\left(\frac{R}{(\mathfrak{m}^{[q_0]}, c)^{[q]}}\right) \leq \lambda\left(\frac{R}{(\mathfrak{m}^{[q_0]}, c)}\right) \mathrm{e}_1 q^d \leq (\mathrm{e}_2 + 1) q_0^{d-1} \mathrm{e}_1 q^d.$$

Dividing by $(q_0 q)^d$ and taking limits shows that $\mathrm{e}_{HK}(I,(I,x)) \leq \dfrac{(\mathrm{e}_2+1)\,\mathrm{e}_1}{q_0}$. Since q_0 may be taken arbitrarily large (this will change the ideal I), we have contradicted the assumption (3). ∎

In later sections we will often want to be able to obtain a minimal reduction of an ideal in a local ring. The standard technique is to pass to a faithfully flat extension. The next remark merely summarizes several well-known facts that we will need.

Remark 2.10. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p.

a) Assume that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local homomorphism with $\mathfrak{n} = \mathfrak{m}S$ (e.g., completion).
   i) For any $\mathfrak{m}$-primary ideal $I \subseteq R$, $\mathrm{e}_{HK}(IS) = \mathrm{e}_{HK}(I)$. In particular, $\mathrm{e}_{HK}(S) = \mathrm{e}_{HK}(R)$.
   ii) If R is CM with canonical module ω_R then S is CM with canonical module $\omega_S = \omega_R \otimes S$. In particular, if R is generically Gorenstein, so that ω_R may be taken to be an ideal of R, then S is also generically Gorenstein.
b) Let Y be an indeterminate over R and set $S = R[Y]_{\mathfrak{m}R[Y]}$. Then S is faithfully flat with maximal ideal extended from R, and residue field isomorphic to $k(Y)$ (so infinite). Part (a) then applies.
c) If R has infinite residue field then $\mathfrak{m}$ has a minimal reduction $\mathbf{x} = x_1, \ldots, x_d$ with $\mathrm{e}(R) = \mathrm{e}((\mathbf{x})) = \mathrm{e}_{HK}((\mathbf{x}))$, and if R is CM then the common value is also equal to $\lambda(R/(\mathbf{x}))$. If R has finite residue field then parts (a) and (b) may be applied in order to change to the situation that the residue field is infinite.

3. Hilbert-Kunz lower bounds via duality

This section will present various lower bounds for the Hilbert-Kunz multplicity of a ring $(R, \mathfrak{m}, k)$ of fixed multiplicity and dimension.

We observe the following:

Lemma 3.1. *If $(R, \mathfrak{m})$ is local of dimension d, $I \subseteq J$ are $\mathfrak{m}$-primary ideals, $c \in R^0$, and M is finitely generated over R, then* $\lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{J^{[q]}M}{(cJ^{[q]}+I^{[q]})M}\right) = 0$.

Proof. Let $n = \mu(M)$ and $k = \mu(J)$. Then one can see that there is a surjection

$$\left(\frac{R}{cR}\right)^{nk} \to \frac{J^{[q]}M}{(cJ^{[q]}+I^{[q]})M} \to 0,$$

and the kernel contains $I^{[q]}\left(\frac{R}{cR}\right)^{nk}$.

Since $\dim R/cR = d-1$, we note that $\lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\left(\frac{R}{cR+I^{[q]}}\right)^{nk}\right) = 0$, which implies our statement. □

We are now ready to formulate an important technical result that will lead to a series of Corollaries which are the main goal of this section.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a generically Gorenstein Cohen-Macaulay ring with system of parameters $\mathbf{x} = x_1, \dots, x_d$. Let $\mathrm{e} = \lambda(R/(\mathbf{x}))$. Suppose that $I \supseteq (\mathbf{x})$ and set $J = (\mathbf{x})^* : I$.*

Let $a = \lambda^*(R/I)$, $f = \lambda^*(R/J)$, and $b = \lambda(((\mathbf{x})^* : I)/(\mathbf{x}))$. Then $\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}}{f+a}$, so, in particular,

$$\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}}{\mathrm{e} - b + a}$$

Proof. Let ω_R be the canonical module of R. Since R is generically Gorenstein, there is a short exact sequence $0 \to \omega_R \to R \to C \to 0$, where $\dim C < \dim(R) = d$.

We can then complete R since by Remark 2.10(a,ii) the canonical ideal completes well, e will be unchanged, and a and f can only decrease. Hence we may assume that R has a q_0-weak test element c.

From the additivity of the Hilbert-Kunz multiplicity on short exact sequences and the fact that $\dim C < d$, we get $\mathrm{e}_{HK}(I;\omega_R) = \mathrm{e}_{HK}(I;R)$. This implies that $\mathrm{e}_{HK}(I_1, I_2;\omega_R) = \mathrm{e}_{HK}(I_1, I_2)$ whenever $I_1 \subseteq I_2$ are $\mathfrak{m}$-primary ideals.

Since $\mathbf{x}$ is a s.o.p., $\mathrm{e}_{HK}((\mathbf{x})) = \mathrm{e}((\mathbf{x})) = e$. Also, $\mathrm{e}_{HK}((\mathbf{x})) = \mathrm{e}_{HK}(J) + \mathrm{e}_{HK}((\mathbf{x}), J)$.

By Proposition 2.7, $\mathrm{e}_{HK}(J) \leq \lambda^*(R/J)\,\mathrm{e}_{HK}(R) = f\,\mathrm{e}_{HK}(R)$.

The heart of the proof is seeing that $\mathrm{e}_{HK}((\mathbf{x}), J;\omega_R) \leq a\,\mathrm{e}_{HK}(R)$, and hence $\mathrm{e}_{HK}((\mathbf{x}), J) = \mathrm{e}_{HK}((\mathbf{x}), J;\omega_R) \leq a\,\mathrm{e}_{HK}(R)$.

Indeed, $\omega_R/(\mathbf{x})^{[q]}\omega_R$ is the canonical module of the Artinian ring $R/(\mathbf{x})^{[q]}$, so it is injective over it. By Matlis duality over complete Artinian rings, we get that $\lambda(R/I^{[q]}) = \lambda\left(\mathrm{Hom}(R/I^{[q]}, \omega_R/(\mathbf{x})^{[q]}\omega_R)\right)$.

Note that by the definition of J, and the fact that c is a q_0-weak test element, we have $cJ^{[q]} \subseteq (\mathbf{x})^{[q]} : I^{[q]}$ for all $q \geq q_0$. Thus for all $q \geq q_0$

$$\frac{(cJ^{[q]} + (\mathbf{x})^{[q]})\omega_R}{(\mathbf{x})^{[q]}\omega_R} \subseteq \frac{(\mathbf{x})^{[q]}\omega_R : I^{[q]}}{(\mathbf{x})^{[q]}\omega_R} = \mathrm{Hom}\left(\frac{R}{I^{[q]}}, \frac{\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right).$$

By the equality

$$\lambda\left(\frac{J^{[q]}\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right)=\lambda\left(\frac{J^{[q]}\omega_R}{(cJ^{[q]}+(\mathbf{x})^{[q]})\omega_R}\right)+\lambda\left(\frac{(cJ^{[q]}+(\mathbf{x})^{[q]})\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right),$$

Lemma 3.1, Matlis duality, and Proposition 2.7, we get

$$\mathrm{e}_{HK}((\mathbf{x}),J;\omega_R)\le \mathrm{e}_{HK}(I;\omega_R)=\mathrm{e}_{HK}(I)\le a\,\mathrm{e}_{HK}(R).$$

In conclusion,

$$e=\mathrm{e}_{HK}((\mathbf{x}),R)=\mathrm{e}_{HK}(J,R)+\mathrm{e}_{HK}((\mathbf{x}),J)\le f\,\mathrm{e}_{HK}(R)+a\,\mathrm{e}_{HK}(R)=(f+a)\,\mathrm{e}_{HK}(R),$$

proving the first inequality stated in the conclusion.

The last inequality follows from the fact that $f=\lambda^*(R/J)\le\lambda(R/J)=e-b$. ☐

The next corollary shows how useful Theorem 3.2 can be when R is not Gorenstein. Note that the lower bound for $\mathrm{e}_{HK}(R)$ does not depend on the dimension of the ring.

Corollary 3.3. *Let $(R,\mathfrak{m})$ be a generically Gorenstein Cohen-Macaulay ring of CM-type t and multiplicity $\mathrm{e}=\mathrm{e}(R)$. Then*

$$\mathrm{e}_{HK}(R)\ge\frac{\mathrm{e}}{\mathrm{e}-t+1}.$$

Proof. By Remark 2.10, we may assume that the residue field is infinite, so there exists a s.o.p. $\mathbf{x}$ with $\mathrm{e}(R)=\lambda(R/(\mathbf{x}))$. Now apply Theorem 3.2 with $I=\mathfrak{m}$ (so $a=1$ and $b=t$). ☐

Corollary 3.4. *Let $(R,\mathfrak{m})$ be a non-regular, generically Gorenstein, Cohen-Macaulay ring of minimal multiplicity. Then $\mathrm{e}_{HK}(R)\ge\mathrm{e}(R)/2$.*

Proof. By the structure theorem of Sally, [12], R has type $t=e-1$. Hence $\mathrm{e}_{HK}(R)\ge\mathrm{e}(R)/(\mathrm{e}(R)-(\mathrm{e}(R)-1)+1)=\mathrm{e}(R)/2$. ☐

Corollary 3.5. *Let $(R,\mathfrak{m},k)$ be a local Cohen-Macaulay ring of characteristic p and dimension d. If $\mathrm{e}_{HK}(R)<\dfrac{\mathrm{e}}{\mathrm{e}-1}$, then R is Gorenstein and F-regular (so strongly F-regular, if R is also excellent).*

Proof. We may assume that R is not regular. Then $\mathrm{e}_{HK}(R)<\dfrac{\mathrm{e}}{\mathrm{e}-1}<2$, so by the associativity formula, Proposition 2.4, R is a domain. If R is not Gorenstein then the type of R, t, is at least 2. Theorem 3.2 then shows that $\mathrm{e}_{HK}\ge\dfrac{\mathrm{e}}{\mathrm{e}-t+1}\ge\dfrac{\mathrm{e}}{\mathrm{e}-1}$. Thus R is Gorenstein, and we are done by Theorem 1.2. ☐

We can now state the desired generalization of Theorem 1.2. The improvement is replacing "F-rational" by an appropriate form of "F-regular" in the conclusion.

Corollary 3.6. *Let $(R,\mathfrak{m},k)$ be an unmixed ring of characteristic p and $\dim(R)=d\ge 2$. Assume either that R is the image of a Cohen-Macaulay ring, or is excellent. If $\mathrm{e}_{HK}(R)\le 1+\max\{1/d!,1/\mathrm{e}(R)\}$, then R is weakly F-regular. If R is excellent then R is strongly F-regular.*

Proof. If R is excellent, then we can complete while preserving the assumption that R is unmixed. Thus, in either case R is the image of a Cohen-Macaulay ring, so we may apply Theorem 1.2, to see that R is Cohen-Macaulay. Therefore its completion is Cohen-Macaulay, and once we have completed the proof of the excellent case, strongly F-regular. Since the map $R \to \widehat{R}$ is pure, R is weakly F-regular.

Assume now that R is excellent. In this case we may pass to the completion without affecting the Hilbert-Kunz multiplicity. Hence by Theorem 1.2 we may assume that R is Cohen-Macaulay.

If R is not strongly F-regular, then $\mathrm{e}_{HK}(R) \geq \mathrm{e}/(\mathrm{e}-1) > 1+1/\mathrm{e}$. So, $1+\frac{1}{d!} \geq \mathrm{e}_{HK} > 1+\frac{1}{\mathrm{e}}$ which implies that $\mathrm{e} > d!$, and therefore $\mathrm{e}_{HK}(R) \geq \dfrac{\mathrm{e}}{d!} > \dfrac{d!+1}{d!}$, which is a contradiction. $\square$

It should be remarked that Corollaries 3.5 and 3.6 are closely related to recent unpublished results of D. Hanes who independently proved in particular that under the assumptions of Corollary 3.6, the ring R is Gorenstein and F-regular.

We get some interesting results from Theorem 3.2 when we can apply it to Gorenstein rings which are not F-regular.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a Gorenstein ring of dimension d and embedding dimension $v = \mu(\mathfrak{m})$. If either R or $\hat{R}$ is not F-regular, then*

$$\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}}{\mathrm{e}-v+d}.$$

Proof. Non F-regularity passes to the completion, so we may assume that R is complete. By Remark 2.10, we may assume that the residue field is infinite, and $\mathbf{x}$ is s.o.p. such that $\mathrm{e}(R) = \lambda(R/(\mathbf{x}))$, while preserving the non-weak-F-regularity of R. If u denotes a socle element modulo $(\mathbf{x})$ then $u \in (\mathbf{x})^*$. We can now apply Theorem 3.2 with $I = \mathfrak{m}$. Then $a = \lambda^*(R/\mathfrak{m}) = 1$, and $b = \lambda(((\mathbf{x})^* : \mathfrak{m})/(\mathbf{x})) \geq v-d+1$, since in the 0-dimensional Gorenstein ring $S = R/(\mathbf{x})$, $(u)S : \mathfrak{m}S = 0 : \mathfrak{m}^2 S$, and $\lambda(0 : \mathfrak{m}^2 S) = \lambda(S/\mathfrak{m}^2 S) = v-d+1$. The corollary now follows. $\square$

Remark 3.8. It is possible, in "pathological" cases (e.g., non-excellent) for a ring to be weakly F-regular, while its completion is not. Loepp and Rotthaus, construct such an example, which is Gorenstein, in [11]. Corollary 3.7 applies in this case.

Corollary 3.7 can be improved, and this improvement, while interesting on its own, will also be useful in section 4. We first establish some notation. For a graded ring $G = \oplus_{i\geq 0} G_i$, finitely generated over G_0 artinian, let $k_i = \lambda(G_i)$. If $\lambda(G) < \infty$, let $r = \max\{i | G_i \neq 0\}$. We note that if $(S, \mathfrak{n})$ is a Gorenstein ring of dimension 0, and G is the associated graded ring of S at $\mathfrak{n}$, then G_r is generated by the image of the socle element, so $k_r = 1$.

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a non F-regular Gorenstein local ring of dimension d and multiplicity $\mathrm{e} = \mathrm{e}(R)$, and let $\mathbf{x} = x_1, \ldots, x_d$ be a minimal reduction of $\mathfrak{m}$. Let G be the associated graded ring of $R/(\mathbf{x})$ (at its maximal ideal), and let r and k_i for $0 \leq i \leq r$ be as above. Then*

$$\mathrm{e}_{HK}(R) \geq \max_{1\leq i\leq r} \left\{ \frac{\mathrm{e}}{\mathrm{e}-k_i} \right\}.$$

As a consequence $\mathrm{e}_{HK}(R) \geq \max\left\{ \dfrac{\mathrm{e}}{\mathrm{e}-\frac{\mathrm{e}-2}{r-1}}, \dfrac{r}{r-1} \right\}$.

Proof. Since R is not F-regular, if u denotes a socle element modulo $(\mathbf{x})$, then $u \in (\mathbf{x})^*$. Thus $(\mathbf{x}) : \mathfrak{m} = (u, \mathbf{x}) \subseteq (\mathbf{x})^*$. We may then apply Theorem 3.2 with $I = \mathfrak{m}^j + (\mathbf{x})$ and

$J = (\mathbf{x})^* : I \supseteq (u, \mathbf{x}) : \mathfrak{m}^j = ((\mathbf{x}) : \mathfrak{m}) : \mathfrak{m}^j = (\mathbf{x}) : \mathfrak{m}^{j+1}$. In this case, $\lambda(R/I) = \sum_{i=0}^{j-1} k_i$ and $\lambda(R/J) = \mathrm{e} - \lambda(J/(\mathbf{x})) \leq \mathrm{e} - \lambda(R/(\mathfrak{m}^{j+1} + (\mathbf{x}))) = \mathrm{e} - (\sum_{i=0}^{j} k_i)$ (Matlis duality and the fact that $J \subset (\mathbf{x}) : \mathfrak{m}^{j+1}$ gives the inequality). Hence

$$\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}}{\lambda^*(R/(\mathfrak{m}^j + (\mathbf{x}))) + \lambda^*(R/J)} \geq \frac{\mathrm{e}}{\sum_{i=0}^{j-1} k_i + \mathrm{e} - (\sum_{i=0}^{j} k_i)} = \frac{\mathrm{e}}{\mathrm{e} - k_j}.$$

Since $k_0 = k_r = 1$, some $k_i \geq \dfrac{\mathrm{e} - 1 - 1}{r - 1}$, thus $\mathrm{e}_{HK}(R) \geq \dfrac{\mathrm{e}}{\mathrm{e} - \frac{\mathrm{e}-2}{r-1}}$. If $r \leq \mathrm{e}/2$ then $\dfrac{\mathrm{e}}{\mathrm{e} - \frac{\mathrm{e}-2}{r-1}} \geq \dfrac{r}{r-1}$.

If R is a hypersurface, then $r = \mathrm{e}$, each $k_i = 1$, and $\mathrm{e}_{HK}(R) \geq \dfrac{\mathrm{e}}{\mathrm{e} - 1} = \dfrac{r}{r-1}$.

Otherwise, $k_1 \geq 2$, and the only case left is when $r > \mathrm{e}/2$. But it is straightforward to see that in this case, $\dfrac{\mathrm{e}}{\mathrm{e} - 2} > \dfrac{r}{r-1}$. □

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a non F-regular Gorenstein ring of dimension $d > 1$. Then $\mathrm{e}_{HK}(R) \geq \dfrac{d}{d-1}$.*

Proof. By Remark 2.10 we may assume that R is complete with infinite residue field and that $\mathbf{x}$ is a s.o.p. which is a minimal reduction of $\mathfrak{m}$.

Let G and r be as in the proof of Corollary 3.9. The result of Corollary 3.9 suffices if $r \leq d$. So we may assume that $r > d$. By the Briançon-Skoda Theorem, $\mathfrak{m}^d \subseteq (\mathbf{x})^*$.

Let $\mathrm{e} = \mathrm{e}(R)$ be the multiplicity. It is easy to see that for any integer $n \leq \mathrm{e}$, $\dfrac{\mathrm{e}}{\mathrm{e} - n} \geq \dfrac{d}{d-1}$ if and only if $n \geq \mathrm{e}/d$. By Corollary 3.9, we are done if some $k_i \geq \mathrm{e}/d$, so assume that each $k_i < \mathrm{e}/d$.

Let $I = \mathfrak{m}^{d-1}$. Then $(\mathbf{x})^* : I \supseteq \mathfrak{m}$ (by the Briançon-Skoda Theorem), so by Theorem 3.2, $\mathrm{e}_{HK}(R) \geq \dfrac{\mathrm{e}}{\mathrm{e} - (\mathrm{e} - 1) + 1 + k_1 + \cdots + k_{d-2}} = \dfrac{\mathrm{e}}{2 + k_1 + \cdots + k_{d-2}}$. Since each $k_i < \mathrm{e}/d$ we get $\mathrm{e}_{HK}(R) > \dfrac{\mathrm{e}}{2 + (d-2)(\mathrm{e}/d)}$, and the right hand side is easily seen to be at least $\dfrac{d}{d-1}$ provided that $\mathrm{e} \geq 2d$.

The only case left is if $\mathrm{e} < 2d$. Then $2d > e > dk_i$ for all k_i implies that each $k_i = 1$, i.e., R is a hypersurface, and $\mathrm{e} = r + 1$ (and, recall, $r > d$). Say $\mathfrak{m} = (z, \mathbf{x})$ minimally. By the Briançon-Skoda theorem, $z^d \in (\mathbf{x})^*$, so $(\mathbf{x})^* : \mathfrak{m} \supseteq (z^d, \mathbf{x}) : z \supseteq (z^{d-1}, \mathbf{x})$. Applying Theorem 3.2 with $I = \mathfrak{m}$ gives $\mathrm{e}_{HK}(R) \geq \dfrac{\mathrm{e}}{1 + d - 1} = \dfrac{\mathrm{e}}{d} \geq \dfrac{d+2}{d} \geq \dfrac{d}{d-1}$ (the last inequality uses that $d > 1$). □

4. Radical extensions and comparison of Hilbert-Kunz multiplicities

In this section, we will develop a technique that, in conjuction with the results obtained so far, will give a lower bound for the Hilbert-Kunz multiplicity of unmixed non-regular local rings of dimension d that depends only on d, and is strictly greater than 1, hence showing that $\epsilon(d) > 0$. This answers one of the open questions mentioned in the Introduction.

We will need to use a result of Watanabe and Yoshida ([15], Theorem 1.6). For a domain R we use $Q(R)$ for the fraction field of R, and R^+ for the absolute integral closure of R (i.e., an integral closure of R in an algebraic closure of $Q(R)$).

Theorem 4.1. *Let $(R,\mathfrak{m})\hookrightarrow(S,\mathfrak{n})$ be a module-finite extension of local domains. Then for every $\mathfrak{m}$-primary ideal I of R,*

$$\mathrm{e}_{HK}(I) = \frac{\mathrm{e}_{HK}(IS)}{[Q(S):Q(R)]}\cdot[S/\mathfrak{n}:R/\mathfrak{m}]. \tag{4.1}$$

We need the following definition.

Definition 4.2. Let $(R,\mathfrak{m})$ be a domain. Let $z\in\mathfrak{m}$, and let n be a positive integer. Let $v\in R^+$ be any root of $f(X)=X^n-z$. We call $S=R[v]$ a *radical extension* for the pair R and z.

Remark 4.3. Whenever S is radical for R and z, then $b:=[Q(S):Q(R)]\le n$. Assume also that R is normal and z is a minimal generator of $\mathfrak{m}$. Then in fact, $b=n$. To see this we need to show that $f(X)=X^n-z$ is the minimal polynomial for $v=z^{1/n}$ over R. Let $g(X)$ be the minimal polynomial of v over $Q(R)[X]$. Since R is normal, $g(X)\in R[X]$. The constant term of $g(X)$ is in $\mathfrak{m}$, since z is not a unit. Then $g(X)|f(X)$ in $R[X]$. Say $f(X)=g(X)h(X)$. Then the constant term of $h(X)$ is a unit (or else $z\in\mathfrak{m}^2$). But mod $\mathfrak{m}$, $g(X)h(X)=X^n$, so in fact, $h(X)$ is a unit constant.

In what follows $\mathfrak{n}$ will denote the maximal ideal of S, whenever S is local. Note that if R is a complete domain and $z\in\mathfrak{m}$, then S must be local.

Theorem 4.4. *Let $(R,\mathfrak{m})$ be a complete local domain of positive prime characteristic having algebraically closed residue field. Let $\mathbf{x}=x_1,\dots,x_d$ be a system of parameters, and set $\mathrm{e}=\mathrm{e}_{HK}((\mathbf{x}))=\mathrm{e}((\mathbf{x}))$, and $a=\lambda(R/(\mathbf{x})^*)$.*

Let $z\in\mathfrak{m}-(\mathbf{x})^*$ be a minimal generator and let $v\in R^+$ be any nth root of z. Let $S=R[v]$ be a radical extension for R and z and denote the maximal ideal of S by $\mathfrak{n}$. Let $b=[Q(S):Q(R)]$. Then

$$\mathrm{e}_{HK}(R)\ge\frac{b(n-1)\,\mathrm{e}+n\,\mathrm{e}_{HK}(S)}{b(a(n-1)+1)}.$$

In the case that $b=n$ this inequality simplifies to

$$\mathrm{e}_{HK}(R)\ge\frac{(b-1)\,\mathrm{e}+\mathrm{e}_{HK}(S)}{a(b-1)+1}.$$

Remark 4.5. If we denote $\mathrm{e}_{HK}(R)=1+\delta_R$ and $\mathrm{e}_{HK}(S)=1+\delta_S$, then the above is equivalent to

$$\delta_R\ge\frac{(n-1)(\mathrm{e}-a+1)-b+1+\delta_S}{b+(n-1)(a-1)},$$

and if $b=n$ this simplifies to $\delta_R\ge\dfrac{(\mathrm{e}-a)+\delta_S}{a(b-1)+1}$.

For the proof of Theorem 4.4 it is helpful to note the following

Remark 4.6. Let $I\subseteq R$ be an ideal in a local ring $(R,\mathfrak{m})$ and $v\in\mathfrak{m}$ an element such that (I,v) is $\mathfrak{m}$-primary. Then for all $n\ge1$, $\mathrm{e}_{HK}((I,v^n),(I,v^{n-1}))\ge\mathrm{e}_{HK}((I,v^{n+1}),(I,v^n))$.

To see this, we observe that for all q, $(I, v^n)^{[q]} : v^{(n-1)q} \subseteq (I, v^{n+1})^{[q]} : v^{nq}$, so

$$\begin{aligned} \mathrm{e}_{HK}((I, v^n), (I, v^{n-1})) &= \lim_{q\to\infty} \frac{1}{q^d} \lambda \left(\frac{(I, v^{n-1})^{[q]}}{(I, v^n)^{[q]}} \right) = \lim_{q\to\infty} \frac{1}{q^d} \lambda \left(\frac{R}{(I, v^n)^{[q]} : v^{(n-1)q}} \right) \\ &\geq \lim_{q\to\infty} \frac{1}{q^d} \lambda \left(\frac{R}{(I, v^{n+1})^{[q]} : v^{nq}} \right) = \mathrm{e}_{HK}((I, v^{n+1}), (I, v^n)). \end{aligned}$$

Proof. Let us write $\mathrm{e}_{HK}(R) = 1 + \delta_R$ and $\mathrm{e}_{HK}(S) = 1 + \delta_S$.

Let $(\mathbf{x})^* = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{a-2} \subsetneq (I_{a-2}, z) = I_{a-1} = \mathfrak{m} \subsetneq R$ be a saturated filtration, and let $w_i \in R$ be an element whose image generates I_i/I_{i-1} (in particular, take $w_{a-1} = z$).

We can then filter $(\mathbf{x})^* S \subseteq S$ by filling in each $I_{i-1}S \subseteq I_i S$ with

$$I_{i-1}S \subseteq (I_{i-1}, v^{n-1}w_i)S \subseteq \cdots \subseteq (I_{i-1}, vw_i)S \subseteq I_i S$$

(where we allow that some of the containments may be equalities).

From Theorem 4.1, and the fact that $[S/\mathfrak{n} : R/\mathfrak{m}] = 1$ ($R/\mathfrak{m}$ is algebraically closed), we have that $\mathrm{e}_{HK}(\mathfrak{m}S) = b\,\mathrm{e}_{HK}(\mathfrak{m}R)$.

Thus, $\mathrm{e}_{HK}(\mathfrak{m}S, \mathfrak{n}) = b(1 + \delta_R) - (1 + \delta_S) = (b-1) + b\delta_R - \delta_S$.

By Remark 4.6, for each $1 \leq j < n$, $\mathrm{e}_{HK}((v^j, \mathfrak{m}S), (v^{j-1}, \mathfrak{m}S) \geq \mathrm{e}_{HK}((v^{j+1}, \mathfrak{m}S), (v^j, \mathfrak{m}S)$. Hence, $\mathrm{e}_{HK}((\mathfrak{m}S), (v^{n-1}, \mathfrak{m}S) \leq \dfrac{\mathrm{e}_{HK}(\mathfrak{m}S, \mathfrak{n})}{n-1} \leq \dfrac{(b-1) + b\delta_R - \delta_S}{n-1}$.

Set $y := \mathrm{e}_{HK}((\mathfrak{m}S), (v^{n-1}, \mathfrak{m}S)$. Consider the filtration

$$\mathfrak{m}S = (z, I_{a-2})S \supseteq (zv, I_{a-2})S \supseteq (zv^2, I_{a-2})S \supseteq \cdots \supseteq (zv^{n-1}, I_{a-2})S \supseteq I_{a-2}S. \tag{4.2}$$

Remark 4.6 applies to each containment in equation 4.2, so each relative Hilbert-Kunz multiplicity is at most $\mathrm{e}_{HK}((zv, I_{a-2})S, \mathfrak{m}S) = \mathrm{e}_{HK}((v^{n+1}, I_{a-2})S, (v^n, I_{a-2})S) \geq y$. Adding them all up we get that $\mathrm{e}_{HK}(I_{a-2}S, \mathfrak{m}S) \leq ny$.

From this it follows that $\mathrm{e}_{HK}(I_{a-2}S, \mathfrak{m}S) \leq n \left(\dfrac{b - 1 + b\delta_R - \delta_S}{n-1} \right)$.

Using Theorem 4.1 to go back to R we have $\mathrm{e}_{HK}(I_{a-2}, \mathfrak{m}) \leq \left(\dfrac{n(b-1+b\delta_R - \delta_S)}{b(n-1)} \right)$. Each of the other $a-1$ terms in the filtration of $(\mathbf{x})^* \subseteq R$ have relative Hilbert-Kunz multiplicity at most $1 + \delta_R$, so we get the inequality

$$\left(\frac{n(b-1+b\delta_R - \delta_S)}{b(n-1)} \right) + (a-1)(1+\delta_R) \geq \mathrm{e}_{HK}((\mathbf{x})^*) = \mathrm{e}\,. \tag{4.3}$$

But $\mathrm{e}_{HK}(R) = 1 + \delta_R$, while $\mathrm{e}_{HK}(S) = 1 + \delta_S$, so equation 4.3 yields

$$\mathrm{e}_{HK}(R) \geq \frac{b(n-1)\,\mathrm{e} + n\,\mathrm{e}_{HK}(S)}{b(a(n-1)+1)}.$$

□

Corollary 4.7. *Let $(R, \mathfrak{m})$ be an F-regular complete local ring of positive prime characteristic having algebraically closed residue field. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters and minimal reduction for $\mathfrak{m}$, and let $\mathrm{e} = \mathrm{e}(R) = \mathrm{e}_{HK}((\mathbf{x})) = \mathrm{e}((\mathbf{x}))$,*

Let $z \in \mathfrak{m} - (\mathbf{x})$ be a minimal generator and let $v \in R^+$ be any nth root of z. Let $S = R[v]$ be a radical extension for R and z and denote its maximal ideal of S by $\mathfrak{n}$. Then

$$\mathrm{e}_{HK}(R) \geq \frac{(n-1)\,\mathrm{e} + \mathrm{e}_{HK}(S)}{\mathrm{e}(n-1)+1}.$$

Proof. By Remark 4.3, $b = [Q(S) : Q(R)] = n$. Since R is F-regular, $(\mathbf{x}) = (\mathbf{x})^*$. Hence one can apply Theorem 4.4 together with the observation that $a = \mathrm{e}$. □

Remark 4.8. Corollary 4.7 can be substantially improved, but the proof is considerably more difficult. We will give improved versions in a later paper, along with improved estimates of lower bounds for $\epsilon(d)$.

Corollary 4.9. *Let $(R, \mathfrak{m})$ be a complete local domain of positive prime characteristic having algebraically closed residue field. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters, and set $\mathrm{e} = \mathrm{e}_{HK}((\mathbf{x})) = \mathrm{e}((\mathbf{x}))$, and $a = \lambda(R/(\mathbf{x})^*)$.*

Asssume that $(\mathbf{x})^* \neq \mathfrak{m}$. Then

$$\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}+1}{a+1}.$$

Proof. Take any minimal generator of $\mathfrak{m}$ not in $(\mathbf{x})^*$ and adjoin a square root of it from R^+. Then apply the previous theorem and note that $2 = n \geq b$ and $\mathrm{e}_{HK}(S) \geq 1$, so

$$\mathrm{e}_{HK}(R) \geq \frac{b(n-1)\mathrm{e}+b}{b(a(n-1)+1)} = \frac{(n-1)\mathrm{e}+1}{a(n-1)+1} = \frac{\mathrm{e}+1}{a+1}.$$ □

Remark 4.10. Assume that $(R, \mathfrak{m})$ is CM of type t, I a parameter ideal such that $I \subsetneq I^* \subsetneq \mathfrak{m}$. Then $e = \lambda(R/I)$, and $t = \lambda((I : \mathfrak{m})/I)$.

The two ideals I^* and $(I : \mathfrak{m})$ are incomparable in many cases.

However, if $I^* \subset (I : \mathfrak{m})$, then $t \leq e - a$ and $\dfrac{e}{e-t+1} \leq \dfrac{e+1}{a+1}$. So, the above corollary improves an earlier result of ours.

We now begin a construction that will yield a lower bound for the Hilbert-Kunz multiplicity of Gorenstein, F-regular, non-regular local rings.

So assume that $(R, \mathfrak{m})$ is a Gorenstein F-regular local ring of multiplicity $\mathrm{e} = \mathrm{e}(R) > 1$. Note that R must be a normal domain. We may complete and by Theorem 3.4 of [1], extend the residue field to assume that it is algebraically closed. Let $\mathbf{x} = x_1, \ldots, x_d$ be a minimal reduction of $\mathfrak{m}$, so that $\lambda(R/(\mathbf{x})) = \mathrm{e}$.

Remark 4.11. Let R and $\mathbf{x}$ be as above, and suppose z, $v = z^{1/n}$ and S are as in Corollary 4.7. Assume, moreover, that $x_1, \ldots, x_{d-1}, z$ is also a minimal reduction of $\mathfrak{m}$. Let $u \in \mathfrak{m}$ denote a socle element modulo $(x_1, \ldots, x_{d-1}, z)$. Then

a) $x_1, \ldots, x_{d-1}, v$ is a minimal reduction of $\mathfrak{n}$ (the maximal ideal of S),
b) u is still a socle element modulo $(x_1, \ldots, x_{d-1}, v)S$, and
c) S is Gorenstein and $\mathrm{e}(S) = \mathrm{e}(R)$.

Proof. Let $\mathbf{x}_{d-1} = x_1, \ldots, x_{d-1}$.

a) If $\mathfrak{m} = (\mathbf{x}_{d-1}, z) + J$, where $\mu(J) = \mu(\mathfrak{m}) - d$, then $\mathfrak{n} = (\mathbf{x}_{d-1}, v)S + JS$. Since J is integral over $(\mathbf{x}_{d-1}, z)R$, the ideal JS is integral over $(\mathbf{x}_{d-1}, z)S$, and hence over the larger ideal $(\mathbf{x}_{d-1}, v)S$. This suffices to show (a).

b) If $u \in (\mathbf{x}_{d-1}, v)S$ then $u \in (\mathbf{x}_{d-1}, z)S \cap R \subseteq ((\mathbf{x}_{d-1}, z)R)^* = (\mathbf{x}_{d-1}, z)R$, a contradiction. With J as in part (a), we have $\mathfrak{n}\, u = ((\mathbf{x}_{d-1}, v)S + JS)u \subseteq JuS + (\mathbf{x}_{d-1}, v)S \subseteq (\mathbf{x}_{d-1}, z)S + (\mathbf{x}_{d-1}, v)S \subseteq (\mathbf{x}_{d-1}, v)S$. Thus u is a socle element.

c) By Remark 4.3, $X^n - z$ is the minimal polynomial of z over R. Hence S is R-free, so flat, with Gorenstein closed fiber. Thus S is Gorenstein. Then $\mathrm{e}(S) = \lambda_S(S/(\mathbf{x}_{d-1}, v)) = \dfrac{1}{n}\lambda_S(S/(\mathbf{x}_{d-1}, v^n)) = \dfrac{1}{n}\lambda_S(S/(\mathbf{x}_{d-1}, z)) = \lambda_R(R/(\mathbf{x}_{d-1}, z)) = \mathrm{e}(R)$. □

Let $d = \dim R$ and $k = \mu(\mathfrak{m}) - d > 1$.

Note that $\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}(R)}{d!}$. Hence whenever $\mathrm{e}(R) \geq d! + 1$, we have that $\mathrm{e}_{HK}(R) \geq 1 + \frac{1}{d!}$.

Therefore, if we want to produce a lower bound for $\mathrm{e}_{HK}(R)$ in terms of only d, there is no harm if we fix $\mathrm{e}(R) = \mathrm{e}$ as well. This is so because we can take the minimum of the lower bounds obtained for fixed d, e while letting e vary between 2 and $d!$.

The residue field of R is infinite, and so we may pick $y_1, \ldots, y_{d+1} \in \mathfrak{m} - \mathfrak{m}^2$ in general position, and therefore, assume that each d-element subset is a minimal reduction of $\mathfrak{m}$ (see, for example, Theorem 8.6.6 of [9], and the comment after it). Let u denote a socle element modulo $(y_1, \ldots, y_d)R$, and let $r = \max\{\, i \mid u \in \mathfrak{m}^i + (y_1, \ldots, y_d)R\}$. We must have $r \geq 2$, since no minimal generator of $\mathfrak{m}$ can be a socle element in a Gorenstein ring which is not a hypersurface. Set $n = \lceil d/r \rceil$ (so $nr \geq d$).

Let $R_0 = R$, and for each $i \geq 1$, let $v_i = y_i^{1/n}$, and set $R_i = R_{i-1}[v_i]$. For each i, write $\mathrm{e}_{HK}(R_i) = 1 + \delta_i$.

For a given $i \geq 1$, if R_{i-1} is F-regular, we may apply Corollary 4.7 to $R_{i-1} \subseteq R_i$ with $\mathbf{x} = v_1, \ldots, v_{i-1}, y_{i+1}, \ldots, y_{d+1}$ and $z = y_i$ ($\mathbf{x}$ is a minimal reduction of R_{i-1} by Remark 4.11(a)). Also, by Remark 4.11(b), u is a socle element modulo $(v_1, \ldots, v_i, y_{i+1}, \ldots, y_d)R_i$. We get, noting that the multiplicity stays the same,

$$1 + \delta_i \geq 1 + \frac{1}{\mathrm{e}(R_{i-1})(n-1)+1}\delta_{i-1} = 1 + \frac{1}{\mathrm{e}(R_0)(n-1)+1}\delta_{i-1}. \tag{4.4}$$

We claim that for some $i \leq d$, R_i is not F-regular. If not, then R_d is F-regular. Let $\mathfrak{m}_{R_0} = (y_1, \ldots, y_d) + J$ with $\mu(J) = \mu(\mathfrak{m}) - d$. It is then clear that $\mathfrak{m}_{R_d} = (v_1, \ldots, v_d) + JR_d$. By the Briançon-Skoda Theorem $\overline{\mathfrak{m}_{R_d}^d} \subseteq ((v_1, \ldots, v_d)R_d)^*$, so

$$\begin{aligned} u \in (JR_0)^r &\subseteq \overline{(y_1, \ldots, y_d)^r R_d} = \overline{(y_1^r, \ldots, y_d^r)R_d} = \overline{(v_1^{rn}, \ldots, v_d^{rn})R_d} \\ &\subseteq \overline{(v_1^d, \ldots, v_d^d)R_d} \subseteq ((v_1, \ldots, v_d)R_d)^* = (v_1, \ldots, v_d)R_d \end{aligned}$$

a contradiction to Remark 4.11(b).

Assume then, that $i_0 = \min\{i \mid R_i \text{ is not F-regular}\}$. By Corollary 3.10, $\mathrm{e}_{HK}(R_i) \geq \dfrac{d}{d-1} = 1 + \dfrac{1}{d-1}$. Repeated application of Equation 4.4 yields

$$\mathrm{e}_{HK}(R) = \mathrm{e}_{HK}(R_0) \geq 1 + \left(\frac{1}{\mathrm{e}(R)(n-1)+1}\right)^{i_0-1} \frac{1}{d-1}.$$

We are now in position to state and prove the main result of the paper.

Theorem 4.12. *Let $(R, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic p and dimension d that is either a homomorphic image of a Cohen-Macaulay ring or excellent. If R is not regular then*

$$\mathrm{e}_{HK}(R) \geq 1 + \frac{1}{(d-1)\cdot(d!(d/2-1)+1)^{d-1}}.$$

Proof. First of all, we can assume that $\mathrm{e}_{HK}(R) < 1 + \frac{1}{d!}$ and hence by Theorem 3.6 we have that R is Cohen-Macaulay and F-rational.

We can make a faithfully flat extension so we can assume that k is algebraically closed and that R is also complete.

If $\mathrm{e} \geq d! + 1$, then $\mathrm{e}_{HK}(R) \geq \frac{\mathrm{e}(R)}{d!} \geq 1 + \frac{1}{d!}$. So, we can assume that $\mathrm{e} \leq d!$.

By Corollaries 3.3 and 3.5, we can assume that R is in fact Gorenstein and strongly F-regular.

Now we are in position to apply the technique described just above the statement of the Theorem and, noting that $n \le d/2$ we obtain that

$$\mathrm{e}_{HK}(R) \ge 1 + \frac{1}{((d-1)\cdot(\mathrm{e}(d/2-1)+1)^{d-1}} \ge 1 + \frac{1}{(d-1)\cdot(d!(d/2-1)+1)^{d-1}}.$$

□

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