THE GROWTH RECURRENCE AND GELFAND-KIRILLOV BASE OF THE ORDINARY CUSP

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ABSTRACT. We introduce the Gelfand-Kirillov base for a numerical semigroup ring over the prime field of characteristic p, where p is prime, and show its existence for the semigroup ring of the ordinary cusp by establishing a growth recurrence with respect to Frobenius.

1. INTRODUCTION

Let R be a commutative ring of characteristic p, where p is a prime integer. Let $F : R \to R$ be the Frobenius homomorphism over R, $F : R \to R$, $F(r) = r^p$, for all $r \in R$. In this paper we will be concerned with the Frobenius skew polynomial ring $R[f; F]$, which is a ring that is a free left R-module generated by $1, f, \ldots, f^n, \ldots$ such that $fr = r^p f$, for all $r \in R$. This ring appears naturally in commutative algebra in positive characteristic, as discussed in works by many authors, such as [5, 8] to name a few. In our recent paper [2], as well as in [1], the concepts of growth recurrence and Gelfand-Kirillov base for $R[f; F]$ were studied. In this paper, we formulate these notions for numerical semigroup rings and we show their existence for the numerical semigroup ring of the ordinary cusp. The computations in this work provide a glimpse of the intricacies of the general case.

1.1. Growth recurrence and Gelfand-Kirillov base.

The Gelfand-Kirillov dimension of an algebra A over a field K is a well studied concept introduced in relation to the growth of the powers of a generating set for A, see [3]. It extends the notion of Krull dimension of a commutative algebra to the noncommutative setting. More recently, the concept of growth recurrence and Gelfand-Kirillov base has been introduced in relation to the Frobenius skew polynomial ring of a commutative ring in [2], where it was studied carefully for the case when R is a polynomial ring in finitely many indeterminates over $\mathbb{F}_p$. Let us review briefly the main points for finitely generated K-algebras, as in [2].

Let A be a K-algebra. We take V to be a finite dimensional generating subspace of A, that is, V is the K-vector space spanned by some elements $v_1, v_2, \cdots, v_h$ of A which are algebra generators of A over K. Now let $V^0 = K$ and then for any $n \in \mathbb{N}$ we let V^n be the K-vector subspace spanned by products of n elements from $\{v_1, v_2, \cdots, v_h\}$. Let $A_n = k + V + V^1 + \cdots + V^n$. Note that $A = \bigcup_{n=0}^{\infty} A_n$. Denote $d_V(n) = \dim_K(A_n)$.

Now let $K = \mathbb{F}_p$ and let $A = R[f; F]$, where R is a numerical semigroup K-algebra (here, the assumption $K = \mathbb{F}_p$ makes $R[f; F]$ a K-algebra). Specifically, let $\mathbb{N} = \mathbb{Z}_{\geq 0}$, S be a numerical semigroup of $\mathbb{N}^n$, and $R = K[t^a : a \in S]$.

Every numerical semigroup has a unique minimal generating set, so let us denote $\{a_1, \dots, a_m\}$ the minimal generators of S.

Let U be the K-subspace of R generated $t^{a_1}, \dots, t^{a_m}$ and recall that the Frobenius homomorphism on R is denoted by F. We will refer to U as a generating subspace for R. Note that the set $V = U + Kf$ is a K-generating subspace for the algebra $R[f; F]$ and its growth function d_V will often denoted by $d_F(-)$.

Inspired by [2], we have the following definition.

Definition 1.1. If $\{d_F(n)\}_n$ satisfies a recurrence $(\mathfrak{R})$, then we refer to the recurrence as a *growth recurrence* for R with respect to F. If the growth recurrence is linear, then its characteristic equation will be called *the growth equation* for R with respect to F given by $(\mathfrak{R})$.

Finally, it is interesting to measure the growth rate of the sequence $\{d_F(n)\}_n$.

Definition 1.2. With the notations introduced above, we call the *Gelfand-Kirillov (GK) base of* R with respect to F the following number

$$\mathrm{GKbase}_F(R) := \inf\{\lambda \in \mathbb{R}_{>0} : d_F(n) = O(\lambda^n)\}$$

if it is finite.

2. The Growth Recurrence for the Ordinary Cusp

In this section, we will study the existence of a growth recurrence and GK base for the Frobenius skew polynomial ring $A = R[f; F]$ where $R = K[t^2, t^3]$, with t indeterminate, is the numerical semigroup ring corresponding to the cusp $x^3 - y^2 = 0$.

2.1. **Skew Monomials.** Since F is the Frobenius endomorphism, we have $ft = t^p f$. We refer to an element $z \in A$ as a **skew monomial** when z is a product of elements from the set $\{t^2, t^3, f\}$. Note that elements that are products of elements from $\{t^2, t^3\}$ are also considered skew monomials when viewed as elements of A.

Definition 2.1. A skew monomial $z \in A$ is in **irreducible form** when it is written as a product with the fewest possible elements from t^2, t^3, f. The number of such terms if considered the *length* of the skew monomial. Obviously, every skew monomial has a length. If z has length n, we will write $\mathrm{length}(z) = n$.

As an example if $p = 2$, $t^4 f = f t^2$ has length 2. And $t^2 f t^3 = f t^4 = f(t^2)^2$ has length 3 although it can be written in irreducible form in two different ways, both of length 3.

2.2. **Direct sum decomposition and growth recurrence.** We let V be the finite dimensional generating K-subspace of $A = R[f;F]$ generated by t^2, t^3, f. Then V^i is spanned by skew monomials that are products of i elements in t^2, t^3, f for $i \geq 1$ with $V^0 = K$. Let $A_n = \sum_{i=0}^{n} V^i$ and note that $A = \bigcup_{n=0}^{\infty} A_n$. We let U denote the generating space of R given by t^2, t^3.

Remark 2.2. Note that the sum decomposition for A_n does not define a graded object. Confusion could arise after the discussion of lengths of the skew monomials that could lead to the expectation of a graded object. However, due to the multiplication in the Frobenius skew polynomial ring, this is not the case. It is also important to notice that the V^i do not form a direct sum decomposition of A_n because $V^i \cap V^j$ is not necessarily trivial. For example, $t^{2p}f = ft^2$ appears in V^{p+1} and V^2. This is the motivation for the next definition.

Definition 2.3. Let $A = R[f;F]$. We define the set W_i to be the K-subspace spanned by irreducible skew monomials of length i. Hence, W_i is spanned by the subset of generators for A_i that are in irreducible form and have length exactly i.

Proposition 2.4. *Let $R = K[t^2, t^3]$ and $A = R[f;F]$ be the Frobenius skew polynomial ring over R. If W_i is the K-subspace spanned by irreducible skew monomials of length i, then $A_n = \bigoplus_{i=0}^{n} W_i$.*

Proof. By definition, every element in A_n is the sum of elements from the subspaces V^i where $0 \leq i \leq n$. The skew monomial generators from V^i can be written in irreducible form, and hence every such element of V^i is in some W_j where $j \leq i$. Hence, $V^j \subset \sum_{i=0}^{n} W_i$ for all $0 \leq j \leq n$. So, $A_n = \sum_{i=0}^{n} W_i$.

To finish we only need to show that

$$W_i \cap \sum_{j \neq i, j=0,\ldots,n} W_j = 0.$$

From the definition of W_i, this assertion is clear since each subspace W_i has basis formed by the irreducible skew monomials of length i and elements in $\sum_{j \neq i} W_j$ are unique K-linear combinations of irreducible skew monomials of length different than i. □

Now let us consider the function $d_V(n) = \dim_K(A_n)$. Since the decomposition of A_n for any n by $\sum_{i=0}^{n} W_i$ is a direct sum decomposition, we can compute $d_V(n) = d_{U,F}(n)$ by the following sum:

$$d_V(n) = d_{U,F}(n) = \dim_K \left(\sum_{i=0}^{n} V^i \right) = \sum_{i=0}^{n} \dim_K(W_i)$$

Hence, to compute $d_V(n)$ we must first develop a method to compute $\dim_K(W_i)$ for any $0 \le i \le n$. This can be done by counting the irreducible skew monomials of length i for any $i \le n$, which is achieved in the next section.

2.3. **Computing the Growth Recurrence.** We now take the direct sum decomposition constructed in Section 2.2 and use it find a recurrence for $d_V(n)$. We can do this by first finding a recurrence for $\dim_K(W_i)$ for all i. By Proposition 2.4 and the comment just following it, we can then determine $d_V(n) = \dim_K(A_n)$.

Remark 2.5. An important observation is that any skew monomial in irreducible form containing f in the product must lead with a skew monomial of form $t^M f$, where M belongs to the semigroup $\langle 2, 3\rangle$. The cardinality of the set of irreducible skew monomials of the form $t^M f$ will allow us to count the irreducible skew monomials containing an f for any length i.

Proposition 2.6. *A term t^m with $m \in S = \langle 2, 3\rangle$ is irreducible of length $d \ge 1$, where $m = 3d - i$, with $i = 0, 1,$ or 2 and $m = 2i + 3(d - i)$. Note that length of t^m is $d = \lceil m/3 \rceil$.*

Proof. Let i, j nonnegative such that $2i + 3j = m, i + j = d$, and d is minimal. These two equations imply i, j are unique with these properties.

Write $m = 2i + 3j = 2i' + 3j'$ with nonnegative i, j, i', j'. Then $2(i - i') = 3(j' - j)$, so $i = i' + 3k$ and $j' = j + 2k$, so $i + j = i' + j + 3k = i' + j' + k$. So, if $i + j$ is minimal, then $k < 0$, which means that $i = 0, 1, 2$. The rest follows now at once. □

Proposition 2.7. *Let d be an integer greater than or equal to 1. A skew monomial is written in irreducible form $t^m f$ of length $d+1$ if and only if $m = 3d-i = 2i+3(d-i) = 2i + 3j$, with $0 \le i \le 2$, $j = d - i$, $m = 2i + 3j \le 2p + 1$, $m = 2i + 3j \ne 2p$. So, if $m \le 2p + 1, m \ne 2p$ then length of $t^m f$ is $\lceil m/3 \rceil + 1$.*

Proof. Clearly, we have that $(t^2)^i(t^3)^j f$ is irreducible of length $d + 1$ implies $(t^2)^i(t^3)^j$ is irreducible of length d and so $0 \le i \le 2$ and $j = d - i$, by Proposition 2.6.

Let $m = 2i + 3j$. If $m \ge 2p + 2$, then $t^m f = t^{m-2p} f t^2$. So, the length of t^{m-2p} is $d-1$, hence $m-2p = 3(d-1)-i'$, form some $i' = 0, 1, 2$. But then $m = 2p+3d-3-i'$ and so $i' = 1$ and $p = 2$. Otherwise, $t^m f$ is reducible.

Now assume $p = 2$ and $m = 4 + 3(d - 1) - 1 = 3d$ such that $3d \ge 2p + 2 = 6$. Then if $d \ge 2$ we see that $t^{3d} = t^{3d-6} f t^3$ and $t^{3(d-2)}$ is in fact of length $d - 2$ which makes $t^{3d-6} f t^3$ of length smaller than $d + 1$. This forces $d = 1$ and $m = 3$, but then $m < 2p + 2$.

If $m = 2p \ge 4$, we claim that $t^m f$ is reducible, since, otherwise, $t^m f = f t^2$ which is of length 2 forcing t^m to have length one and hence $m = 1, 2,$ or 3, impossible.

Finally, let us argue that if $m \le 2p + 1, m \ne 2p$, then $t^m f$ is irreducible. Since $f t^2 = t^{2p} f$, then we have $t^m f = t^{m-2p} f t^2$, but since $m - 2p \le 1 \notin S$, this reduction does not work in our ring. □

Definition 2.8. Let $k \geq 1$. We define

$$\mathcal{B}_k = \{(i,j) \in \mathbb{Z}^2_{\geq 0} : \text{length}((t^2)^i (t^3)^j f) = i + j + 1 = k\}.$$

Denote by β_k the cardinality of $\mathcal{B}_k$.

Proposition 2.9. *Let $k \geq 1$ and i, j nonnegative integers. Then $(i,j) \in \mathcal{B}_k$ if and only if $i + j + 1 = k$, $2i + 3j \leq 2p + 1, 2i + 3j \neq 2p$ and $0 \leq i \leq 2$. Also, $\mathcal{B}_k \neq \emptyset$ if and only if $3k \leq 2p + 6$, or equivalently $k \leq \lfloor \frac{2p}{3} \rfloor + 2$.*

1. *Moreover, $\mathcal{B}_1 = \{(0,0)\}$ and the corresponding skew monomial is f, while $\mathcal{B}_2 = \{(0,1),(1,0)\}$ and corresponding skew monomials are $t^2 f, t^3 f$. In conclusion, $\beta_1 = 1, \beta_2 = 2, \beta_k \leq 3$.*
2. *For $3 \leq k, 3k \leq 2p + 2$, $\mathcal{B}_k = \{(0, k-1), (1, k-2), (2, k-3)\}$ and $\beta_k = 3$;*
3. *If $3k = 2p + 3$, then $\mathcal{B}_k = \{(1, k-2), (2, k-3)\}$ and $\beta_k = 2$;*
4. *If $3k = 2p + 4$, then $\mathcal{B}_k = \{(0, k-1), (2, k-3)\}$ and $\beta_k = 2$;*
5. *If $3k = 2p + 5$, then $\mathcal{B}_k = \{(1, k-2)\}$ and $\beta_k = 1$:*
6. *If $3k = 2p + 6$, then $\mathcal{B}_k = \{(2, k-3)\}$ and $\beta_k = 1$.*

Proof. Applying Proposition 2.7 to $(t^2)^i(t^3)^j f$, we conclude that $\text{length}((t^2)^i(t^3)^j f) = \lceil \frac{2i+3j}{3} \rceil + 1$ and $0 \leq i \leq 2$. Additionally, $2i+3j \leq 2p+1, 2i+3j \neq 2p$. Also, $i+j+1 = k$.

Note that $\lceil \frac{2i+3j}{3} \rceil + 1 = i+j+1$ is always satisfied if $i = 0,1,2$. Hence, $\mathcal{B}_k \subseteq \{(0, k-1), (1, k-2), (2, k-3)\}$. In fact, we can use the conditions $2i+3j \leq 2p+1, 2i+3j \neq 2p$ to list precisely what elements are in $\mathcal{B}_k$.

Clearly, $\mathcal{B}_1 = \{(0,0)\}$, while $\mathcal{B}_2 = \{(0,1),(1,0)\}$

For $k \geq 3$, $(0, k-1) \in \mathcal{B}_k$ if and only if $3k \leq 2p+4, 3k \neq 2p+3$; $(1, k-2) \in \mathcal{B}_k$ if and only if $3k \leq 2p+5, 3k \neq 2p+4$; $(2, k-3) \in \mathcal{B}_k$ if and only if $3k \leq 2p+6, 3k \neq 2p+5$. □

Theorem 2.10. (1) *Assume that there exist nonnegative integers n, i_l, j_l, k_l such that $n > 0, k_l > 0$, $i_l + j_l > 0$, for all $l = 1, \ldots, n$, and nonnegative integers i_0, j_0 such that z is written in irreducible form as*

$$z = (t^2)^{i_n}(t^3)^{j_n} f^{k_n} \cdots (t^2)^{i_1}(t^3)^{j_1} f^{k_1} (t^2)^{i_0}(t^3)^{j_0}.$$

Then for all $l = 1, \ldots, n$, we have $2i_l + 3j_l \leq 2p + 1$, $2i_l + 3j_l \neq 2p$.

In fact, if we let $M_l = 2i_l + 3j_l$ and $d_l = \lceil M_l/3 \rceil$, then $0 \leq i_l \leq 2$ and $j_l = d_l - i_l$, for all $l = 1, \ldots, n$

(2) *Let $p \geq 7$. Using the notations above, then for every $1 \leq l$ such that $k_l = 1$we have that either $2i_l + 3j_l \leq p + 1$ or $2i + 3j_l \neq p$.*

(3) *Let $p \geq 7$. Then any skew monomial has a unique irreducible form where, for $1 \leq l$, if $k_l = 1$, $M_l \leq p + 1, M_l \neq p$ and, otherwise, $M_l \leq 2p + 1, M_l \neq 2p$.*

Proof. (1) Consider the element z in irreducible form written as in the hypothesis of the theorem. Note that the length of z equals $i_0 + j_0 + \sum_{l=1}^{n}(i_l + j_l + k_l)$.

Assume that $2i_l + 3j_l \geq 2p + r$, for some $1 \leq l \leq n$ with $r \geq 2$, or $r = 0$. Then $t^{2i_l+3j_l} f = t^r f t^2$. Recall that the length of t^r is $\lceil r/3 \rceil$, by Proposition 2.6.

Hence the element $z = (t^2)^{i_n}(t^3)^{j_n} f^{k_n} \cdots (t^2)^{i_l}(t^3)^{j_l} f^{k_l} \cdots (t^2)^{i_1}(t^3)^{j_1} f^{k_1} (t^2)^{i_0}(t^3)^{j_0}$ can have the term $(t^2)^{i_l}(t^3)^{j_l} f^{k_l}$ swapped out by $t^r f t^2 f^{k_l - 1}$. This shows that the length of z is bounded above by

$$i_0 + j_0 + \sum_{h=1, h\neq l}^{n} (i_h + j_h + k_h) + \lceil r/3 \rceil + 1 + k_l - 1 < i_0 + j_0 + \sum_{h=1}^{n} (i_h + j_h + k_h),$$

because

$$\lceil \frac{r}{3} \rceil = \lceil \frac{2i_l + 3j_l - 2p}{3} \rceil < i_l + j_l,$$

when $p \geq 2$. This contradicts the fact that the length of z is $i_0 + j_0 + \sum_{l=1}^{n} (i_l + j_l + k_l)$.

So the irreducible form of z must have $2i_l + 3j_l \leq 2p + 1, 2i_l + 3j_l \neq 2p$, for all $l = 1, \ldots, n$. Additionally, a similar argument as above shows that $x^{2i_l + 3j_l}$ must be in irreducible form and hence Proposition 2.6 applies.

(2) Let $1 \leq l$ such that $k_l = 1$. In that case, $t^{M_l} f t^{M_{l-1}} = t^{M_l - p} f t^{1 + M_{l-1}}$. Note that $\lceil \frac{M_l}{3} \rceil - \lceil \frac{M_l - p}{3} \rceil \geq 2$ since $p \geq 7$, while $\lceil \frac{1 + M_{l-1}}{3} \rceil - \lceil \frac{M_{l-1}}{3} \rceil \leq 1$, which means that applying the above identity would lower the length of z, unless $M_l \leq p + 1$ and $M_l \neq p$.

(3) This is a direct consequence of second part proved above.

□

Example 2.11. Let $p = 5$. Note that $z = t^9 f t^3 f = t^4 f t^4 f$ can be written in two distinct irreducible forms (of length 6).

Definition 2.12. Let $p \geq 7$. Let z be a monomial in R such that f appears in its expression. Write z in irreducible form. The leading form of z are the unique i, j such that $z = (t^2)^i (t^3)^j f \cdot z'$ is its irreducible form. Note that by Theorem 2.10 the numbers i, j as well as the element z' is unique. We will write $\text{lead}(z) = (t^2)^i \cdot (t^3)^j$.

Proposition 2.13. *Let $p \geq 7$. Let z be a monomial in R such that f appears in its expression. If $z = (t^2)^i (t^3)^j f^k \cdot z'$ and $2i + 3j \leq p + 1, 2i + 3j \neq p, k \geq 1$, then* $\text{lead}(z) = (t^2)^i (t^3)^j$.

Proof. This is clear since no reduction affecting the leading term can be performed using $t^p f = ft$. □

Theorem 2.14. *Let $p \geq 7$. Let $z, t^m f$ be elements in R. Assume that f is a factor in z, z has length l, and $m \leq 2p + 1, m \neq 2p$. Then*

(1) If $\text{lead}(z) = f$, then length of $t^m f \cdot z$ is $l + \lceil m/3 \rceil + 1$;
(2) If $\text{lead}(z) \neq f$, then the length of $t^m f \cdot z$ is $l + \lceil m/3 \rceil + 1$ if and only if $m \leq p + 1, m \neq p$.

Proof. (1) This statement is clear since $t^m f$ has length $\lceil m/3 \rceil + 1$, and $x^m f \cdot z$ cannot be reduced using $t^p f = ft$ since $t \notin R$.

(2) We can write $z = t^{m'} \cdot fz'$, with $m' \geq 2$, $m' = 2i + 3j \leq 2p + 1, m' \neq 2p$ and $i = 0, 1$, or 2. If $m \leq p + 1, m \neq p$, then Theorem 2.10 applies. Conversely, if $m \geq p + 2$ or $m = p$, then, by the same Theorem 2.10, $t^m f \cdot z$ is not in irreducible form and, hence, its length must be less than $l + \lceil m/3 \rceil + 1$.

□

Definition 2.15. The following definitions and notations provide the building blocks to determine $\dim_K(W_d)$.

- We will denote $\dim_K(W_d)$ by C_d. Hence, C_d is equal to the number of monomials of length d.
- Let $W_{i,j,d}$ denote the K-vector space generated the skew monomials of length d with leading form $(t^2)^i(t^3)^j f$. Let $\dim W_{i,j,d} = c_{i,j,d}$.
- We will make the convention that $C_i = 0$ for negative i and $c_{i,j,d} = 0$ for $d \leq i + j$.

Proposition 2.16. *Let $d \geq 1$ and $p \geq 7$. Let C_d =$dim_K(W_d)$ and $c_{i,j,d}$ be defined as in 2.15. Then we have*

$$C_d = \sum_{(i,j)\in\mathcal{B}_k,\ 1\leq k\leq d} c_{i,j,d} + 3.$$

Proof. By Theorem 2.10 we know that every z of length d, containing a factor of f, must have $\mathrm{lead}(z) = (t^2)^i(t^3)^j$, so $z \in W_{i,j,d}$ and $(i,j) \in \mathcal{B}_k$ and $1 \leq k \leq d$. If z has no factor of f, then there are exactly three such terms by Proposition 2.6. □

Example 2.17. Let $R = K[t^2, t^3]$ where $K = \mathbb{F}_7$, and $A = R[f; F]$ the Frobenius skew polynomial ring over R. Let $\underline{a} = (1, 2)$ and consider $c_{\underline{a},7}$. Then every monomial counted in $c_{\underline{a},7}$ is of the form $(t^2)(t^3)^2 fz = t^8 fz$.

We can list all the skew monomials in $W_{1,2,7}$ and we get $t^8ft^2ft^3, t^8ft^3ft^2, t^8t^2ft^2, t^8ft^4f$, $t^8ft^5f, t^8ft^6f, t^8f^2t^4, t^8f^2t^5, t^8f^2t^6, t^8ft^2f, t^8t^3f, t^8ff^2t^2, t^8ff^2t^3, t^8ff^3, t^8ft^8, t^8ft^8, t^8ft^7$.

Hence, comparing this list to $t^8f \cdot z$, we see that z is in irreducible form and, hence, it is counted in C_3. On the other hand, we can attach t^8f to any skew monomial in W_3, and the resulting skew monomial will still be in irreducible form, and hence, is among the elements of $W_{\underline{a},7}$. Thus, $c_{\underline{a},7} = C_3$.

The conclusion of the example above holds in general and is shown in the next propositions. This result will be useful in developing a recurrence to compute C_d. Note that we are only counting skew monomials that contain an f in the product with the values $c_{\underline{a},d}$. The skew monomials that do not contain an f are counted in the same manner as counting monomials of degree d in R.

Proposition 2.18. *Let $p \geq 7$. Let i, j, d be nonnegative integers such that $i+j+1 \leq d$ and $(t^2)^i(t^3)^j f$ has length $i + j + 1$. Assume $2i + 3j \leq p + 1, 2i + 3j \neq p$.*

Then there exists a one-to-one correspondence between $W_{i,j,d}$ and $W_{d-(i+j+1)}$. In particular $c_{i,j,d} = C_{d-(i+j+1)}$.

Proof. Consider the map
$$T : W_{d-(i+j+1)} \longrightarrow W_{i,j,d},$$
defined by $T(z) = (t^2)^i(t^3)^j f \cdot z$. This function is well defined by Theorem 2.14. It is clearly injective because the ring R is an integral domain. For the surjectivity, we apply Theorem 2.10. □

Proposition 2.19. *Let i, j, d be nonnegative integers such that $i + j + 2 \leq d$ and $(t^2)^i(t^3)^j f$ has length $i + j + 1$. Assume that $2i + 3j = p$ or $p + 2 \leq 2i + 3j \leq 2p + 1$. Assume further that $2i + 3j \neq 2p$.*

Then there exists a one-to-one correspondence between $W_{i,j,d}$ and $W_{d-(i+j+2)}$. In particular $c_{i,j,d} = C_{d-(i+j+2)}$.

Proof. We now consider the map
$$T : W_{d-(i+j+2)} \longrightarrow W_{i,j,d},$$
defined by $T(z) = (t^2)^i(t^3)^j f^2 \cdot z$. This function is well defined by Theorem 2.14. It is clearly injective because the ring R is an integral domain. For the surjectivity, we again apply Theorem 2.10 and Theorem 2.14. □

Definition 2.20. Let $k \geq 1$. We define
$$\Gamma_k = \{(i, j) \in \mathcal{B}_k : 2i + 3j \leq p + 1, 2i + 3j \neq p\}.$$
Similarly,
$$\Delta_k = \{(i, j) \in \mathcal{B}_k : p + 2 \leq 2i + 3j \leq 2p + 1 \text{ and } 2i + 3j \neq 2p, \text{ or } 2i + 3j = p\}.$$
Denote the cardinality of Γ_k by γ_k and the cardinality of Δ_k by δ_k. Note that, when $\mathcal{B}_k \neq \emptyset$, the sets Γ_k and Δ_k form a partition for $\mathcal{B}_k$.

If $(i, j) \in \Gamma_k$, then $t^{2i+3j}f$ has length k, so $k - 1 = \lceil \frac{2i+3j}{3} \rceil$ by Proposition 2.7 and so $1 \leq k \leq \lceil \frac{p+1}{3} \rceil + 1$. Similarly, for $(i, j) \in \Delta_k$, we have $\lceil \frac{p+2}{3} \rceil + 1 \leq k \leq \lceil \frac{2p+1}{3} \rceil + 1$, unless $k = \lceil \frac{p}{3} \rceil + 1$.

We can refine these inequalities as in the following proposition, for $k \geq 3$.

Proposition 2.21. *Let $k \geq 3$. If $k > \lfloor \frac{2p}{3} \rfloor + 2$, then $\gamma_k = \delta_k = 0$. Additionally,*

(1) If $k \geq \lceil \frac{p+1}{3} \rceil + 2$, then $\gamma_k = 0$. If $k \leq \lfloor \frac{p+2}{3} \rfloor$, then $\delta_k = 0$.
(2) If $3 \leq k$ and $3k \leq p + 2$, then $\gamma_k = 3$ and $\delta_k = 0$.
(3) If $p + 7 \leq 3k \leq 2p + 2$, then $\gamma_k = 0$ and $\delta_k = 3$.

Proof. We have seen in Proposition 2.9 that $\mathcal{B}_k = \emptyset$ if and only if $k > \lfloor \frac{2p}{3} \rfloor + 2$ and that $\mathcal{B}_k = \{(0, k-1), (1, k-2), (2, k-3)\}$ for $3 \leq k, 3k \leq 2p + 2$. Additionally, $(i, j) \in \Gamma_k$ if and only if $2i + 3j \leq p + 1, 2i + 3j \neq p$, so by direct inspection $(0, k - 1) \in \Gamma_k$ if and only if $3k \leq p + 4, 3k \neq p + 3$; $(1, k - 2) \in \Gamma_k$ if and only if $3k \leq p + 5, 3k \neq p + 4$; $(2, k - 3) \in \Gamma_k$ if and only if $3k \leq p + 6, 3k \neq p + 5$. So, if $3k \leq p + 2$, then $\gamma_k = 3$ and $\delta_k = 0$. Also, if $3k \geq p + 7$, then $\gamma_k = 0$.

Similarly, $(0, k-1) \in \Delta_k$ if and only if $3k = p+3$ or $p+5 \leq 3k \leq 2p+4, 3k \neq 2p+3$; $(1, k-2) \in \Delta_k$ if and only if $3k = p+4$ or $p+6 \leq 3k \leq 2p+5, 3k \neq 2p+4$; $(2, k-3) \in \Delta_k$ if and only if $3k = p+5$ or $p+7 \leq 3k \leq 2p+6, 3k \neq 2p+5$. So if $p+7 \leq 3k \leq 2p+2$, then $\delta_k = 3$. □

It is useful to see the values for these numbers in an example. We will record below the values obtained for $p = 7$ and $p = 11$.

k	γ_k	δ_k	β_k
1	1	0	1
2	2	0	2
3	3	0	3
4	1	2	3
5	0	3	3
6	0	2	2
7	0	0	0
8	0	0	0
9	0	0	0
10	0	0	0

k	γ_k	δ_k	β_k
1	1	0	1
2	2	0	2
3	3	0	3
4	3	0	3
5	2	1	3
6	0	3	3
7	0	3	3
8	0	3	3
9	0	1	1
10	0	0	0

TABLE 1. the cases $p = 7$ and, respectively, $p = 11$

Theorem 2.22. *Let $p \geq 7$ and $d \geq 1$. Let $C_d = \dim_K(W_d)$. Then we have*

$$C_d = \sum_{1 \leq k \leq d-1,\ \Gamma_k \neq \emptyset} \gamma_k \cdot C_{d-k} + \sum_{1 \leq k \leq d-1,\ \Delta_k \neq \emptyset} \delta_k \cdot C_{d-k-1} + \beta_d + 3.$$

or, equivalently,

$$C_d = \sum_{1 \leq k \leq d} (\gamma_k + \delta_{k-1}) \cdot C_{d-k} + \delta_d + 3. \tag{2.3.1}$$

Proof. For a positive integer $1 \leq k \leq d$, let $(i, j) \in \mathcal{B}_k$. Note that $i + j + 1 = k \leq d$. If $k = d$, then $c_{i,j,d} = 1$. If $k \leq d-1$ and $(i, j) \in \Gamma_k \subseteq \mathcal{B}_k$, then $c_{i,j,d} = C_{d-k}$ by Proposition 2.18. Similarly, for $k \leq d-1$ and $(i, j) \in \Delta_k \subseteq \mathcal{B}_k$, then $c_{i,j,d} = C_{d-k-1}$ by Proposition 2.19. Finally, apply Proposition 2.16 and we obtain the statement of the theorem. Recall that $C_0 = 1$ and $\beta_d = \gamma_d + \delta_d$.

□

Corollary 2.23. *Let $p \geq 7$. Let $A = R[f; F]$ where $R = \mathbb{Z}_p[t^2, t^3]$, with t indeterminate.*

(1) *Let* $d \geq \lfloor \frac{2p}{3} \rfloor + 3$. *Then*

$$C_d = \sum_{1 \leq k \leq \lfloor \frac{p+2}{3} \rfloor + 1} \gamma_k \cdot C_{d-k} + \sum_{\lfloor \frac{p+2}{3} \rfloor + 2 \leq k \leq \lfloor \frac{2p}{3} \rfloor + 3} \delta_{k-1} \cdot C_{d-k} + 3 \tag{2.3.2}$$

is a linear recurrence relation of order $\lfloor \frac{2p}{3} \rfloor + 3$.

(2) *Let* $d_F(n) = d_n$. *A growth recurrence for* R *with respect to* F *is*

$$d_n = \sum_{1 \leq k \leq \lfloor \frac{p+2}{3} \rfloor + 1} \gamma_k \cdot d_{n-k} + \sum_{\lfloor \frac{p+2}{3} \rfloor + 2 \leq k \leq \lfloor \frac{2p}{3} \rfloor + 3} \delta_{k-1} \cdot d_{n-k} + 3n - K,$$

where K *is a constant. This is a linear recurrence relation of order* $\lfloor \frac{2p}{3} \rfloor + 3$.

(3) *The growth equation for* R *with respect to* F *is*

$$X^N = \sum_{1 \leq k \leq \lfloor \frac{p+2}{3} \rfloor + 1} \gamma_k \cdot X^{N-k} + \sum_{\lfloor \frac{p+2}{3} \rfloor + 2 \leq k \leq \lfloor \frac{2p}{3} \rfloor + 3} \delta_{k-1} \cdot X^{N-k},$$

where $N = \lfloor \frac{2p}{3} \rfloor + 3$.

Proof. For (1), using Proposition 2.21, we can specify the values of k for which γ_k and δ_k are zero and use this information in conjunction with the equation 2.3.1. First start by noting that for p prime with $p \geq 7$, we have $\lceil \frac{p+1}{3} \rceil = \lfloor \frac{p+2}{3} \rfloor$. It is important to note that, by Proposition 2.21, if $k > \lfloor \frac{2p}{3} \rfloor + 2$, then $\gamma_k = \delta_k = 0$; also, $\delta_k \neq 0$ if $k = \lfloor \frac{2p}{3} \rfloor + 2$, which gives the order of the recurrence: to see this, note that if $3k \leq 2p + 6$ then, by Proposition 2.9, $\mathcal{B}_k$ is non empty and so $\beta_k \neq 0$. But Proposition 2.21 applies to $k = \lfloor \frac{2p}{3} \rfloor + 2 \geq \lceil \frac{p+1}{3} \rceil + 2 = \lfloor \frac{p+2}{3} \rfloor + 2$, and hence for $k = \lfloor \frac{2p}{3} \rfloor + 2$, $\gamma_k = 0$ and so $\delta_k = \beta_k \neq 0$, because in general $\beta_k = \gamma_k + \delta_k$.

For (2), simply use that $\sum_{d=0}^{n} C_d = d_n$ in the recurrence obtained in (1). □

This next result will be helpful in computing the Gelfand-Kirillov of the cusp.

Proposition 2.24. *Let* $a_1, \ldots, a_{n_0}$ *and* $\{x_n\}_{n \geq 0}$ *satisfying the relation*

$$x_n = a_1 x_{n-1} + \cdots + a_{n_0} x_{n-n_0}, \tag{2.3.3}$$

for all $n \geq n_0$.

Denote

$$b_l = x_l - \sum_{i=1}^{n_0 - 1} a_i x_{l-i} \tag{2.3.4}$$

for $l = 1, \ldots, n_0 - 1$ *and* $b_0 = x_0$.

Then

$$\sum_{n \geq 0} x_n t^n = \frac{b_0 + b_1 t + \cdots + b_{n_0 - 1} t^{n_0 - 1}}{1 - a_1 t - \cdots - a_{n_0} t^{n_0}}.$$

Proof. Write the equation 2.3.3 for $n \geq n_0$ and multiply it by t^n. We get

$$x_n t^n = a_1 t \cdot x_{n-1} t^{n-1} + \cdots + a_{n_0} t^{n_0} \cdot x_{n-n_0} t^{n-n_0}.$$

Adding all these equations we obtain

$$\sum_{n \geq n_0} x_n t^n = a_1 t \sum_{n \geq n_0 - 1} x_n t^n + \cdots + a_{n_0} t^{n_0} \sum_{n \geq 0} x_n t^n.$$

Denote $F(t) = \sum_{n \geq 0} x_n t^n$ and $F_k(t) = \sum_{0 \leq n \leq k} x_n t^n$ for $k \geq 0$. From the last equation we get

$$F(t) - F_{n_0-1}(t) = a_1 t \cdot F(t) - a_1 t F_{n_0-2}(t) + \cdots + a_{n_0-1} t^{n_0-1} \cdot F(t) - a_{n_0-1} t^{n_0-1} \cdot F_0(t) - a_{n_0} t^{n_0} F(t).$$

Rearranging the terms we get

$$F(t)(1 - a_1 t - \cdots - a_{n_0} t^{n_0}) = F_{n_0-1}(t) - a_1 t F_{n_0-2}(t) - \cdots - a_{n_0-1} t^{n_0-1} \cdot F_0(t).$$

Now, we can use equation 2.3.4 to simplify the right hand side. This results in

$$F(t)(1 - a_1 t - \cdots - a_{n_0} t^{n_0}) = b_0 + b_1 t + \cdots + b_{n_0-1} t^{n_0-1},$$

hence the statement.

□

Theorem 2.25. *Let $p \geq 7$. Let $A = R[f; F]$ where $R = \mathbb{Z}_p[t^2, t^3]$, with t indeterminate. The following assertions are true.*

1. *The growth equation for R with respect to F has a unique simple positive root λ and all other roots have absolute value less than λ.*
2. *There exists nonzero ρ such that $C_d = \rho \cdot \lambda^d + o(\lambda^d)$.*
3. $$\mathrm{GKbase}_F(R) = \lambda.$$

Proof. We have, from Equation 2.3.1,

$$C_d = \sum_{1 \leq k \leq d} (\gamma_k + \delta_{k-1}) \cdot C_{d-k} + \delta_d + 3.$$

Set $a_n = \gamma_n + \delta_{n-1}$ for $n \geq 1$. We can make a change of variable $x_d = C_d - c$, for a constant c to obtain

$$x_d = \sum_{1 \leq k \leq d} (\gamma_k + \delta_{k-1}) \cdot x_{d-k} + \delta_d.$$

Note that $-1 < c < 0$, $x_0 = C_0 - c = 1 - c$ with $1 < 1 - c < 2$. Also, $a_i = \gamma_i$, for $1 \leq i \leq \lfloor \frac{p+2}{3} \rfloor + 1$ and $a_i = \delta_{i-1}$, for $\lfloor \frac{p+2}{3} \rfloor + 2 \leq i \leq \lfloor \frac{2p}{3} \rfloor + 3$, as we have seen in Corollary 2.23 (1) and we have for $d \geq d_0 = \lfloor \frac{2p}{3} \rfloor + 3$:

$$x_d = a_1 x_{d-1} + \cdots + a_{d_0} x_{d-d_0}.$$

Let $F(t) = \sum_{d \geq 0} x_d t^d$. By Proposition 2.24 we end up with

$$F(t) = \frac{(1-c) + \sum_{\lfloor \frac{p+2}{3} \rfloor + 2 \le i \le \lfloor \frac{2p}{3} \rfloor + 3} \delta_{i-1} t^i}{1 - \sum_{1 \le i \le \lfloor \frac{p+2}{3} \rfloor + 1} \gamma_i t^i - \sum_{\lfloor \frac{p+2}{3} \rfloor + 2 \le i \le \lfloor \frac{2p}{3} \rfloor + 3} \delta_{i-1} t^i}.$$

Due to the particular form of the growth equation which has nonzero coefficients of index $N-1, N-2$, where $N = \lfloor \frac{2p}{3} \rfloor + 3$ is the degree of the equation, we can apply Ostrowsky's Theorem in [4]. So the growth equation for R with respect to F has a unique simple positive root λ and all other roots have absolute value less than λ. This gives (1).

Now, for (2), note that due to the remarks above about the growth equation, the denominator in the rational function giving $F(t)$ has a unique simple positive root of smallest magnitude among all its roots. The numerator has only negative roots because its coefficients are nonnegative. In conclusion, the irreducible rational function describing $F(t)$ will have $1/\lambda$ as the unique positive simple root of smallest magnitude.

In conclusion, by standard facts about linear recurrences as in Theorem 4.1.1 and Corollary 4.2.1 in [6], the coefficients of the generating function $F(t)$ will satisfy $x_d = \rho \cdot \lambda^d + o(\lambda^d)$, with nonzero ρ. This gives the same conclusion about C_d.

Part (3) follows at once since $d_n = \sum_{d=0}^n C_d$.

□

Example 2.26. (1) Using the values from Table 2.3, we can find that the growth equation for R for $p = 7$ is

$$x^7 = x^6 + 2x^5 + 3x^4 + x^3 + 2x^2 + 3x + 2,$$

and the growth equation for R for $p = 11$ is

$$x^{10} = x^9 + 2x^8 + 3x^7 + 3x^6 + 2x^5 + x^4 + 3x^3 + 3x^2 + 3x + 1.$$

(2) Using Wolfram Alpha [7], we can compute the roots of the respective growth equations and establish that for $p = 7$, $\mathrm{GKbase}_F(R) \approx 2.46646$, while for $p = 11$, $\mathrm{GKbase}_F(R) \approx 2.52555$.

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References

[1] Dills, Alan, *Counting Generating Sets in Frobenius Skew Polynomial Rings*, Dissertation, Georgia State University, 2021. doi: https://doi.org/10.57709/23991366. 1, 12

[2] Dills, Alan; Enescu, Florian, *Counting skew monomials in the Frobenius skew polynomial ring,* Comm. Algebra, **51** (2022), no. 5, 1955–1968. doi: https://doi.org/10.1080/00927872.2022.2148256. 1, 2

[3] Krause, Günter R.; Lenagan, Thomas H. *Growth of algebras and Gelfand-Kirillov dimension.* Revised edition. Graduate Studies in Mathematics, **22**. American Mathematical Society, Providence, RI, 2000. x+212 pp. ISBN: 0-8218-0859-1. 1

[4] Prasolov, Victor V. *Polynomials.* Translated from the 2001 Russian second edition by Dimitry Leites. Algorithms and Computation in Mathematics, **11**. Springer-Verlag, Berlin, 2004. xiv+301 pp. ISBN: 3-540-40714-6 12

[5] Sharp, Rodney Y., *Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure.* Trans. Amer. Math. Soc. **359** (2007), no. 9, 4237–4258. 1

[6] Stanley, Richard. P., *Enumerative Combinatorics*, volume 1, 2nd edition, Cambridge Studies in Advances Mathematics 49, Cambridge Univ. Press, ISBN 978-1107-60262-5, 2012. 12

[7] Wolfram Alpha LLC, *wolframalpha.com*, 2023. 12

[8] Yoshino, Yuji, *Skew-polynomial rings of Frobenius type and the theory of tight closure.* Comm. Algebra **22** (1994), no. 7, 2473–2502. 1

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